Introduction to (easy) quantum groups

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Table of contents

§1 Introduction

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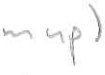
1.1 Example (Warm up): The study of symmetries arising in mathematics is an important tool in order to learn about the geometry of the considered objects, or to distinguish them (see for instance the fundamental group in topology). Let us consider three examples.

(a) The finite set $X_n = \{x_1, \dots, x_n\}$ consisting of n points has $\mathrm{Aut}(X_n) := \{f: X_n \to X_n \text{ bijective}\} = S_n$ as its automorphism group, the permutation group.

(b) The sphere $S^{\mathbb{C}}_{n-1} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\} \subseteq \mathbb{R}^n$ has O_n as its isometry group, i.e. $\mathrm{ISO}(S^{\mathbb{C}}_{n-1}) := \{A: S^{\mathbb{C}}_{n-1} \to S^{\mathbb{C}}_{n-1} \mid \langle Ax, Ay \rangle = \langle x, y \rangle\} = O_n$

(c) The cube  may be seen as the graph having 8 vertices and 12 edges. Its automorphism group is the group of all bijections α:  $\to$ , i.e. if the vertices v_i and v_j are connected by an edge e_{ij}, then $\alpha(e_{ij})$ has to connect $\alpha(v_i)$ and $\alpha(v_j)$. The automorphism group is $\mathbb{Z}_2^{\oplus 3} \rtimes S_3 =: \mathbb{Z}_2 \wr S_3$ (wreath product), or more generally $\mathbb{Z}_2 \wr S_n$ for the hypercube.

Let us check this for $n=2$, i.e. for $\square$. The automorphisms are:

e, α, β, σ, $\alpha\beta\sigma$, $\alpha\beta$, $\alpha\sigma$, $\beta\sigma$

Now $(\underset{\alpha}{\mathbb{Z}_2} \oplus \underset{\beta}{\mathbb{Z}_2}) \rtimes \underset{\sigma}{S_2} = \langle \alpha, \beta, \sigma \mid \alpha^2 = \beta^2 = e,\ \alpha\beta = \beta\alpha,\ \sigma^2 = e,\ \sigma\alpha\sigma = \beta \rangle$

has elements $e, \alpha, \beta, \sigma, \alpha\beta, \alpha\sigma, \beta\sigma, \alpha\beta\sigma$, and the above automorphisms fulfill the relations of $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes S_2$.
Note that the graph o—o o—o ... o—o has the same symmetry group.

1.2 <u>Reminder:</u> We want to study symmetries in an operator algebraic context. Recall the philosophy of C^*-/Von-Neumann-algebras:

X topl., compact / measurable $\xleftrightarrow{\text{dualize}}$ $\mathcal{C}(X) := \{f : X \to \mathbb{C} \text{ cont.}\}$ / $L^\infty(X) := \{f : X \to \mathbb{C} \text{ measurable, bounded}\}$ $\overset{fg \neq gf}{\rightsquigarrow}$ C^*-algebras / von N. algebras

$A \subseteq B(H)$ C^*-alg. $:\Leftrightarrow$ A norm closed $*$-subalgebra
$A \subseteq B(H)$ vNalg. $:\Leftrightarrow$ A weak/strong closed $*$-subalgebra
(H Hilbert space, $B(H)$ = bounded linear operators)

Fact: X topl., compact $\Rightarrow$ $\mathcal{C}(X)$ commutative C^*-algebra, unital
A comm. C^*-algebra, unital $\Rightarrow$ $\exists X$ $\big(= \operatorname{Spec} A := \{\varphi : A \to \mathbb{C} \text{ hom.}, \varphi \neq 0\}\big)$ topl., compact $A \cong \mathcal{C}(X)$

Hence: C^*-algebras $\widehat{=}$ „noncommutative topology" (see [13, Ch. 1])
C^*-alg. may be seen as (noncomm.) function algebras over „quantum spaces".

Likewise: von Neumann algebras $\widehat{=}$ „noncomm. measure theory"
There is also an abstract definition of C^*-algebras as a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$. But we have:

Fact (GNS construction): Let A be an (abstractly defined) C^*-algebra and let φ be a positive linear functional on A. Then there is a representation $\pi_\varphi : A \to B(H_\varphi)$ such that $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$ for some vector $\xi_\varphi \in H_\varphi$.

As a consequence, every C^*-algebra has a faithful representation on some Hilbert space, i.e. the abstract and the concrete definition coincide.
(Gelfand-Naimark)

The abstract definition of a C^*-algebra however enables us to construct C^*-algebras in a purely abstract way.

universal C^*-algebras: Let $E = \{x_i \mid i \in I\}$ be a set of generators.

Consider $P(E)$ the $*$-algebra of polynomials in $E \cup E^*$ $\left(P(E) = \mathrm{span}\{x_{i_1}^{a_1} x_{i_2}^{*a_2} \cdots x_{i_k}^{a_k(*)}\}\right)$.

Let $R \subseteq P(E)$ be a set of relations. Consider $J(R) \subseteq P(E)$ the ideal generated by R. Put $A(E,R) := \frac{P(E)}{J(R)}$. This is the „universal $*$-algebra generated by the generators E and the relations R".

We want to turn A into a C^*-algebra.

Put $\|x\| := \sup\{p(x) \mid p \text{ is } C^*\text{-seminorm on } A(E,R)\}$ for $x \in A(E,R)$

$$\left(p(\lambda x) = |\lambda| p(x),\ p(x+y) \leq p(x) + p(y),\ p(xy) \leq p(x)p(y),\ p(x^*x) = p(x)^2\right)$$

If now $\|x\| < \infty$ for all $x \in A(E,R)$, we put

$$C^*(E \mid R) := \overline{\frac{A(E,R)}{\{x \in A(E,R) \mid \|x\| = 0\}}}^{\|\cdot\|} \quad \text{(completion)}$$

This is the „universal C^*-algebra with generators E and relations R".

It has the „universal property" (more important to know than the precise construction): If B is any C^*-algebras with elements $\{y_i \in B \mid i \in I\}$ fulfilling the relations R, then there is a unique $*$-homomorphism $\varphi: C^*(E \mid R) \to B$, $x_i \mapsto y_i$

Example: $C^*(u, 1 \mid u^*u = uu^* = 1) \cong \mathcal{C}(S^1)$, $S^1 \subseteq \mathbb{C}$ sphere.

1.3 <u>Remark:</u> Coming back to viewing C^*-algebras as functions over „noncommutative spaces". What are arising symmetries?

Let X be a topol., compact space and let $(G, \circ)$ be a compact group acting on X. Applying the machinery of 1.2, we may let $\mathcal{C}(G)$ act on $\mathcal{C}(X)$, and we then turn $\mathcal{C}(X)$ noncommutative. How to turn $\mathcal{C}(G)$ noncommutative? Just replace it by a noncommutative C^*-algebra. But: $(G, \circ)$ is more than a set, it comes with a group law

$$\circ: G \times G \to G, \quad (s,t) \mapsto st.$$

Dualizing it to $\mathcal{C}(G)$ yields:

$$\Delta: \mathcal{C}(G) \to \mathcal{C}(G \times G), \quad f \mapsto ((s,t) \mapsto f(st))$$

Observing $\mathcal{C}(G \times G) \cong \mathcal{C}(G) \otimes \mathcal{C}(G)$, $((s,t) \mapsto f(s)g(t)) \longleftrightarrow f \otimes g$, we conclude that a definition of a quantum group should be a C^*-algebra A with some map $\Delta: A \to A \otimes A$.

1.4 <u>Definition</u> (Woronowicz 87/95) [4,5]: A <u>compact quantum group</u> is a unital C^*-algebra A together with a unital $*$-homomorphism $\Delta: A \to A \otimes_{min} A$ such that $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ (coassociativity) and $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are linearly dense in $A \otimes_{min} A$, i.e. $\text{Span}\{\Delta(a)(1\otimes b) \mid a,b \in A\} \subseteq A \otimes A$ dense.
We also write $A = \mathcal{C}(G)$ and call G the quantum group.

1.5 <u>Remarks</u>: (a) A quantum group is not a group!

(b) If G is a compact group, then $\mathcal{C}(G)$ is a unital C^*-algebra and $\Delta: \mathcal{C}(G) \to \mathcal{C}(G \times G) \cong \mathcal{C}(G) \otimes \mathcal{C}(G)$ is coassociative since:
$f \longmapsto ((s,t) \mapsto f(st))$

$(\Delta \otimes id)\Delta(f)(s,t,u) = \Delta(f)((st),u) = f((st)u) = f(s(tu)) = (id \otimes \Delta)\Delta(f)(s,t,u)$
(check for $\Delta(f) = f_1 \otimes f_2$, then linear combinations etc)

(c) This turns G into a compact quantum semigroup (via $(\mathcal{C}(G), \Delta)$) but what is the density condition for?

Fact (1): Let G be a compact semigroup. If it has cancellation, i.e. if $st = su$ implies $t = u$, then G is a group.

Fact (2): Let G be a compact semigroup and let $\Delta: \mathcal{C}(G) \to \mathcal{C}(G) \otimes \mathcal{C}(G)$ be defined as before. If now $\Delta(\mathcal{C}(G))(\mathcal{C}(G) \otimes 1)$ and $\Delta(\mathcal{C}(G))(1 \otimes \mathcal{C}(G))$ are dense in $\mathcal{C}(G) \otimes \mathcal{C}(G)$, then G has cancellation and is hence a group. [see 1, Prop. 3.2 & 3.3]

Thus, the density condition characterizes the step from a semigroup to a group.

(c) Every compact group is a compact quantum group (simply check that the density condition is fulfilled).
Conversely, let (A, Δ) be a compact quantum group such that A is commutative. Then $A \cong \mathcal{C}(G)$ for $G = \text{Spec}\, A = \{\varphi: A \to \mathbb{C} \text{ hom.}, \varphi \neq 0\}$ by Gelfand's theorem (see 1.2). Furthermore, $\Delta: \mathcal{C}(G) \to \mathcal{C}(G) \otimes \mathcal{C}(G) \cong \mathcal{C}(G \times G)$ yields a map $m: G \times G \to G$ by $m: \text{Spec}(A \otimes A) \longrightarrow \text{Spec}\, A$.
$\varphi \longmapsto \varphi \circ \Delta$

Since Δ is coassociative, m is associative and hence G is a compact semigroup which is a group by Fact (2) of (b). [see 3, Prop. 5.1.3]

We conclude: Compact quantum groups generalize compact groups.

1.6 Example: (a) Let G be a discrete group. The group algebra $\mathbb{C}G$ is given by $\mathbb{C}G = \{\sum_{g\in G} \alpha_g g \mid \alpha_g \neq 0$ only for finitely many $g \in G\}$ with $(\sum \alpha_g g)(\sum \beta_h h) := \sum \alpha_g \beta_h gh$ and $(\sum \alpha_g g)^* := \sum \overline{\alpha_g} g^{-1}$. It is a *-algebra. The <u>full group C^*-algebra</u> of G is the completion $C^*_{max}(G)$ of $\mathbb{C}G$ with respect to $\|x\| := \sup\{\|\pi(x)\| \mid \pi : \mathbb{C}G \to B(H)\ ^*\text{-rep.}\}$.

We can also write $C^*_{max}(G)$ as a universal C^*-algebra:

$$C^*_{max}(G) = C^*(u_g, g \in G \mid u_g u_h = u_{gh},\ u_{g^{-1}} = u_g^*) \quad (\text{note } u_e = 1)$$

As an example, check that $C^*_{max}(\mathbb{Z}) = C^*(u \text{ unitary}) = C(S^1)$ (see 1.2).

Now, $C^*_{max}(G)$ becomes a quantum group via $\Delta : C^*_{max}(G) \to C^*_{max}(G) \otimes C^*_{max}(G)$, $u_g \longmapsto u_g \otimes u_g$

(use the universal property of $C^*_{max}(G)$ to check the existence of Δ).

(b) The left regular representation of G is given by $\lambda : G \to B(\ell^2(G))$, $g \mapsto (\delta_h \mapsto \delta_{gh})$ for the orthonormal basis $(\delta_h)_{h\in G}$ of $\ell^2(G)$. It can be extended to $\lambda : \mathbb{C}G \to B(\ell^2(G))$ and λ is faithful. Thus $\mathbb{C}G$ is isomorphic to $\lambda(\mathbb{C}G)$ and we define the <u>reduced group C^*-algebra</u> of G by $C^*_{red}(G) := \overline{\lambda(\mathbb{C}G)} \subseteq B(\ell^2(G))$.

There is a natural *-homomorphism $C^*_{max}(G) \to C^*_{red}(G)$ letting the following diagram commute:

$$\begin{array}{ccc} C^*_{max}(G) & \xrightarrow{\Delta} & C^*_{max}(G) \otimes C^*_{max}(G) \\ \downarrow & & \downarrow \\ C^*_{red}(G) & \xrightarrow{\Delta_{red}} & C^*_{red}(G) \otimes C^*_{red}(G) \\ \lambda(g) & \longmapsto & \lambda(g) \otimes \lambda(g) \end{array}$$

Hence, also $(C^*_{red}(G), \Delta_{red})$ is a compact quantum group. [3, Ex. 5.1.2]

The examples $(C(G), \Delta)$, $(C^*_{max}(G), \Delta)$ and $(C^*_{red}(G), \Delta)$ all come from classical groups. Let us now come to honest quantum examples. Before doing so, let us define actions of quantum groups (also called coactions).

1.7 <u>Example</u>: Let $\alpha : G \times X \to X$, $(s, x) \mapsto sx$ be an action of a (classical) compact group G on a topol., compact space X. We thus obtain a *-homomorphism

$\tilde{\alpha} : C(X) \to C(G \times X) \cong C(G) \otimes C(X)$, $f \mapsto f \circ \alpha$. It satisfies $(\mathrm{id} \otimes \tilde{\alpha})\tilde{\alpha} = (\Delta \otimes \mathrm{id})\tilde{\alpha}$,

i.e. $(\mathrm{id} \otimes \tilde{\alpha})\tilde{\alpha}(f)(s,t,x) = \tilde{\alpha}(f)(s, tx) = f(s(tx)) \underset{\alpha \text{ associative}}{=} f((st)x) = \tilde{\alpha}(f)(st, x) = (\Delta \otimes \mathrm{id})\tilde{\alpha}(f)(s,t,x)$

1.8 Definition: An action of a compact quantum group (A,Δ) on a C^*-algebra B is a unital *-homomorphism $\alpha: B \to B \otimes_{min} A$ such that $(id \otimes \alpha)\alpha = (\Delta \otimes id)\alpha$ (coassociative).

1.9 Remark: Note that we may differ between left and right actions. Furthermore, note that some authors require further properties of α, see for instance [7, Def. 2.1]. However, we will always only need this „weak" version of an action.

1.10 Example: In Example 1.1(a), we computed $Aut(X_n) = S_n$ for $X_n = \{x_1, \dots, x_n\}$. We ~~know~~ may write the functions $C(X_n)$ over X_n as a universal C^*-algebra by $C(X_n) = C^*(p_1, \dots, p_n \mid p_i = p_i^* = p_i^2, \sum_{i=1}^n p_i = 1)$.
Note that the relations imply that the projections p_i are orthogonal, hence they commute.
Now, since the p_i form a basis of $C(X_n)$, every action on $C(X_n)$ is of the form $\alpha: C(X_n) \to C(X_n) \otimes A$, $\alpha(p_j) = \sum_{i=1}^n p_i \otimes a_{ij}$ for some $a_{ij} \in A$.
Since $\alpha(p_j) = \alpha(p_j)^*$, we have $a_{ij} = a_{ij}^*$ and $\alpha(p_j)^2 = \alpha(p_j)$ implies $a_{ij}^2 = a_{ij}$, as $\sum_{i,k} p_i p_k \otimes a_{ij} a_{kj} = \sum_i p_i \otimes a_{ij}^2$. Furthermore:

$$1 \otimes 1 = \alpha(\sum_j p_j) = \sum_j \alpha(p_j) = \sum_{ij} p_i \otimes a_{ij} = \sum_i (p_i \otimes (\sum_j a_{ij}))$$

Thus $\sum_j a_{ij} = 1$ for all i.
This lead Wang to the definition of the quantum permutation group S_n^+ via $C(S_n^+) := A_s(n) := C^*(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1)$ and $\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}$. Put $u_{ij}' := \sum_k u_{ik} \otimes u_{kj}$

Δ exists: $u_{ij}' = u_{ij}'^*$ and $u_{ij}'^2 = \sum_{kl} \underbrace{u_{ik} u_{il}}_{\delta_{kl}} \otimes u_{kj} u_{lj} = \sum_k u_{ik} \otimes u_{kj} = u_{ij}'$

$\sum_k u_{ik}' = \sum_k \sum_l u_{il} \otimes u_{lk} = \sum_l u_{il} \otimes (\sum_k u_{lk}) = \sum_l u_{il} \otimes 1 = 1 \otimes 1 = \sum_k u_{kj}'$

Δ coassoc.: $(\Delta \otimes id)\Delta(u_{ij}) = \sum_k \Delta(u_{ik}) \otimes u_{kj} = \sum_{kl} u_{il} \otimes u_{lk} \otimes u_{kj}$
$(id \otimes \Delta)\Delta(u_{ij}) = \sum_l u_{il} \otimes \Delta(u_{lj}) = \sum_{kl} u_{il} \otimes u_{lk} \otimes u_{kj}$ "

density: $\Delta(u_{ij})(1 \otimes u_{mj}) = \sum_k u_{ik} \otimes u_{kj} u_{mj} = u_{im} \otimes u_{mj}$.
Thus $u_{im} \otimes 1 = \sum_j u_{im} \otimes u_{mj} \in \Delta(A_s(n))(1 \otimes A_s(n))$ and $1 \otimes u_{mj} \in \Delta(\dots)(\dots)$.
Hence S_n^+ is a compact quantum group acting on $C(X_n)$. Wang showed that it is the quantum automorphism group of X_n in a reasonable sense. [7]

Note that $S_n \subseteq S_n^+$ in the following sense. Firstly, we have $C(S_n) = C(S_n^+)/\langle u_{ij}u_{kl} = u_{kl}u_{ij}\rangle$ by Gelfand's Theorem. Indeed, the coordinate functions $\tilde{u}_{ij} : S_n \to \mathbb{C}$, $g \mapsto g_{ij}$, where $S_n \subseteq M_n(\mathbb{C})$, fulfill the relations of $C(S_n^+)/\langle u_{ij}u_{kl} = u_{kl}u_{ij}\rangle$. This is since any permutation matrix $g = (g_{ij}) \in S_n \subseteq M_n(\mathbb{C})$ consists of entries $g_{ij} \in \{0,1\}$ (hence $g_{ij} = g_{ij}^* = g_{ij}^2$) and each row and each column sums up to 1. As the $\tilde{u}_{ij}$ generate $C(S_n)$ as a C^*-algebra, we have a surjection from $C(S_n^+)/\langle\ldots\rangle$ onto $C(S_n)$. Moreover, to any character $\varphi : C(S_n^+)/\langle\ldots\rangle \to \mathbb{C}$ we may associate a permutation matrix $(\varphi(\tilde{u}_{ij}))_{1\le i,j\le n}$; on the other hand any permutation matrix gives rise to a character this way. Thus $\mathrm{Spec}\, C(S_n^+)/\langle\ldots\rangle$ is homeomorphic to S_n and hence $C(S_n^+)/\langle\ldots\rangle \cong C(\mathrm{Spec}\, C(S_n^+)/\langle\ldots\rangle) \cong C(S_n)$.

The comultiplication on $C(S_n)$ is given by $\Delta : C(S_n) \to C(S_n \times S_n)$, $f \mapsto ((s,t) \mapsto f(st))$.

But the group multiplication on S_n is just matrix multiplication. Hence $\Delta(\tilde{u}_{ij})(s,t) = \tilde{u}_{ij}(st) = \sum_k \tilde{u}_{ik}(s)\tilde{u}_{kj}(t)$. Under the identification $C(S_n \times S_n) \cong C(S_n) \otimes C(S_n)$, we have $\tilde{u}_{ik}\tilde{u}_{kj} \leftrightarrow \tilde{u}_{ik} \otimes \tilde{u}_{kj}$.

We thus have a $*$-homomorphism $\varphi : C(S_n^+) \to C(S_n)$ such that $\Delta \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$. As φ is surjective, we have $S_n \subseteq S_n^+$ as quantum subgroups. [7]

1.11 Definition: (A, Δ_A) is a quantum subgroup of (B, Δ_B), if there is a surjection $\varphi : B \twoheadrightarrow A$ such that $\Delta_A \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_B$. [6, Def. 2.13]

Actually, S_n^+ fits into the setting of compact matrix quantum groups, which were defined by Woronowicz before compact quantum groups. At the time he called them compact matrix pseudogroups.

1.12 Definition [4]: A compact matrix quantum group (CMQG) is a unital C^*-algebra A which is generated by elements $u_{ij} \in A$, $1 \le i,j \le n$ (such that $*\text{-alg}(u_{ij}, 1) \subseteq A$ dense). Furthermore, the $*$-homomorphism $\Delta : A \to A \otimes_{\min} A$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ must exist and $u = (u_{ij})$ and $u^t = (u_{ji})$ must be invertible.

1.13 Proposition [10, Remark 2]: CMQG $\Rightarrow$ CQG (see also Cor. 3.4)

1.14 Example: (a) In the same spirit as the step from S_n to S_n^+, we may define a quantum version of the orthogonal group $O_n \subseteq M_n(\mathbb{C})$. Note that $C(O_n) = C^*(u_{ij}, 1\le i,j\le n \mid u_{ij}=u_{ij}^*, u=(u_{ij}) \text{ orth.}, u_{ij}u_{kl}=u_{kl}u_{ij})$, again by comparison of the space of characters. We thus define the <u>free orthogonal quantum group</u> O_n^+ by $C(O_n^+) := A_o(n) := C^*(u_{ij} \mid u_{ij}=u_{ij}^*, u \text{ orth.})$.

Note that „u orth." means $\sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij}$, which is just $uu^t = u^t u = 1$. O_n^+ is a CMQG and $O_n \subseteq O_n^+$. Moreover $S_n^+ \subseteq O_n^+$.

In fact, by the theory of quantum isometry groups (Goswami, Bhowmick, ...), one can show that O_n^+ is the quantum isometry group of the quantum sphere. [11]

Note that $C(S_{n-1}) \cong C^*(x_1,\dots,x_n \mid x_i=x_i^*, \sum_i x_i^2=1, x_ix_j=x_jx_i)$ since Spec of the latter is homeomorphic to S_{n-1}. Define the quantum sphere by $C^*(x_1,\dots,x_n \mid x_i=x_i^*, \sum_i x_i^2=1)$ and check that O_n^+ acts on it via

$$\alpha: C^*(x_1,\dots,x_n\mid\dots) \longrightarrow C^*(x_1,\dots,x_n\mid\dots)\otimes C(O_n^+), \qquad x_i \longmapsto \sum_j x_j\otimes u_{ij}$$

Now $QISO(C^*(x_1,\dots,x_n\mid\dots)) = O_n^+$ [11, Th. 7.2]

(b) Wang and Van Daele also defined a deformed version of O_n^+ (see [8]). It has been changed a bit ([7, Appendix]), but the definition used nowadays is the one from Banica [12]. Let $Q\in GL_n(\mathbb{C})$ with $Q\bar Q = c1_n$, $c\in\mathbb{R}$. We define $O_n^+(Q)$ by $A_o(Q) := C^*(u_{ij}, 1\le i,j\le n \mid u \text{ unitary}, u = Q\bar u Q^{-1} \text{ for } \bar u := (u_{ij}^*))$.
Note that for $Q=1_n$, we have $O_n^+(1_n)=O_n^+$ and that $u_{ij}\neq u_{ij}^*$ for general Q.
Furthermore $u^t = \bar u^*$ is invertible: $\bar u(Q^{-1}u^*Q) = (Q^{-1}uQ)(Q^{-1}u^*Q)=1$.
(Note: $u = Q\bar u Q^{-1} = Q(\bar Q u \bar Q^{-1})Q^{-1}$, hence the requirement $Q\bar Q = c1_n$)

1.15 Example: (a) The <u>free unitary quantum group</u> U_n^+ is defined as the CMQG given by $C(U_n^+) := C^*(u_{ij}, 1\le i,j\le n \mid u, u^t \text{ are unitaries})$. We have $U_n \subseteq U_n^+$. [6]
For $Q\in GL_n(\mathbb{C})$, we define $U_n^+(Q)$ by $A_u(Q) := C^*(u_{ij}\mid u, Q\bar u Q^{-1} \text{ unitary})$. [8,12]
In order to fit with Definition 1.12, we really need the requirement for both, u and u^t to be unitaries. This is a bit surprising, because
$C(U_n) = C^*(u_{ij}\mid u,u^t \text{ unitaries}, u_{ij} \text{ commute}) = C^*(u_{ij}\mid u \text{ unitary}, u_{ij} \text{ commute})$, since:

$uu^*=1: \sum_k u_{ik}u_{jk}^* = \delta_{ij}$ $\qquad$ $u^{t*}u^t=1: \sum_k u_{ik}^*u_{jk}=\delta_{ij}$

$u^*u=1: \sum_k u_{ki}^*u_{kj}=\delta_{ij}$ $\qquad$ $u^tu^{t*}=1: \sum_k u_{ki}u_{kj}^*=\delta_{ij}$

Hence, for commuting generators, $uu^* = 1 \iff u^{t*}u^t = 1$, but for noncommuting this is not true. Brown gave a definition in 1981 of the C^*-algebra $C^*(u_{ij}, 1 \leq i,j \leq n \mid u \text{ is a unitary})$. It admits a comultiplication
$\Delta: C^*(u_{ij}) \to C^*(u_{ij}) \otimes C^*(u_{ij})$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$, but u^t is not invertible. Let $n=2$.

Then $u = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ is unitary, but $u^t = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$ is not invertible. [6, Ex. 4.1]

1.16 Example: Consider $SU(2) := \{u \in M_2(\mathbb{C}) \mid u \text{ unitary}, \det u = 1\} \subseteq U_2$.
We may write it as $SU(2) = \{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \mid a, c \in \mathbb{C}, |a|^2 + |c|^2 = 1\}$.
For $q \in [-1,1] \setminus \{0\}$, Woronowicz defined the quantum group $SU_q(2)$ by
$C^*(\alpha, \gamma \mid \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is a unitary})$ and $\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$,
$\Delta(\gamma) = \gamma \otimes \alpha - \alpha^* \otimes \gamma$. This is a very early and important example. [9] $SU_q(2) \cong O^+\begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}$ [3, Sect. 6.2]

1.17 Example: In Ex. 1.1(c) we saw that $H_n := \mathbb{Z}_2 \wr S_n = \mathbb{Z}_2^{\oplus n} \rtimes S_n$ is the symmetry group of a hypercube ($n=1$: —, $n=2$: $\square$, $n=3$:) as well as of the graph o–o o–o o–o …
How to construct a quantum version of it? We first represent
$H_n = \langle a_1, \dots, a_n, \sigma \in S_n \mid a_i^2 = e, a_i a_j = a_j a_i, \sigma a_i \sigma^{-1} = a_{\sigma(i)} \rangle$ as a matrix group
$H_n \cong \{(u_{ij}) \mid u_{ik}u_{jk} = u_{ki}u_{kj} = 0 \text{ for } i \neq j, u \text{ orth.}\} \subseteq M_n(\mathbb{C})$
via $\tilde{a}_i := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix}$, $S_n \subseteq M_n(\mathbb{C})$. It is clear that
$H_n \twoheadrightarrow \langle \tilde{a}_1, \dots \tilde{a}_n, \hat{S}_n \rangle \subseteq M_n(\mathbb{C})$ exists, is surjective, and (by dimension) injective.
Now $\langle \tilde{a}_1, \dots \tilde{a}_n, S_n \rangle = \{(u_{ij}) \mid \dots\}$, since $\{(u_{ij}) \mid \dots\}$ is a group (check it) containing $\tilde{a}_1, \dots \tilde{a}_n, S_n$, and thus „$\subseteq$". As for „$\supseteq$", check that $u_{ik}u_{jk} = 0$ for $i \neq j$ implies that at most one $u_{ik} \neq 0$ in each column. Together with $\sum_i u_{ik}^2 = 1$ from orthogonality, this implies $u_{ik}^2 = 1$ for exactly one entry in the column.
Hence $u_{ik} \in \{-1, 1\}$, i.e. matrices in $\{(u_{ij}) \mid \dots\}$ are permutation matrices with some 1 potentially being replaced by -1. All these matrices may be constructed in $\langle \tilde{a}_1, \dots \tilde{a}_n, S_n \rangle$.
Now, it is easy to define H_n^+ via $C(H_n^+) := C^*(u_{ij} \mid \overset{u_{ij} = u_{ij}^*}{u_{ik}u_{jk} = u_{ki}u_{kj} = 0}, i \neq j, u \text{ orth.})$.
This is the <u>free hyperoctahedral quantum group</u>, as defined in [14].
Note that the crucial step is to find a representation of H_n as a matrix group in terms of algebraic relations on the u_{ij}. In [14], H_n^+ is <u>not</u> constructed as the quantum symmetry of a quantum cube [14, Th. 4.2, Prop. 5.3, end of Sect. 5] but of the graph
(However, we have $H_n^+ = \mathbb{Z}_2 \wr_* S_n^+$, i.e. a free wreath product [14, Th. 5.5], [15], i.e. $C(H_n^+) = C^*(\mathbb{Z}_2)^{*n} * C(S_n^+) \,/\, \langle [a^{(i)}, u_{ij}] = 0 \rangle$.)

(For a graph G, taking its n-fold copies, we have $\mathrm{Aut}(G^{\sqcup n}) = \mathrm{Aut}(G) \wr S_n$. In this sense, H^+ is the quantum autom. group of the quantum graph o–o ... o–o.) In EQG 1–10

We see $\mathrm{Aut}(\square) = \mathrm{Aut}(\text{o–o}^{\sqcup n})$ but $\mathrm{QAut}(\square) \neq \mathrm{QAut}(\text{o–o}^{\sqcup n})$. Hence, the quantum world is richer! We will see other such examples soon.

$\mathrm{QAut}(\square) = C^*(u_{ij} \mid u_{ij}u_{kl} = u_{kl}u_{ij}$ if $k \neq \ldots$, $u_{ij} = u_{ij}^*$, $u_{ij}u_{ik} = -u_{ik}u_{ij}$ if $j \neq \ldots$, $u_{ji}u_{ki} = -u_{ki}u_{ji}$ if $j \neq \ldots$, u orth.$)$

1.18 Remark: Let us end this introductory section with some remarks about the history of quantum groups and other directions.

(a) Our motivation for C^*-algebraic quantum groups came from a purely operator algebraic point of view: We wanted to find good symmetries for "noncommutative spaces" which were given via their function algebras. But "the origin of the study of quantum groups should be found in physics. Due to the preface of [16], the term „quantum groups" was coined by Drinfeld on the ICM in Berkeley 1986. He and Jimbo are certainly two of the pioneers. Due to [17, preface] quantum groups have „interrelations with the quantum inverse scattering method, the theory of integrable models, elementary particle physics, conformal and quantum field theories." They should serve as „the concept of symmetry in physics". Also Woronowicz had applications to physics in mind, when he introduced his quantum groups. In [4, Introduction] he writes that „in the existing theory [in physics] it is known that the symmetry described by the considered group is broken" referring to elementary particle physics.

(b) Another main motivation to define and study quantum groups comes from the attempt to generalize Pontrjagin duality (see [19], [3, Ch. 1]). Recall: If G is a locally compact abelian group, we associate its dual group $\hat{G}$ given by all characters $\varphi: G \to \mathbb{T}$ (continuous group hom.) of G endowed with pointwise multiplication. Then, $\hat{G}$ is again locally compact and abelian and we have $\hat{\hat{G}} \cong G$, i.e. G and $\hat{G}$ are dual to each other. Now, what is $\hat{G}$ for non-abelian groups? Enock and Schwartz solved this question working with Kac algebras (von Neumann algebraic Hopf algebras), see for instance [3, Th. 8.3.15].

(c) Many attempts in giving a meaning to the word „quantum group" are purely algebraic ([16][17][18][3, first chapters]). They are based on the theory of Hopf algebras. The idea is, that a group does not only come with its group law $\circ: G \times G \to G$ (which dualizes to a comultiplication $\Delta: A \to A \otimes A$) but also with inverses $^{-1}: G \to G$ (being dualized to an antipode $S: A \to A$) and a neutral element $e: \{e\} \to G$ (dualized to a counit $\varepsilon: A \to \mathbb{C}$). All these maps should fit together, i.e. we require $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$, $(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id$, $m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta$, where $\eta: \underset{1 \mapsto 1}{\mathbb{C} \to A}$ and $m: \underset{a \otimes b \mapsto ab}{A \otimes A \to A}$.

(see Def. 1, Def. 2, Def. 3, Def. 4 of [17, 1.1]). Note that $S: A \to A^{op}$ is an antihomomorphism into the opposite algebra of A, i.e. $S(ab) = S(b)S(a)$.

The Hopf algebra structure of a compact quantum group is a bit subtle. It is automatic on a certain dense *-subalgebra of the C^*-algebra ([5, Th. 2.2]), but on the C^*-algebra itself, the antipode is not bounded in general. But note that by Remark 1.5, Woronowicz' definition is a reasonable generalization of compact groups even without having a counit and an antipode.

Furthermore, one could equip Hopf algebras either with a topological and hence C^*-algebraic structure ([17],[3]) or with a measurable and thus von Neumann algebraic one ([3],[19]). In Woronowicz' opinion, the C^*-algebraic framework is „the right one" if one wants to deal with compact (and hence topological) groups ([9, Introduction]).

(d) There are locally compact quantum groups ([19]) generalizing locally compact groups. (See also [3]) But we will only treat the compact ones.

(e) The main approaches to obtain examples of compact quantum groups are

   (i) liberations („let drop commutativity $fg=gf$"), Wang, Banica and others [6,7,14]

   (ii) deformations („deform $fg=qgf$, $q\in\mathbb{C}$"), Woronowicz' $SU_q(2)$, Wang's $O^+(Q)$ etc. [9,8]

   (iii) quantum isometry groups („quantized group of Riemannian isometries on a noncommutative manifold"), Goswami, Bhowmick and others [11 and its references 8-11, 21]

We will only treat concepts evolving from (i).

(little overview, 3 pages: [3, Sect. 4.3] or [3, Introduction]

§2 Haar state and other basics on quantum groups

2.1 Remark: One of the very striking features of Woronowicz' definition of a quantum group is the existence of a Haar state. Let us first consider the classical case, i.e. let G be a compact group. Then there is a unique Haar measure μ_G on G such that $\int_G f(ts)\,d\mu_G(s) = \int_G f(s)\,d\mu_G(s)$ for all $f \in \mathcal{C}(G)$, $t \in G$ (Riesz theorem).
Hence we have a positive linear functional $h : \mathcal{C}(G) \to \mathbb{C}$, $h(f) := \int f\,d\mu_G$.
Thus, we should be able to find a functional on the C^*-algebra associated to our quantum group, which „looks like" integrating with respect to μ_G.
How to capture the left invariance? We need to express it in „quantum terms".

Observe: $(id \otimes h)\Delta(f)(t) = \int \Delta f(t,s)\,d\mu_G(s) = \int f(ts)\,d\mu(s) = \int f(s)\,d\mu(s) = h(f)(t)$

2.2 Theorem [4, Th. 4.2][5, Th. 2.3], [20]: Let (A, Δ) be a compact quantum group. There is a unique state $h : A \to \mathbb{C}$ („the Haar state") such that $(id \otimes h)\Delta = (h \otimes id)\Delta = h$.

Proof: Put $g * h := (g \otimes h) \circ \Delta$ for linear functionals $h, g \in A'$, and $a * h := (id \otimes h)\Delta(a)$ for $a \in A$.
We thus want to prove $a * h = h * a = h(a)$ for a unique state $h \in A'$.
Uniqueness: If h' is another such state, then

$$h'(a) = h(h'(a)1) = h((id \otimes h')\Delta(a)) = h \otimes h'\Delta(a) = h'((h \otimes id)\Delta(a)) = h'(h(a)) = h(a)$$

(check for $\Delta(a) = a_1 \otimes a_2$ etc)

Existence: For $\omega \in A'$ pos., define $K_\omega := \{ \varrho \in A' \text{ state} \mid \varrho * \omega = \omega * \varrho = \omega(1)\varrho \} \subseteq \{ \varrho \in A' \text{ pos.}, \|\varrho\| \leq 1 \} =: A'_1$

1.) A'_1 is compact, $K_\omega \neq \emptyset$, $K_\omega \subseteq A'_1$ closed.
2.) $\bigcap_{i=1}^n K_{\omega_i} \neq \emptyset$ for all $\omega_i \in A'$ pos., $n \in \mathbb{N}$.
3.) By Cantor's intersection principle and 2.), we have $\exists h \in \bigcap_{\omega \in A' \text{ pos.}} K_\omega \neq \emptyset$. It satisfies $a * h = h * a = h(a)$.

Proof of 1.): Let $\omega \in A'$ be a state. Put $\omega_n := \frac{1}{n} \sum_{k=1}^n \omega^{*k}$. As A'_1 is compact, $(\omega_n)_{n \in \mathbb{N}}$ has an accumulation point ϱ. Since $\|\omega * \omega_n - \omega_n\| = \frac{1}{n}\|(\omega^{*(n+1)} - \omega)\| \leq \frac{2}{n}$, we have $\omega * \varrho = \varrho$, likewise $\varrho * \omega = \varrho$.
Proof of 2.): Let $\omega_1, \omega_2 \in A'$ be positive, $\omega := \omega_1 + \omega_2$. We prove $K_\omega \subseteq K_{\omega_1}$. Hence also $\emptyset \neq K_\omega \subseteq K_{\omega_1} \cap K_{\omega_2}$ and iteratively $\emptyset \neq K_{\Sigma \omega_i} \subseteq \bigcap K_{\omega_i}$.

Lit.: [20] Van Daele, The Haar measure on a compact quantum group, 1995
[21] Wang, Tensor products and crossed products of CQG, 1995
[22] Bichon, Quantum automorphism groups of finite graphs, ~~[illegible]~~ 2003

Assume that ω is a state. Let $\rho \in K_\omega$ and put

$L_{\rho\otimes\omega} := \{ q \in A \otimes A \mid \rho\otimes\omega(q^* q) = 0 \} \underset{(\omega_1 \le \omega)}{\subseteq} L_{\rho\otimes\omega_1} \underset{\text{Cauchy-Schwarz}}{\subseteq} \ker \rho\otimes\omega_1$.

One can show that $(id \otimes \Psi_L)\Delta(A) \subseteq L_{\rho\otimes\omega}$ for $\Psi_L : A \to A$, $a \mapsto \rho * a - 1\rho(a)$.

Thus $1 \otimes \Psi_L(A) \underset{[\dots]}{\subseteq} (A \otimes 1)(id \otimes \Psi_L)\Delta(A) \subseteq (A \otimes 1) L_{\rho\otimes\omega} \underset{L_{\rho\otimes\omega} \text{ is a left ideal}}{\subseteq} \ker \rho\otimes\omega_1$

Hence $0 = \rho\otimes\omega_1(1 \otimes \Psi_L(a)) = \omega_1((\rho \otimes id)\Delta(a) - 1\rho(a)) = \rho * \omega_1(a) - \omega_1(1)\rho(a)$
$\Rightarrow \rho \in K_{\omega_1}$

Proof of 3.): For $h \in \bigcap K_\omega$, we have $h * \omega = \omega * h = \omega(1) h$ $\forall \omega \in A'$ states, i.e. if $a \in A$, then we have $\omega(b) = 0$ $\forall \omega \in A'$ states, where $b := h(a) - id \otimes h\Delta(a) \in A$. Hence $b = 0$. We conclude $a * h = h * a = h(a)$ $\forall a \in A$.
[3, Th. 5.1.6] $\square$

2.3 Remark: The Haar state is not always faithful. Let G be a discrete group. Then $(C^*_{max}(G), \Delta)$ is a compact quantum group accordingly to Ex. 1.6.
Its Haar state is $h_{max} : C^*_{max}(G) \to \mathbb{C}$, $h(u_g) = \delta_{g,e}$, since
$(id \otimes h_{max})\Delta(u_g) = (id \otimes h_{max})(u_g \otimes u_g) = \delta_{g,e} u_e = \delta_{g,e} 1 = h_{max}(u_g)$.
It is tracial, i.e. $h(ab) = h(ba)$ $\forall a, b \in C^*_{max}(G)$.
The Haar state on $(C^*_{red}(G), \Delta)$ is given by $h_{red} : C^*_{red}(G) \to \mathbb{C}$, $h_{red}(x) := \langle x\delta_e, \delta_e \rangle$ for $C^*_{red}(G) \subseteq B(\ell^2(G))$ with ONB $(\delta_g)_{g \in G}$. Indeed for $\alpha : C^*_{max}(G) \to C^*_{red}(G)$ we have:
$(id \otimes h_{red})\Delta_{red}(\alpha(u_g)) = (id \otimes h_{red})(\alpha \otimes \alpha)\Delta_{max}(u_g) = \alpha(u_g) h_{red}(\alpha(u_g)) = \alpha(u_g)\langle \delta_g, \delta_e \rangle$
$= \delta_{g,e} = h_{red}(\alpha(u_g))$

Thus
$$\begin{tikzcd} C^*_{max}(G) \arrow[r, "\alpha"] \arrow[rr, bend right, "h_{max}"'] & C^*_{red}(G) \arrow[r, "h_{red}"] & \mathbb{C} \end{tikzcd}$$
is a commutative diagram.

Now, if α is not an isomorphism (i.e. if G is not amenable), then h_{max} is not faithful. On the other hand, h_{red} is faithful on $C^*_{red}(G)$:
Let $R_s \in B(\ell^2(G))$ be the right shift, defined by $R_s\delta_t := \delta_{ts}$.
Then $R_s x = x R_s$ $\forall x \in C^*_{red}(G)$ as $R_s \lambda(u_g)\delta_t = \delta_{gts} = \lambda(u_g) R_s \delta_t$.
Now $h_{red}(x^* x) = 0$ implies $\langle x\delta_e, x\delta_e \rangle = 0$ and hence $x\delta_e = 0$.
But $x\delta_g = x R_g \delta_e = R_g x \delta_e = 0$ $\forall g \in G$, thus $x = 0$.

2.4 Definition: Let $G = (A, \Delta)$ be a compact quantum group and let h be its Haar state. The <u>reduced version</u> of G is given by (A_{red}, Δ_{red}), where $A_{red} := \pi_h(A) \subseteq B(H_h)$ by GNS construction (π_h, H_h) with respect to h, and $\Delta_{red}(\pi_h(x)) := (\pi_h \otimes \pi_h)\Delta(x)$. We may also <u>associate a von Neumann algebra</u> $L(G)$ to G.

2.5 Prop.: Δ_{red} in the previous definition is well defined and

$$\begin{tikzcd} A \arrow[r, "\pi_h"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & A_{red} \arrow[d, "\Delta_{red}"] \\ A\otimes A \arrow[r, "\pi_h\otimes\pi_h"] & A_{red}\otimes A_{red} \end{tikzcd}$$

is commutative. Moreover, $h_{red}: A_{red}\to\mathbb{C}$ defined by $h_{red}(\pi_h(x)):=h(x)$ is the Haar state on (A_{red},Δ_{red}) and h is faithful.

Proof: Let $\pi_h(x)=0$, thus $h(x^*x)=0$. But then
$(h\otimes h)\Delta(x^*x) = h((id\otimes h)\Delta(x^*x)) = h(h(x^*x)) = h(x^*x) = 0$ and
thus $\Delta(x)\in\ker\pi_{h\otimes h} = \ker\pi_h\otimes\pi_h$. Hence Δ_{red} is well defined.
Moreover $(id\otimes h_{red})\Delta_{red}(\pi_h(x)) = (id\otimes h_{red})(\pi_h\otimes\pi_h)\Delta(x) = \pi_h(id\otimes h)\Delta(x) = \pi_h(h(x)) = h(x)$
$= h_{red}(\pi_h(x))$,
hence h_{red} is the Haar state.
Finally $h_{red}(\pi_h(x^*x))=0 \Rightarrow h(x^*x)=0 \Rightarrow \pi_h(x^*x)=0$, i.e. h_{red} is faithful. □

2.6 Definition: A compact quantum group is of Kac type, if the Haar state is a trace, i.e. if $h(ab)=h(ba)\ \forall a,b$.

~~2.7 Prop.: If G is a compact quantum group~~

Later, we will see other characterizations of Kac type, and examples of such QG.
We end with a few constructions of new QG's out of given ones.

2.7 Prop.: (Free product) Let (A,Δ_A), (B,Δ_B) be CQG. Then the free prod. $(A,\Delta_A)*(B,\Delta_B)$ is given by $(A*_{\mathbb{C}}B,\Delta)$, where

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "\iota_A\otimes\iota_A\circ\Delta_A"'] & A*_{\mathbb{C}}B \arrow[d, "\Delta"] & B \arrow[l, hook', "\iota_B"'] \arrow[dl, "\iota_B\otimes\iota_B\circ\Delta_B"] \\ & (A*B)\otimes(A*B) & \end{tikzcd}$$

Proof: Δ exists by the universal property of a free product: If C is any unital C^*-algebra with unital $*$-hom's $f: A\to C$, $g: B\to C$, then there is a unique unital $*$-hom. $f*g: A*B\to C$ such that

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "f"'] & A*_{\mathbb{C}}B \arrow[d, "f*g"] & B \arrow[l, hook', "\iota_B"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

↪ [6]
[3, Sect. 6.3]

2.8 Remark: (a) If (A,Δ_A), (B,Δ_B) are CMQG, so is $(A,\Delta_A)*(B,\Delta_B)$.
(b) The Haar state h on $(A,\Delta_A)*(B,\Delta_B)$ is the free product of the Haar states h_A of (A,Δ_A) and h_B of (B,Δ_B) in the sense of free probability, i.e. $h\circ\iota_A=h_A$, $h\circ\iota_B=h_B$, $\forall\, c_1,\dots,c_n\in A$ or B such that $h_A(c_i)=0$ resp. $h_B(c_i)=0$ and c_i, c_{i+1} come from different algebras: then $h(c_1\cdots c_n)=0$.

2.9 <u>Prop.:</u> (Tensor product) Let (A, Δ_A), (B, Δ_B) be CQG. Then

$(A, \Delta_A) \otimes (B, \Delta_B) := (A \otimes_{max} B, \, \varsigma \circ (\Delta_A \otimes \Delta_B))$, where $\varsigma : (A \otimes A) \otimes_{max} (B \otimes B) \to (A \otimes_{max} B) \otimes (A \otimes_{max} B)$ is a CQG.

[21][3, Sect. 6.3]

The Haar state is $h_A \otimes h_B$ and it is a CMQG if the original ones are.

2.10 <u>Prop.:</u> (free wreath product) Let (A, Δ_A) be a CQG. The free wreath product with S_n^+ is given by $B := \dfrac{A^{*n} * C(S_n^+)}{\langle \iota_k(a) u_{ki} - u_{ki} \iota_k(a), \; 1 \le i, k \le n \rangle}$

where $\iota_k : A \to A^{*n}$ is the embedding as the k-th copy. The comultiplication is given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ and $\Delta(\iota_k(a)) = \sum_\ell (\iota_k \otimes \iota_\ell)(\Delta_A(a))(u_{k\ell} \otimes 1)$.

2.11 <u>Example:</u> If Γ is a graph, then $\Gamma^{\sqcup n}$ is the graph obtained from copying Γ n-times. For instance $(\circ\!-\!\circ)^{\sqcup n} = \circ\!-\!\circ \;\; \circ\!-\!\circ \; \dots \; \circ\!-\!\circ$. In the classical theory, we have $Aut(\Gamma^{\sqcup n}) = Aut(\Gamma) \wr S_n$.

[15]

Now, H_n^+ of Example 1.17 may be written as $H_n^+ = \mathbb{Z}_2 \wr_* S_n^+$.
Since $\mathbb{Z}_2 = Aut(\circ\!-\!\circ)$, we have that H_n^+ is the quantum automorphism group of $\circ\!-\!\circ \dots \circ\!-\!\circ$.

[15][14, Th. 5.5]

2.12 <u>Remark:</u> See [21] for how to equip the C^*-algebra $A \rtimes_\alpha \Gamma$ with a CQG structure.

[22] ~~…~~ [7]