§2 Haar state and other basics on quantum groups

2.1 <u>Remark</u>: One of the very striking features of Woronowicz' definition of a quantum group is the existence of a Haar state. Let us first consider the classical case, i.e. let G be a compact group. Then there is a unique Haar measure μ_G on G such that $\int_G f(ts)\,d\mu_G(s) = \int_G f(s)\,d\mu_G(s)$ for all $f \in \mathcal{C}(G)$, $t \in G$ (Riesz' theorem).

Hence we have a positive linear functional $h: \mathcal{C}(G) \to \mathbb{C}$, $h(f) := \int f\,d\mu_G$.

Thus, we should be able to find a functional on the C^*-algebra associated to our quantum group, which „looks like" integrating with respect to μ_G.

How to capture the left invariance? We need to express it in „quantum terms".

Observe: $(id \otimes h)\Delta(f)(t) = \int \Delta f(t,s)\,d\mu_G(s) = \int f(ts)\,d\mu(s) = \int f(s)\,d\mu(s) = h(f)(t)$

2.2 <u>Theorem</u> [4, Th. 4.2] [5, Th. 2.3], [20]: Let (A, Δ) be a compact quantum group. There is a unique state $h: A \to \mathbb{C}$ („the <u>Haar state</u>") such that $(id \otimes h)\Delta = (h \otimes id)\Delta = h$.

Proof: Put $g * h := (g \otimes h) \circ \Delta$ for linear functionals $h, g \in A'$, and $a * h := (id \otimes h)\Delta(a)$ for $a \in A$.

We thus want to prove $a * h = h * a = h(a)$ for a unique state $h \in A'$.

Uniqueness: If h' is another such state, then

$$h'(a) = h(h'(a)1) = h((id \otimes h')\Delta(a)) \underset{\uparrow}{=} h \otimes h'\Delta(a) \underset{\uparrow}{=} h'((h \otimes id)\Delta(a)) = h'(h(a)) = h(a)$$

(check for $\Delta(a) = a_1 \otimes a_2$ etc)

Existence: For $\omega \in A'$ pos., define $K_\omega := \{\varrho \in A' \text{ state} \mid \varrho * \omega = \omega * \varrho = \omega(1)\varrho\} \subseteq \{\varrho \in A' \text{ pos.}, \|\varrho\| \leq 1\} =: A'_1$

1.) A'_1 is compact (weak*-topol.), $K_\omega \neq \emptyset$, $K_\omega \subseteq A'_1$ closed.

2.) $\bigcap_{i=1}^n K_{\omega_i} \neq \emptyset$ for all $\omega_i \in A'$ pos., $n \in \mathbb{N}$.

3.) By Cantor's intersection principle and 2.), we have $\exists h \in \bigcap_{\omega \in A' \text{ pos.}} K_\omega \neq \emptyset$. It satisfies $a * h = h * a = h(a)$.

Proof of 1.): Let $\omega \in A'$ be a state. Put $\omega_n := \frac{1}{n}\sum_{k=1}^n \omega^{*k}$. As A'_1 is compact, $(\omega_n)_{n \in \mathbb{N}}$ has an accumulation point ϱ. Since $\|\omega * \omega_n - \omega_n\| = \frac{1}{n}\|(\omega^{*(n+1)} - \omega)\| \leq \frac{2}{n}$, we have $\omega * \varrho = \varrho$, likewise $\varrho * \omega = \varrho$.

Proof of 2.): Let $\omega_1, \omega_2 \in A'$ be positive, $\omega := \omega_1 + \omega_2$. We prove $K_\omega \subseteq K_{\omega_1}$. Hence also $\emptyset \neq K_\omega \subseteq K_{\omega_1} \cap K_{\omega_2}$ and iteratively $\emptyset \neq K_{\Sigma\omega_i} \subseteq \bigcap K_{\omega_i}$.

Lit.: [20] Van Daele, The Haar measure on a compact quantum group, 1995
[21] Wang, Tensor products and crossed products of CQG, 1995
[22] Bichon, Quantum automorphism groups of finite graphs, 2003

Assume that ω is a state. Let $\rho \in K_\omega$ and put

$$L_{\rho\otimes\omega} := \{q \in A\otimes A \mid \rho\otimes\omega(q^*q)=0\} \underset{(\omega_1\leq\omega)}{\subseteq} L_{\rho\otimes\omega_1} \underset{\text{Cauchy-Schwarz}}{\subseteq} \ker \rho\otimes\omega_1 .$$

One can show that $(\mathrm{id}\otimes\Psi_L)\Delta(A) \subseteq L_{\rho\otimes\omega}$ for $\Psi_L : A \to A$, $a \mapsto \rho * a - 1\rho(a)$.

Thus $1\otimes\Psi_L(A) \underset{[\dots]}{\subseteq} (A\otimes 1)(\mathrm{id}\otimes\Psi_L)\Delta(A) \subseteq (A\otimes 1)L_{\rho\otimes\omega} \underset{L_{\rho\otimes\omega} \text{ is a left ideal}}{\subseteq} \ker \rho\otimes\omega_1$

Hence $0 = \rho\otimes\omega_1(1\otimes\Psi_L(a)) = \omega_1((\rho\otimes\mathrm{id})\Delta(a) - 1\rho(a)) = \rho*\omega_1(a) - \omega_1(1)\rho(a)$
$\Rightarrow \rho \in K_{\omega_1}$

Proof of 3.): For $h \in \bigcap K_\omega$, we have $h*\omega = \omega*h = \cancel{\omega(1)}h$ $\forall \omega\in A'$ states, i.e. if $a\in A$, then we have $\omega(b)=0$ $\forall\omega\in A'$ states, where $b := h(a) - \mathrm{id}\otimes h\,\Delta(a) \in A$. Hence $b=0$. We conclude $a*h = h*a = h(a)$ $\forall a\in A$.
[3, Th. 5.1.6] □

2.3 Remark: The Haar state is not always faithful. Let G be a discrete group. Then $(C^*_{max}(G),\Delta)$ is a compact quantum group according to Ex. 1.6.
Its Haar state is $h_{max}: C^*_{max}(G)\to\mathbb{C}$, $h_{max}(u_g)=\delta_{g,e}$, since
$(\mathrm{id}\otimes h_{max})\Delta(u_g) = (\mathrm{id}\otimes h_{max})(u_g\otimes u_g) = \delta_{g,e}u_e = \delta_{g,e}1 = h_{max}(u_g)$.
It is tracial, i.e. $h(ab)=h(ba)$ $\forall a,b\in C^*_{max}(G)$.
The Haar state on $(C^*_{red}(G),\Delta)$ is given by $h_{red}: C^*_{red}(G)\to\mathbb{C}$, $h_{red}(x) := \langle x\delta_e,\delta_e\rangle$ for $C^*_{red}(G)\subseteq B(\ell^2(G))$ with ONB $(\delta_g)_{g\in G}$. Indeed for $\alpha: C^*_{max}(G)\to C^*_{red}(G)$ we have:
$(\mathrm{id}\otimes h_{red})\Delta_{red}(\alpha(u_g)) = (\mathrm{id}\otimes h_{red})(\alpha\otimes\alpha)\Delta_{max}(u_g) = \alpha(u_g)h_{red}(\alpha(u_g)) = \alpha(u_g)\langle\delta_g,\delta_e\rangle$
$= \delta_{g,e} = h_{red}(\alpha(u_g))$

Thus
$$\begin{tikzcd} C^*_{max}(G) \arrow[r, "\alpha"] \arrow[rr, bend right, "h_{max}"'] & C^*_{red}(G) \arrow[r, "h_{red}"] & \mathbb{C} \end{tikzcd}$$
is a commutative diagram.

Now, if α is not an isomorphism (i.e. if G is not amenable), then h_{max} is not faithful. On the other hand, h_{red} is faithful on $C^*_{red}(G)$:
Let $R_s\in\mathbb{B}(\ell^2(G))$ be the right shift, defined by $R_s\delta_t := \delta_{ts}$.
Then $R_s x = xR_s$ $\forall x\in C^*_{red}(G)$ as $R_s\lambda(u_g)\delta_t = \delta_{gts} = \lambda(u_g)R_s\delta_t$.
Now $h_{red}(x^*x)=0$ implies $\langle x\delta_e, x\delta_e\rangle = 0$ and hence $x\delta_e = 0$.
But $x\delta_g = xR_g\delta_e = R_g x\delta_e = 0$ $\forall g\in G$, thus $x=0$.

2.4 Definition: Let $G(A,\Delta)$ be a compact quantum group and let h be its Haar state. The <u>reduced version</u> of G is given by (A_{red},Δ_{red}), where $A_{red} := \pi_h(A)\subseteq B(H_h)$ by GNS construction (π_h,H_h) with respect to h, and $\Delta_{red}(\pi_h(x)) := (\pi_h\otimes\pi_h)\Delta(x)$. We may also <u>associate a von Neumann algebra</u> $L(G)$ to G.

2.5 Prop.: Δ_{red} in the previous definition is well-defined and

$$\begin{tikzcd} A \arrow[r, "\pi_h"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A_{red} \arrow[d, "\Delta_{red}"] \\ A\otimes A \arrow[r, "\pi_h\otimes\pi_h"] & A_{red}\otimes A_{red} \end{tikzcd}$$

is commutative. Moreover, $h_{red}: A_{red}\to\mathbb{C}$ defined by $h_{red}(\pi_h(x)) := h(x)$ is the Haar state on (A_{red},Δ_{red}) and it is faithful.

Proof: Let $\pi_h(x)=0$, thus $h(x^*x)=0$. But then
$(h\otimes h)\Delta(x^*x) = h(id\otimes h)\Delta(x^*x) = h(h(x^*x)) = h(x^*x) = 0$ and
thus $\Delta(x)\in\ker\pi_{h\otimes h} = \ker\pi_h\otimes\pi_h$. Hence Δ_{red} is well-defined.
Moreover $(id\otimes h_{red})\Delta_{red}(\pi_h(x)) = (id\otimes h_{red})(\pi_h\otimes\pi_h)\Delta(x) = \pi_h(id\otimes h)\Delta(x) = \pi_h(h(x)) = h(x)$
$= h_{red}(\pi_h(x))$,
faithfulness hence h_{red} is the Haar state.
~~Finally $h_{red}(\pi_h(x^*x))=0 \Rightarrow h(x^*x)=0 \Rightarrow \pi_h(x^*x)=0$, i.e. h_{red} is faithful.~~
□

2.6 Definition: A compact quantum group is of Kac type, if the Haar state is a trace, i.e. if $h(ab)=h(ba)$ $\forall a,b$. [36, Def. 1.7.4]

~~2.7 Prop.: If G is a compact quantum group~~

Later, we will see other characterizations of Kac type, and examples of such QG.
We end with a few constructions of new QG's out of given ones.

2.7 Prop.: (Free product) Let (A,Δ_A), (B,Δ_B) be CQG. Then the free prod. $(A,\Delta_A)*(B,\Delta_B)$ is given by $(A*_{\mathbb{C}}B,\Delta)$, where

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "\iota_A\otimes\iota_A\circ\Delta_A"'] & A*_{\mathbb{C}}B \arrow[d, "\Delta"] & B \arrow[l, hook', "\iota_B"'] \arrow[dl, "\iota_B\otimes\iota_B\circ\Delta_B"] \\ & (A*B)\otimes(A*B) & \end{tikzcd}$$

Proof: Δ exists by the universal property of a free product: If C is any unital C^*-algebra with unital $*$-hom's $f: A\to C$, $g: B\to C$, then there is a unique unital $*$-hom. $f*g: A*B\to C$ such that

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "f"'] & A*_{\mathbb{C}}B \arrow[d, "f*g"] & B \arrow[l, hook', "\iota_B"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

⇝ [6]
[3, Sect. 6.3]

2.8 Remark: (a) If (A,Δ_A), (B,Δ_B) are CMQG, so is $(A,\Delta_A)*(B,\Delta_B)$.
(b) The Haar state h on $(A,\Delta_A)*(B,\Delta_B)$ is the free product of the Haar states h_A of (A,Δ_A) and h_B of (B,Δ_B) in the sense of free probability, i.e. $h\circ\iota_A = h_A$, $h\circ\iota_B = h_B$, $\forall c_1,\dots,c_n\in A$ or B such that $h_A(c_i)=0$ resp. $h_B(c_i)=0$ and c_i, c_{i+1} come from different algebras: then $h(c_1\cdots c_n)=0$.

2.9 <u>Prop.:</u> (Tensor product) Let (A, Δ_A), (B, Δ_B) be CQG. Then

$(A, \Delta_A) \otimes (B, \Delta_B) := (A \otimes_{max} B, \varsigma \circ (\Delta_A \otimes \Delta_B))$, where $\varsigma : (A \otimes A) \otimes_{max} (B \otimes B) \longrightarrow (A \otimes_{max} B) \otimes (A \otimes_{max} B)$ is a CQG. [21][3, Sect. 6.3]

The Haar state is $h_A \otimes h_B$ and it is a CMQG if the original ones are.

2.10 <u>Prop.:</u> (free wreath product) Let (A, Δ_A) be a CQG. The free wreath product with S_n^+ is given by

$$B := \frac{A^{*n} * C(S_n^+)}{\langle \iota_k(a) u_{ki} - u_{ki} \iota_k(a),\ 1 \leq i, k \leq n \rangle}$$

where $\iota_k : A \longrightarrow A^{*n}$ is the embedding as the k-th copy. The comultiplication is given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $\Delta(\iota_k(a)) = \sum_\ell (\iota_k \otimes \iota_\ell)(\Delta_A(a))(u_{k\ell} \otimes 1)$.

2.11 <u>Example:</u> If Γ is a graph, then $\Gamma^{\sqcup n}$ is the graph obtained from copying Γ n-times. For instance $(\circ\!-\!\circ)^{\sqcup n} = \circ\!-\!\circ\ \circ\!-\!\circ \ldots \circ\!-\!\circ$. In the classical theory, we have $Aut(\Gamma^{\sqcup n}) = Aut(\Gamma) \wr S_n$. [15]

Now, H_n^+ of Example 1.17 may be written as $H_n^+ = \mathbb{Z}_2 \wr_* S_n^+$.
Since $\mathbb{Z}_2 = Aut(\circ\!-\!\circ)$, we have that H_n^+ is the quantum automorphism group of $\circ\!-\!\circ \ldots \circ\!-\!\circ$. [15][14, Th. 5.5]

2.12 <u>Remark:</u> See [21] for how to equip the C^*-algebra $A \rtimes_\alpha \Gamma$ with a CQG structure. [22] [7]

§ 3 Representation Theory and Tannaka-Krein

Lit.: [23] Van Daele, Dual pairs of Hopf $*$-algebras, 1993
[24] Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups..., 1988

3.1 Remark: In order to motivate the definition of a representation of a quantum group, let us first consider the classical case. Let G be a compact group and let $U: G \to B(H)$ be a finite-dimensional representation, i.e. $\dim H = n$ and U is continuous. Thus $U \in C(G, M_n(\mathbb{C})) = C(G) \otimes M_n(\mathbb{C})$, hence $U = \sum_{\alpha\beta=1}^{n} u_{\alpha\beta} \otimes e_{\alpha\beta}$, where $e_{\alpha\beta}$ are the matrix units of $M_n(\mathbb{C})$. Now, U is a group homomorphism, i.e.

$$u(gh) = \sum_{\alpha\beta} u_{\alpha\beta}(gh) \otimes e_{\alpha\beta} = \sum_{\alpha\beta} \Delta(u_{\alpha\beta})(g,h) \otimes e_{\alpha\beta} \qquad (e_{\alpha\gamma} e_{\gamma'\beta} = \delta_{\gamma\gamma'} e_{\alpha\beta})$$

$$\| $$

$$u(g)u(h) = \Big(\sum_{\alpha\gamma} u_{\alpha\gamma}(g) \otimes e_{\alpha\gamma}\Big)\Big(\sum_{\gamma'\beta} u_{\gamma'\beta}(h) \otimes e_{\gamma'\beta}\Big) = \sum_{\alpha\beta} \underbrace{\Big(\sum_\gamma u_{\alpha\gamma}(g) u_{\gamma\beta}(h)\Big)}_{= \sum_\gamma (u_{\alpha\gamma} \otimes u_{\gamma\beta})(g,h)} \otimes e_{\alpha\beta}$$

$$\Rightarrow \Delta(u_{\alpha\beta}) = \sum_\gamma u_{\alpha\gamma} \otimes u_{\gamma\beta}$$

3.2 Definition: Let (A,Δ) be a compact quantum group (or simply a unital C^*-algebra with a $*$-hom. $\Delta: A \to A \otimes A$). A <u>finite-dimensional representation</u> is a matrix $(u_{\alpha\beta}) \in M_n(A)$ such that $\Delta(u_{\alpha\beta}) = \sum_\gamma u_{\alpha\gamma} \otimes u_{\gamma\beta}$. If the matrix has an inverse, then it is called <u>non-degenerate</u>, if it is unitary, then $(u_{\alpha\beta})$ is a <u>unitary representation</u>. (We omit the def. of an infinite-dim. unitary repr., see [1, Def. 5.1] (uses multiplier alg.)) [1, Def. 3.5]

3.3 Theorem: Let A be a unital C^*-algebra with a unital $*$-homomorphism $\Delta: A \to A \otimes A$.

(a) If A is generated (as a normed algebra) by the matrix elements of its non-degenerate finite-dimensional representations, then (A,Δ) is a compact quantum group.

(b) If A is generated (as a C^*-algebra) by elements (u_{ij}) such that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $u = (u_{ij})$, $u^t = (u_{ji})$ are invertible, then (A,Δ) is a compact quantum group.

Proof: (a) Δ is coassociative: $(\mathrm{id} \otimes \Delta)\Delta(u_{\alpha\beta}) = (\mathrm{id} \otimes \Delta)(\sum_\gamma u_{\alpha\gamma} \otimes u_{\gamma\beta}) = \sum_{\gamma\delta} u_{\alpha\gamma} \otimes u_{\gamma\delta} \otimes u_{\delta\beta} = (\Delta \otimes \mathrm{id})\Delta(u_{\alpha\beta})$ for all non-degenerate fin.-dim. repr., hence also on the closure.

Now, let $(w_{\alpha\beta})$ be the inverse of a non-degenerate fin.-dim. repr. $(u_{\alpha\beta})$.

Then $\sum_\gamma \Delta(u_{\alpha\gamma})(1 \otimes w_{\gamma\beta}) = \sum_{\gamma\varepsilon} u_{\alpha\varepsilon} \otimes u_{\varepsilon\gamma} w_{\gamma\beta} = \sum_\varepsilon u_{\alpha\varepsilon} \otimes \delta_{\varepsilon\beta} = u_{\alpha\beta} \otimes 1$, i.e. $u_{\alpha\beta} \otimes 1 \in \Delta(A)(1 \otimes A)$.

Thus, if $a \otimes 1, b \otimes 1 \in \Delta(A)(1 \otimes A)$, then also $ab \otimes 1 \in \Delta(A)(1 \otimes A)$ by checking:

$$a \otimes 1 = \sum_k \Delta(x_k)(1 \otimes y_k) \quad \Rightarrow \quad ab \otimes 1 = (a \otimes 1)(b \otimes 1) = \sum_k \Delta(x_k)(b \otimes 1)(1 \otimes y_k)$$

$$b \otimes 1 = \sum_l \Delta(z_l)(1 \otimes t_l) \qquad\qquad = \sum_{kl} \Delta(x_k z_l)(1 \otimes t_l y_k) \in \Delta(A)(1 \otimes A)$$

Therefore the algebra generated by $(u_{\alpha\beta})$ is in $\Delta(A)(1 \otimes A)$, i.e. $A \otimes 1 \subseteq \overline{\Delta(A)(1 \otimes A)} \subseteq A \otimes A$ $\Rightarrow$ "$=$".

(b) $\bar{u} = (u_{ij}^*) = (u^t)^*$ is invertible since u^t is, and $\Delta(u_{ij}^*) = \Delta(u_{ij})^* = \sum_k u_{ik}^* \otimes u_{kj}^*$.

Thus, u and $\bar{u}$ are non-degenerate representations and we may use (a). $\square$

[1, Sect. 3]

3.4 <u>Corollary</u>: Every compact matrix quantum group is a compact quantum group.

3.5 <u>Remark</u>: In fact, the converse of Th. 3.3 is also true: We can find a dense $*$-subalgebra of A which is generated by the matrix elements of fin.-dim. unitary representations of (A,Δ). This yields an algebraic picture of the quantum group, a Hopf algebra. In [17, 11.3.1], Th. 3.3 is taken as definitions for CQG and CMQG in the algebraic setting.

3.7

3.6 <u>Theorem</u>: Representation theory of a compact quantum group (A,Δ).

(a) Every nondegenerate fin.-dim. representation is equivalent (i.e. $\exists\, x \in M_{n(u)\times n(v)}(\mathbb{C})$ invertible: $xu = vx$) to a unitary representation.

(b) Every unitary representation decomposes into a direct sum of irreducible finite-dim. representations.

(c) There is a (right) regular representation of (A,Δ) containing all irreducible unitary representations.

Proof: (a) Let v be a non-deg. fin.-dim. repr. We put $y := (\mathrm{id}\otimes \underset{\text{Haar state}}{h})(\underbrace{v^*v}_{\in M_n(A)}) \in M_n(\mathbb{C})$ and check $y \geq 0$ and $y \geq \varepsilon 1$. Thus $w := (y^{\frac{1}{2}}\otimes 1)v(y^{-\frac{1}{2}}\otimes 1)$ is well-defined and equivalent to v. Furthermore $w^*w = 1$, w invertible, hence unitary.

Check $w^*w = 1$ by checking $v^*(y\otimes 1)v = y\otimes 1$:

$$(\mathrm{id}\otimes h\otimes \mathrm{id})(\mathrm{id}\otimes\Delta)(v^*v) = \mathrm{id}\otimes((h\otimes \mathrm{id})\circ\Delta)(v^*v) = \mathrm{id}\otimes h(v^*v) = y\otimes 1$$

$$\shortparallel$$

$$(\mathrm{id}\otimes h\otimes \mathrm{id})(\mathrm{id}\otimes\Delta)\Big(\sum_{ijk} e_{ij}\otimes v_{ki}^* v_{kj}\Big) = (\mathrm{id}\otimes h\otimes \mathrm{id})\Big(\sum_{ijk\alpha\beta} e_{ij}\otimes v_{k\alpha}^* v_{k\beta}\otimes v_{\alpha i}^* v_{\beta j}\Big) = \sum_{ij\alpha\beta} e_{ij}\otimes \underbrace{\Big(\sum_k h(v_{k\alpha}^* v_{k\beta})\Big)}_{y_{\alpha\beta}} v_{\alpha i}^* v_{\beta j}$$

$$= v^*(y\otimes 1)v$$

(b) $B := \{y \in M_n(\mathbb{C}) \mid v(y\otimes 1) = (y\otimes 1)v\} \subseteq M_n(\mathbb{C})$ C^*-subalgebra for $v \in M_n(A)$ unitary. (Use $v^*v = 1$ to prove that it is $*$-closed.) Take a maximal family of mutually orthogonal, minimal projections. If v is not fin.-dim., work harder...

(c) Put $H := H_h$ the Hilbert space obtained from the GNS representation (H_h, π_h, ξ_0) with respect to the Haar measure. Let K be a Hilbert space on which A acts faithfully, non-degenerate via $\sigma: A \hookrightarrow B(K)$. Define $u \in B(H\otimes K)$ via $u(\pi_h(a)\xi_0\otimes\eta) := (\pi_h\otimes\sigma)\circ\Delta(a)(\xi_0\otimes\eta)$, $a\in A$, $\eta\in K$.

Check that this is a unitary representation, the right regular representation.

Let v be an irreducible unitary repr. acting on the Hilbert space L.

Let $x \in K(H\otimes K, L)$ and put $y := (\mathrm{id}\otimes h)(v^*(x\otimes 1)u)$. Similar to (a), we can check that $v^*(y\otimes 1)u = y\otimes 1$. If e is the projection onto the range of y, we have $(e\otimes 1)v(y\otimes 1) = (e\otimes 1)(y\otimes 1)u = (y\otimes 1)u = v(y\otimes 1)$ and hence $(e\otimes 1)v(e\otimes 1) = v(e\otimes 1)$; likewise $(e\otimes 1)v^*(e\otimes 1) = v^*(e\otimes 1)$ and thus $v(e\otimes 1) = (e\otimes 1)v$. Since v is irreducible, we have $e = 0$ or $e = 1$. Prove that the latter is the case (work to do...), hence y is surjective and we have $(y\otimes 1)u = v(y\otimes 1)$. Then v is equivalent to a subrepresentation of u.

$\square$

[1, Sect. G]

3.7 <u>Theorem</u>: Let (A, Δ) be a compact quantum group. Let A_0 be the subspace of A spanned by the matrix elements of all finite-dimensional unitary representations.
Then: (a) $A_0 \subseteq A$ is a dense $*$-subalgebra.

(b) $\Delta(A_0) \subseteq A_0 \odot A_0$ for $A_0 \odot A_0$ the algebraic tensor product.

(c) $(A_0, \Delta_{|A_0})$ is a Hopf $*$-algebra.

(d) The restriction $h_{|A_0}$ of the Haar measure of A is faithful on A_0.

We thus have a canonical algebraic quantum group sitting inside our C^*-algebraic one. Likewise we can pass from an algebraic q.g. to a C^*-algebraic one by standard completion ($\|x\| := \sup\{p(x) \mid p$ C^*-seminorm on $A_0\} \ldots$), see also [3, Sect. 5.4].

Proof: (a) <u>If</u> u_{ij}^α and u_{kl}^β are matrix elements of fin.-dim. unitary rep.s, their product is a matrix element of the tensor product of the two rep.s. Moreover, the adjoint of a fin.-dim. unitary rep. is equivalent to a unitary rep. Hence A_0 is a $*$-subalg. Density is a bit technical, but not too complicated.

(b) Same for this item

(c) By 3.6, we may restrict to irred. rep.s and it can be proven that their matrix elements form a basis for A_0, denoted by u_{ij}^α. Define linear maps

$$\varepsilon: A_0 \to \mathbb{C}, \quad u_{ij}^\alpha \mapsto \delta_{ij} \quad \text{and} \quad \cancel{S}\; S: A_0 \to A_0, \quad u_{ij}^\alpha \mapsto (u_{ji}^\alpha)^*. \quad \text{Then:}$$

$(\varepsilon \otimes id) \circ \Delta(a) = (id \otimes \varepsilon) \circ \Delta(a) = a$ since $id \otimes \varepsilon(\Delta(u_{ij}^\alpha)) = \sum_k u_{ik}^\alpha \varepsilon(u_{kj}^\alpha) = u_{ij}^\alpha$,

and $m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \varepsilon$ since

$$m \circ (id \otimes S) \circ \Delta(u_{ij}^\alpha) = \sum_k u_{ik}^\alpha S(u_{kj}^\alpha) = \sum_k u_{ik}^\alpha (u_{jk}^\alpha)^* = \delta_{ij} = \varepsilon(u_{ij}^\alpha) \text{ as } u^\alpha \text{ is unitary.}$$

(d) Let $a \in A_0$. Then $a^* \overset{\varrho \otimes id(u^\alpha)}{=} \sum_{ij} \varrho(e_{ij}) u_{ij}^\alpha$ for some fin. dim. unitary rep. $u^\alpha \in M_n(A)$ and some linear functional ϱ on $M_n(\mathbb{C})$. ($\varrho(e_{ij}) = \underset{ev_{kl}(e_{ij})}{\underset{\parallel}{\delta_{ki}\delta_{lj}}}$ for $a = u_{kl}^{\alpha *}$ for instance)

Assume $h(a^*a) = 0$. Then $h(a^*b) = 0$ $\forall b \in A_0$ by Cauchy-Schwarz.

Thus $\varrho((id \otimes h)(u^\alpha(1 \otimes b))) = \varrho(id \otimes h(\sum_{ij} e_{ij} \otimes u_{ij}^\alpha b)) = \sum_{ij} \varrho(e_{ij}) h(u_{ij}^\alpha b) = h(a^*b) = 0$.

One can show that $B := \overline{\{(id \otimes h)(u^\alpha(1 \otimes b)) \mid b \in A\}}^{\|\cdot\|} \subseteq M_n(\mathbb{C})$ is a C^*-algebra and $u^\alpha \in B \otimes A$. Hence $\varrho(B) = 0$ and $a^* = \cancel{\varrho(u^\alpha) = 0}$ $\varrho \otimes id(u^\alpha) = 0$. $\square$

[1, Sect. 7]

3.8 Remark: (a) Counit ε and antipode S of a Hopf *-algebra are uniquely determined, ε is a *-homomorphism and S is an antihom.
(Furthermore $S(S(a^*)^*) = a \; \forall a$ and $\Delta \circ S = S \otimes S \circ \Delta^{op}$)

(b) If (A_0, Δ) is a Hopf-*-algebra such that A_0 is generated as an algebra by the matrix coefficients of fin.dim. unitary repr., then there is a CQG (A, Δ) such that A_0 occurs in A as in Th. 3.7. [3b, Sect. 1.6] [23, 2.3, 2.4]

(c) In general, the antipode on the C*-algebra needs not to be bounded. [3, p.105]

3.9 Prop.: Let (A, Δ) be a CQG and let (A_0, Δ_0) be the Hopf *-algebra inside of A according to Th. 3.7, with antipode S on A_0. TFAE: [3b, Th. 1.6.6]
(i) The Haar state is a trace (i.e. (A, Δ) is of Kac type)
(ii) $S^2 = id$
(iii) S is *-preserving

3.10 Examples: (a) Let $G \subseteq U_n^+$ be a CMQG and a sub QG. Hence $A = C(G)$ is generated by some u_{ij} ~~fulfilling~~ such that u, u^t are unitaries (plus other conditions). [3b, 1.7.7] see also [17, 11, 3, and page 15]
If now $\underset{u_{ij} \mapsto \delta_{ij}}{\varepsilon : A \to \mathbb{C}}$ and $\underset{u_{ij} \mapsto u_{ji}^*}{S : A \to A^{op}}$ are *-hom., then

$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id$: $\quad (\varepsilon \otimes id)\Delta(u_{ij}) = \sum_k \underbrace{\varepsilon(u_{ik})}_{\delta_{ik}} u_{kj} = u_{ij}$

$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = \varepsilon$: $\quad m \circ (S \otimes id) \circ \Delta(u_{ij}) = \sum_k S(u_{ik}) u_{kj} = \sum_k u_{ki}^* u_{kj} \overset{u^*u = 1}{=} \delta_{ij} = \varepsilon(u_{ij})$

$S^2 = id$: $\quad S \circ S(u_{ij}) = S(u_{ji}^*) = S(u_{ji})^* = u_{ij}$

This way, we may check that S_n^+, O_n^+, U_n^+ and H_n^+ are of Kac type, simply by checking that δ_{ij} and u_{ji}^* fulfill the relations of $C(S_n^+)$, $C(O_n^+)$, ... and using the universal property.
(For S_n^+: $C^*(u_{ij} \mid u_{ij}$ proj., $\sum_k u_{ik} = \sum_k u_{kj} = 1)$ okay for δ_{ij} and $u_{ji}^* = u_{ji}$)

(b) The CMQG $SU_q(2)$, $O_n^+(Q)$, $U_n^+(Q)$ are not of Kac type in general. This was one of the reasons for Woronowicz to introduce $SU_q(2)$ and his approach to QG: for Kac algebras we always have $S^2 = id$. Furthermore, recall that the Haar state on $(C^*_{max/red}(G), \Delta)$ for a discrete group G is a trace (Rem. 2.3). In this sense, Kac type QG are a bit closer to the classical world.

Why is $U_n^+(Q)$ not Kac? Recall that u and $Q\bar{u}Q^{-1}$ are unitaries for $U_n^+(Q)$. This implies $(Q^*Q)^{-1} u^t (Q^*Q)\bar{u} = 1$ since $1 = (Q\bar{u}Q^{-1})^*(Q\bar{u}Q^{-1}) = Q^{*-1} u^t (Q^*Q)\bar{u}Q^{-1}$.
Now $\varepsilon(u_{ij}) := \delta_{ij}$ and $S(u_{ij}) := u_{ji}^*$, $S(u_{ij}^*) := \sum_{lm} b_{il} u_{ml} a_{mj}$ for
$u \mapsto u^*$ $\qquad \bar{u} \mapsto (Q^*Q)^{-1} u^t (Q^*Q)$

$Q^*Q = (a_{ij})$, $(Q^*Q)^{-1} = (b_{ij})$ satisfy the Hopf algebra relations, e.g.
$m\circ(S\otimes \mathrm{id})\circ\Delta(u_{ij}^*) = \sum_k S(u_{ik}^*)u_{kj}^* = \sum_{klm} b_{il}\,u_{ml}\,a_{mk}\,u_{kj}^* = \delta_{ij} = \varepsilon(u_{ij})$.
this is $(Q^*Q)^{-1}u^t(Q^*Q)\bar{u}$
And S is not *-preserving.

3.11 Remark: We may also define the dual of a CQG in the spirit of a generalization of the Pontryagin duality. The role of characters (used to define the dual of an abelian compact group) is now played by unitary irred. reps.
Let (A,Δ) be a CQG and let $\{u^\alpha \mid \alpha\in I\}$ be its mutually non-equivalent unitary irred. reps. Let $A_0 \subseteq A$ be the Hopf algebra of Th. 3.7. Let B_0 be the space of linear functionals on A given by $x \mapsto h(ax)$ for $a\in A_0$. Then B_0 is a subalgebra of the dual A^* of A and it is of the form $B_0 = \bigoplus_{\alpha\in I} M_{n(\alpha)}(\mathbb{C})$.
We may find a comultiplication $\Delta_0 : B_0 \to M(B_0\odot B_0)$ into the multiplier algebra (or its completion).
Thus, B_0 is a discrete QG which is by definition a pair $(\bigoplus M_{n(\alpha)}(\mathbb{C}), \Delta : A\to M(A\otimes A))$ such that A is a multiplier Hopf *-algebra (see [VanDaele, 94] for definitions).
This generalizes Pontryagin duality: If G is a compact abelian group, then the unitary irred. reps of $(C(G),\Delta)$ are given by the characters of G and the dual QG B_0 from above is $C(\hat{G})$ for the group $\hat{G}$ of characters of G. [1, Sect. 8]

3.12 Definition: Let u and v be two representations of a CQG (A,Δ), i.e.
$u = \sum_{\alpha\beta} e_{\alpha\beta}\otimes u_{\alpha\beta} \in M_{n_u}(\mathbb{C})\otimes A$, $v = \sum_{\gamma\delta} e_{\gamma\delta}\otimes v_{\gamma\delta} \in M_{n_v}(\mathbb{C})\otimes A$. ~~We define~~ We define:

(a) $u\otimes v := \sum_{\alpha\beta\gamma\delta} e_{\alpha\beta}\otimes e_{\gamma\delta}\otimes u_{\alpha\beta}v_{\gamma\delta} \in (M_{n_u}(\mathbb{C})\otimes M_{n_v}(\mathbb{C}))\otimes A$.

(b) $\bar{u} := \sum_{\alpha\beta} e_{\alpha\beta}\otimes u_{\alpha\beta}^*$ "conjugate representation"

3.13 Remark: (a) u irred. $\Rightarrow$ $\bar{u}$ irred. [1, 6.9]
(b) u irred. and unitary $\Rightarrow$ $\bar{u}$ is equivalent with a unitary repr. [1, 6.10]

Pontryagin duality states that a locally comp. abelian group G may be reconstructed from its dual group $\hat{G}$. The classical Tannaka-Krein generalizes to compact (non-abelian) groups: They may be reconstructed from the finite-dimensional unitary representations. The space of representations may be presented by the intertwiner space $\{T \text{ linear} \mid Tu = vT$, where u,v are representations of the group$\}$.
Now, Woronowicz' Tannaka-Krein result [24] extends this fact to CQG's. We will sketch it now.

3.14 Prop.: Let $G=(A,u)$ be a CMQG (i.e. $u=(u_{ij})$ is the generating matrix, the "fundamental rep."). Denote by $\mathrm{Rep}\,G := \{$fin.-dim. unitary rep. of $G\}$, $\mathrm{Mor}(v,w) := \{T : H_v \to H_w \text{ lin.} \mid Tv = wT\}$ space of intertwiners, for $v,w \in \mathrm{Rep}\,G$ and $v \in B(H_v)\otimes A$, $w \in B(H_w)\otimes A$.

Then $\mathrm{Rep}\,G$ together with Mor is a concrete $^{(ii-v)}$ monoidal $^{(i,vi,vii)}$ W^*-category, i.e.

(i) Mor is closed under $\otimes$, i.e. $T_1 \in \mathrm{Mor}(u_1,v_1),\ T_2 \in \mathrm{Mor}(u_2,v_2) \Rightarrow T_1\otimes T_2 \in \mathrm{Mor}(u_1\otimes u_2, v_1\otimes v_2)$

(ii) Mor is closed under composition, i.e. $T_1 \in \mathrm{Mor}(u_1,u_2),\ T_2 \in \mathrm{Mor}(u_2,u_3) \Rightarrow T_2T_1 \in \mathrm{Mor}(u_1,u_3)$

(iii) Mor is closed under involution, i.e. $T \in \mathrm{Mor}(v,w) \Rightarrow T^* \in \mathrm{Mor}(w,v)$

(iv) $\mathrm{id} \in \mathrm{Mor}(v,v)\quad \forall v$

(v) $H_v = H_w,\ \mathrm{id} \in \mathrm{Mor}(v,w) \Rightarrow v = w$

(vi) $(u\otimes v)\otimes w = u\otimes(v\otimes w)\quad \forall u,v,w$

(vii) $\exists\, 1 \in \mathrm{Rep}\,G : H_1 = \mathbb{C},\ 1v = v1 = v\quad \forall v$

Furthermore, for all $v \in \mathrm{Rep}\,G$, there exists $\bar v \in \mathrm{Rep}\,G$ conjugate $\left("(v_{ij}^*)"\right)$, and the fundamental rep. u together with $\bar u$ generate $\mathrm{Rep}\,G$.

Proof: For (i–vii) simply check $(T_1\otimes T_2)(u_1\otimes u_2) = (T_1u_1)\otimes(T_2u_2) = (v_1T_1)\otimes(v_2T_2)$ etc. u and $\bar u$ generate $\mathrm{Rep}\,G$ since any irred. rep. is contained in $u\otimes u\otimes \bar u\otimes \bar u\otimes u\otimes \ldots \otimes \bar u$. By 3.6(3) this yields all of $\mathrm{Rep}\,G$. $\square$ [24, Th. 1.2]

3.15 Theorem ("Tannaka-Krein for CMQG"): If (R, Mor) is any concrete monoidal W^*-category such that some $u \in R$ exists and $\{u,\bar u\}$ generate R, then there is a CMQG $G=(A,u)$ such that $R = \mathrm{Rep}\,G$. Moreover $\underset{u \mapsto u'}{A \to B}$, if (B,u') is a CMQG such that $R \subseteq \mathrm{Rep}(B,u')$.

Proof: First, (R,Mor) may be "completed" in a suitable sense (taking direct sums, equivalences of rep.'s, subrepresentations), with same (A,u). [24, Prop. 2.7]

Hence $\hat R$ contains sufficiently many irreducible objects R_{irr} ([24, Prop. 2.4]) and we define $A_0 := \bigoplus_{\alpha\in R_{irr}} B(H_\alpha)'$, i.e. $a \in A_0$ is of the form $a = \sum_{\alpha\in R_{irr}} u^\alpha_\xi$, $\xi \in B(H_\alpha)'$ ($\leftarrow$ lin. functionals on $B(H_\alpha)$), finite sum.

Using $\otimes$ on R, we define a product on A_0, and $v \mapsto \bar v$ defines an involution. (Complex conjugation exists by [24, Prop. 2.3]). We define the norm as the supremum of all C^*-seminorms on A_0, which is in fact a norm as there is a faithful Haar state on A_0. Completion yields a C^*-algebra A, and $A_0 \subseteq A$ is exactly the Hopf algebra of 3.7. $\square$ [24, Sect. 3] and Th. 1.3