Schmidt rank of generalized Werner states.

$\mathcal{H}$ $\quad \dim \mathcal{H} = n \quad \{|i\rangle, 1 \leq i \leq n\}$ ONB

U unitary operator in $\mathcal{H}$.

$\pi(U) : U \to U \otimes \bar{U}$ a representation of $\mathcal{U}(\mathcal{H})$

J conjugation w.r.t. $\{|i\rangle\}$

$\Gamma(J) : |u\rangle|v\rangle \longrightarrow |u\rangle\langle Jv|$

$$\Gamma(J)\, \pi(U)\, \Gamma(J)^{-1} = U \cdot U^{\dagger} = \tilde{\pi}(U)(\cdot)$$

<u>Remark</u> $\{\tilde{\pi}(U)\}'$ is spanned by

id and $X \to \mathrm{Tr}(X)\, I = \langle I|X\rangle |I\rangle$

Thus $\{\pi(U), U \in \mathcal{U}(\mathcal{H})\}'$ is spanned by

$I_{\mathcal{H}\otimes\mathcal{H}}$ and $|\psi_0\rangle\langle\psi_0|$, $\quad |\psi_0\rangle = \frac{1}{\sqrt{n}} \sum |ii\rangle$

<u>Proposition 1</u> Any state ρ commuting every $U \otimes \bar{U}$ has the form

$$\rho = \rho_F = F\, |\psi_0\rangle\langle\psi_0| + (1-F)\, \frac{(I - |\psi_0\rangle\langle\psi_0|)}{n^2 - 1}$$

where

$$|\psi_0\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |ii\rangle$$

(the maximum entangled ~~bas~~ pure state in the canonical basis)

$\bar{U}$ being conjugate in the canonical basis.

ρ_F = generalized Werner state.

<u>Goal</u> : Evaluation of Schmidt rank ρ_F.

<u>Proposition 2</u> For any state $|\psi\rangle$

$$\langle \psi | \rho_F | \psi \rangle = \frac{1}{n^2-1} \left\{ (n^2 F - 1) \, |\langle \psi_0 | \psi \rangle|^2 + 1 - F \right\}$$

If $F > \frac{1}{n^2}$

$\langle \psi | \rho_F | \psi \rangle$ attains its maximum when $\psi = \psi_0$

and the maximum value $= F$.

If $F \leq \frac{1}{n^2}$ the maximum value is $\frac{1-F}{n^2-1}$. It is attained when $\psi \perp \psi_0$.

<u>PROPOSITION 3</u>: Suppose for a state ρ and a maximum entangled state ψ

$$\langle \psi | \rho | \psi \rangle > \frac{k}{n}.$$

Then

$$\text{Schmidt rank } \rho \geq k+1.$$

Pf: We divide it into 2 lemmas.

<u>Lemma A</u> Let $|\psi\rangle = \sum_{j=1}^{n} \lambda_j |u_j\rangle |v_j\rangle$ be a state in Schmidt form. Then

$$\max \left\{ |\langle \psi | \psi' \rangle| \, , \, \psi' \text{ max. entangled state} \right\}$$

$$= \frac{1}{\sqrt{n}} \sum_j \lambda_j .$$

<u>Pf</u>: Choose

$$|\psi'\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |\xi_j\rangle |\eta_j\rangle$$

$\{\xi_j\}, \{\eta_j\}$ ONB's.

Then

$$\langle \psi | \psi' \rangle = \frac{1}{\sqrt{n}} \sum_{j,\ell} \lambda_j \langle u_j | \xi_\ell \rangle \langle v_j | \eta_\ell \rangle$$

Choose unitaries U, V s.t. $U\xi_\ell = |\ell\rangle$, $V\eta_\ell = |\ell\rangle$, C conjugation w.r.t $\{|\ell\rangle\}$. Then

$$\langle \psi | \psi' \rangle = \frac{1}{\sqrt{n}} \sum_{j,\ell} \lambda_j \langle U u_j | \ell \rangle \langle V v_j | \ell \rangle$$

$$= \frac{1}{\sqrt{n}} \sum_{j,\ell} \lambda_j \langle U u_j | \ell \rangle \langle \ell | C V v_j \rangle$$

$$= \frac{1}{\sqrt{n}} \sum_{j} \lambda_j \langle U u_j | C V v_j \rangle$$

Since Uu_j, CVv_j are unit vectors and $\lambda_j \geq 0$

$$|\langle \psi | \psi' \rangle| \leq \frac{1}{\sqrt{n}} \sum \lambda_j$$

In the top line choose $|\xi_\ell\rangle = |\ell\rangle$, consider the unitary V satisfying $VCv_j = u_j \ \forall j$ and put $C\eta_\ell = V^\dagger \ell$, i.e., $\eta_\ell = CV^\dagger|\ell\rangle$, so that

$$\psi' = \frac{1}{\sqrt{n}} \sum |\xi_\ell\rangle |\eta_\ell\rangle.$$

Then

$$\langle \psi | \psi' \rangle = \frac{1}{\sqrt{n}} \sum \lambda_j \langle u_j | \ell \rangle \langle C\eta_\ell | C v_j \rangle$$

$$= \frac{1}{\sqrt{n}} \sum \lambda_j \langle u_j | \ell \rangle \langle VC\eta_\ell | VC v_j \rangle$$

$$= \frac{1}{\sqrt{n}} \sum_{j,\ell} \lambda_j \langle u_j | \ell \rangle \langle \ell | u_j \rangle$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \lambda_j. \qquad \blacksquare$$

<u>Corollary to Lemma A</u> Let Schmidt number $(\psi) = k$.
Then
$$\max\{ |\langle\psi|\psi'\rangle| \mid \psi' \text{ max entangled state}\} \leq \sqrt{\tfrac{k}{n}}. \qquad (1)$$

<u>Pf</u> : $|\psi\rangle = \sum_{j=1}^{k} \lambda_j |u_j\rangle|v_j\rangle$, in Schmidt form

$\lambda_j = 0$ if $j \geq k+1$. By Lemma A, LHS of (1)

$$= \frac{1}{\sqrt{n}} (\lambda_1 + \lambda_2 + \cdots + \lambda_k)$$
$$\leq \frac{1}{\sqrt{n}} k^{1/2} (\lambda_1^2 + \cdots + \lambda_k^2)^{1/2} = \sqrt{\tfrac{k}{n}}. \quad \blacksquare$$

Proof of PROPOSITION 3. If Schmidt rank $\rho \leq k$
then
$$\rho = \int_{S_k} |\psi\rangle\langle\psi| \, \mu(d\psi)$$

If $|\psi'\rangle$ is max. entangled,
$$\langle\psi'|\rho|\psi'\rangle = \int_{S_k} |\langle\psi'|\psi\rangle|^2 \mu(d\psi) \leq \frac{k}{n}. \quad \blacksquare$$

(Exercise) <u>PROPOSITION 4</u> (Integral formula). Let
$$|\psi_k\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |i\, i\rangle \quad (\text{with Sch No. } k).$$
$$\int U \otimes \bar{U} \, |\psi_k\rangle\langle\psi_k| (U \otimes \bar{U})^\dagger \, dU = \rho_{k/n}.$$

In, particular, Sch. ~~No~~ Rank $(\rho_{k/n}) \leq k$.

COROLLARY Sch. Rank $\rho_{k/n} = k \qquad 1 \le k \le n$

Pf By PROP. 2

$$\max\{\langle \psi | \rho_{k/n} | \psi \rangle,\ \psi \text{ max. ent. state}\}$$

$$= \frac{k}{n} > \frac{k-1}{n}.$$

So by PROP. 3

$$\text{Sch. Rank } \rho_{k/n} \ge k.$$

By PROP. 4

$$\text{Sch. Rank } \rho_{k/n} \le k.$$

Thus

$$\text{Sch. Rank } \rho_{k/n} = k. \quad \blacksquare$$

PROP. 5 Let $\frac{1}{n^2} \le F \le \frac{k}{n}$. Then ρ_F is a convex combination of I/n^2 and $\rho_{k/n}$. (I in $\mathcal{H} \otimes \mathcal{H}$)

Pf: Express

$$\rho_F = (1-\theta)\, I/n^2 + \theta\, \rho_{k/n} \quad \text{and solve for } \theta.$$

i.e.,
$$\rho_F = (1-\theta)\, I/n^2 + \theta \left\{ (1-k/n) \frac{\left(I - |\psi_0\rangle\langle\psi_0|\right)}{n^2-1} + \frac{k}{n} |\psi_0\rangle\langle\psi_0| \right\}$$

$$= \left(\frac{1-\theta}{n^2} + \theta \frac{(1-k/n)}{n^2-1} \right) \left(I - |\psi_0\rangle\langle\psi_0| \right)$$

$$+ \left(\frac{1-\theta}{n^2} + \theta \frac{k}{n} \right) |\psi_0\rangle\langle\psi_0|$$

a spectral decomposition where

$$F = \frac{1-\theta}{n^2} + \theta \frac{k}{n}$$

or
$$\theta = \frac{F - 1/n^2}{k/n - 1/n^2}.$$

Clearly, $0 \le \theta \le 1$ if $\frac{1}{n^2} \le F \le \frac{k}{n}$. $\blacksquare$

COR: Here Sch. Rank $\rho_F \le k$ if $\frac{1}{n^2} \le F \le \frac{k}{n}$

<u>Theorem</u>: Let $\frac{k-1}{n} < F \leqslant k/n$, $F \geqslant \frac{1}{n^2}$.

Then Sch. Rank $\rho_F = k$.

<u>Pf</u> : Since $F > \frac{k-1}{n}$ Sch. Rank $\rho_F \geqslant k$ (PROP 3)

By PROP 5, Sch Rank $\rho_F \leqslant k$. ■.

Q: What happens when $0 < F < \frac{1}{n^2}$?

One expects Sch rank $\rho_F = 1$.