Hamel bases and measurability

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Abstract. This is a note - set in the background of some historic comments - discussing the relationship between measurability and Hamel bases for $\mathbb{R}$ over $\mathbb{Q}$. We explicitly note that such a basis must necessarily fail to be Borel measurable (or even 'analytic' in the sense of descriptive set theory). We also discuss some constructions in the literature which yield Hamel bases which even fail to be Lebesgue measurable, and discuss an elementary construction of a Hamel basis which is Lebesgue measurable.

It is customary in a course on measure theory, after the construction of Lebesgue measure, to show that not all sets are Lebesgue measurable by giving Vitali's example of non-measurable set. One way of describing this set is as follows: consider the dense subgroup $D = \{m + \sqrt{2}n : m, n \in \mathbb{Z}\}$ of the additive group $\mathbb{R}$ of real numbers and 'pick one point from each coset' to form a set V. Performing the directions within quotes requires the use of 'axiom of choice'. Now $\cup_{d\in D}(V + d) = \mathbb{R}$, and the sets $\{V + d : d \in D\}$, are pairwise disjoint.

Assertion: The set V is not Lebesgue measurable.

(*Reason:* If V were Lebesgue measurable, then each $V+d$ would also be Lebesgue measurable; and $\{V + d : d \in D\}$ is would be a countable partition of $\mathbb{R}$ into Lebesgue measurable sets. Then at least one $V + d$, and hence V itself, would have positive Lebesgue measure (by translation invariance of the Lebesgue measure.) Since the 'difference set' of a set with positive Lebesgue measure contains an open interval around the origin (see J. C. Oxtoby [2], Theorem 4.8), the set $\{x - y : x, y \in V\} = V - V$ must contain an open interval. This would imply,since D is dense in $\mathbb{R}$, that there are x, y in V such that $x - y \neq 0$ and $x - y \in D$, i.e., x, y are distinct and belong to the same coset of D contrary to the way V was defined. So V is indeed not Lebesgue measurable and the assertion is proved.

Note: Zermelo's axiom of choice is used to construct the set V, so *there is no concrete description of* V.

Around the same time that Vitali obtained his example, independently, Paul Lévy told Lebesgue his discovery of a similar example. Moreover Lévy was able to enlarge the class of sets to which Lebesgue measure can be extended in a translation invariant manner. Lebesgue, who was not interested in arguments depending on Zermelo's axiom of choice, led Lévy away from further study of these sets. (Lévy also later came to know of Vitali's construction (Paul Lévy [1]).)

Lévy returned to these ideas nearly forty years later, equipped with the knowledge of Hamel bases and some results of H. Cartan and G. Choquet (Paul Lévy [1]). Recall that every vector space V over a field k, possesses a *Hamel basis*: i.e., there is a set $B \subset V$ such that any element of V is uniquely expressible as a finite linear combination of elements from B. The proof of this statement, in its full generality, requires the axiom of choice (or one of its manifold equivalent forms).

In particular, let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, which may be chosen to contain the number 1.

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Clearly, the set H cannot be countable, since $\mathbb{R}$ is not. Let us index elements of H as ω_ν, $\nu \in [0,1]$, $\omega_0 = 1$. Lévy's description of a non-Lebesgue measurable set using Hamel basis (Paul Lévy [1]) goes thus:

For $x \in \mathbb{R}$, let $a_\nu(x)$ be the co-efficient of ω_ν in the expansion of x as a finite rational combination of elements from H. For a rational r write

$$A_r = \{x : a_0(x) = r\}.$$

Note that $\omega_0 = 1$, and $a_0(x)$ is the co-efficient of ω_0 in the expansion of x. It is trivial to verify that (1) $A_r = A_0 + r$, (2) $A_r \cap A_s = \emptyset$ if $r \neq s$, (3) $\cup_{r\in\mathbb{Q}} A_r = \mathbb{R}$. It follows as in Vitali's example that all the A_r's are not Lebesgue measurable sets.

What should be noted is that once the existence of Hamel basis is granted, the description of A_r does not use the axiom of choice. Indeed, given the set H, the description of A_0 is even quite 'concrete', so one might wonder how 'good' the set H could be. The next proposition asserts that H cannot be very 'good'.

Before stating the proposition, we would like to recall that a subset of $\mathbb{R}$ is said to be an analytic set (in the sense of descriptive set theory) if it is the continuous image of a Borel measurable set. It is a fact that analytic sets are Lebesgue measurable - see J. C. Oxtoby [2], p 22 - and of course Borel sets are analytic.

PROPOSITION 0..1. *No Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ is Borel measurable or even analytic.*

Proof: Let H be any Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Without loss of generality we may assume that $1 \in H$. Let $K = H \setminus \{1\}$, K^n the n-fold Cartesian product of K with itself. For a positive integer n and an n-tuple $\mathbf{r} = (r_1, r_2, \cdots, r_n)$ of rational numbers, define a function $f_\mathbf{r} : K^n \to \mathbb{R}$ by

$$f_\mathbf{r}(k_1, k_2, \cdots, k_n) = \sum_{i=1}^{n} r_i k_i.$$

Suppose H is Borel measurable or analytic; then so are K and K^n. Since $f_\mathbf{r}$ is continuous, we see that the image $f_\mathbf{r}(K^n)$ is analytic. The assumption that H is a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, and the fact that $K = H \setminus \{1\}$, imply that the set we called A_0 earlier is nothing but the (countable) union of the analytic sets $f_\mathbf{r}(K^n)$, and consequently also analytic. Since analytic sets are Lebesgue measurable, we see that A_0, hence each A_r, should also be Lebesgue measurable. The contradiction shows that H could not have been analytic, thereby proving the proposition.

□

We conclude with an assorted collection of remarks regarding Hamel bases and their measurability properties or otherwise:

(1) The question of whether a Hamel basis can be the complement of an analytic set (a so called coanalytic set) or a continuous image of one cannot be settled by the above method; indeed this question seems to be tied to deep set theory, since a result of Gödel says that the statement 'there is a non-Lebesgue-measurable continuous image of a co-analytic set' is consistent with axioms of Zermelo-Frankel set theory, provided these axioms are consistent among themselves. (J. C. Oxtoby [2], p 22)

(2) A Hamel basis H can not contain a subset of positive Lebesgue measure. For if it did, the difference set $H - H$ would contain an open interval. Every real number can then be written as $r(x - y)$ with r rational and $x, y \in H$, which is not possible.

(3) H. Cartan and G. Choquet show, - using the 'well-ordering theorem' (which is another variant of the axiom of choice), that a Hamel basis can be chosen to intersect every perfect subset of $\mathbb{R}$. They further show, as a consequence, that the additive group of real numbers can be written as a direct sum of countable number of subgroups $R_n, n = 1, 2, 3 \ldots$ such that each R_n

has outer measure $b - a$ in each interval (a, b). (Paul Lévy [1]).

(4) *A Hamel basis can be Lebesgue measurable*. In order to see this, it will suffice for us to show that the usual Cantor ternary set C contains a Hamel basis - since every set contained in a set of Lebesgue measure zero is Lebesgue measurable. It is not hard to see that

$$C + C \ (= \{x + y : x, y \in C\} \ ) = [0, 2],$$

so C spans $\mathbb{R}$ as a vector space over $\mathbb{Q}$, and hence C must contain a Hamel basis (which (is nothing but a minimal spanning set of $\mathbb{R}$).

References

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