Symmetries and independence in noncommutative probability

Claus Köstler
Institute of Mathematics and Physics
Aberystwyth University

cck@aber.ac.uk

ICM Satellite Conference on Operator Algebras
The Institute of Mathematical Sciences (IMSc)
Chennai

August 10, 2010

Motivation

Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeablity** *[and* **rotatability***] - stand out as especially interesting and important in several ways: Their study leads to some* **deep structural theorems** *of great beauty and significance [...].*

Olav Kallenberg (2005)

Motivation

Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeablity** *[and* **rotatability***] - stand out as especially interesting and important in several ways: Their study leads to some* **deep structural theorems** *of great beauty and significance [...].*

Olav Kallenberg (2005)

Quest

- Transfer related concepts to **noncommutative probability**
- Show that these concepts are fruitful in the study of **operator algebras** and **quantum dynamics**

Motivation

Though many probabilistic symmetries are conceivable [...], four of them - **stationarity**, **contractability**, **exchangeablity** *[and* **rotatability***] - stand out as especially interesting and important in several ways: Their study leads to some* **deep structural theorems** *of great beauty and significance [...].*

Olav Kallenberg (2005)

Quest

- Transfer related concepts to **noncommutative probability**
- Show that these concepts are fruitful in the study of **operator algebras** and **quantum dynamics**

Remark

Noncommutative probability = classical & quantum probability

Foundational result on distributional symmetries and invariance principles in classical probability

The random variables $(X_n)_{n \geq 0}$ are said to be **exchangeable** if

$$\mathbb{E}\big(X_{\mathbf{i}(1)} \cdots X_{\mathbf{i}(n)}\big) = \mathbb{E}\big(X_{\sigma(\mathbf{i}(1))} \cdots X_{\sigma(\mathbf{i}(n))}\big) \qquad (\sigma \in \mathbb{S}_\infty)$$

for all n-tuples $\mathbf{i}\colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$ and $n \in \mathbb{N}$.

Foundational result on distributional symmetries and invariance principles in classical probability

The random variables $(X_n)_{n\geq 0}$ are said to be **exchangeable** if

$$\mathbb{E}\big(X_{\mathbf{i}(1)}\cdots X_{\mathbf{i}(n)}\big) = \mathbb{E}\big(X_{\sigma(\mathbf{i}(1))}\cdots X_{\sigma(\mathbf{i}(n))}\big) \qquad (\sigma \in \mathbb{S}_\infty)$$

for all n-tuples $\mathbf{i}\colon \{1,2,\ldots,n\} \to \mathbb{N}_0$ and $n \in \mathbb{N}$.

Theorem (De Finetti 1931,. . .)

The law of an exchangeable sequence $(X_n)_{n\geq 0}$ is given by a unique convex combination of infinite product measures.

Foundational result on distributional symmetries and invariance principles in classical probability

The random variables $(X_n)_{n\geq 0}$ are said to be **exchangeable** if

$$\mathbb{E}\big(X_{\mathbf{i}(1)} \cdots X_{\mathbf{i}(n)}\big) = \mathbb{E}\big(X_{\sigma(\mathbf{i}(1))} \cdots X_{\sigma(\mathbf{i}(n))}\big) \qquad (\sigma \in \mathbb{S}_\infty)$$

for all n-tuples $\mathbf{i}\colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$ and $n \in \mathbb{N}$.

Theorem (De Finetti 1931,. . .)

The law of an exchangeable sequence $(X_n)_{n\geq 0}$ is given by a unique convex combination of infinite product measures.

"Any exchangeable process is an average of i.i.d. processes."

Foundational result on distributional symmetries and invariance principles in free probability

Replacing permutation groups by Wang's quantum permutation groups . . .

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables $x_1, x_2, \ldots$ in a W*-algebraic probability space $(\mathcal{A}, \varphi)$:

Foundational result on distributional symmetries and invariance principles in free probability

Replacing permutation groups by Wang's quantum permutation groups . . .

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables $x_1, x_2, \ldots$ in a W*-algebraic probability space $(\mathcal{A}, \varphi)$:

(a) the sequence is **quantum exchangeable**

Foundational result on distributional symmetries and invariance principles in free probability

Replacing permutation groups by Wang's quantum permutation groups . . .

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables $x_1, x_2, \ldots$ in a W*-algebraic probability space $(\mathcal{A}, \varphi)$:

(a) the sequence is **quantum exchangeable**

(b) the sequence is identically distributed and **freely independent with amalgamation over $\mathcal{T}$**

Here $\mathcal{T}$ denotes the **tail algebra** $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathrm{vN}(x_k | k \geq n)$.

Foundational result on distributional symmetries and invariance principles in free probability

Replacing permutation groups by Wang's quantum permutation groups . . .

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables $x_1, x_2, \dots$ in a W*-algebraic probability space $(\mathcal{A}, \varphi)$:

(a) the sequence is **quantum exchangeable**

(b) the sequence is identically distributed and **freely independent with amalgamation over** $\mathcal{T}$

Here $\mathcal{T}$ denotes the **tail algebra** $\mathcal{T} = \bigcap_{n\in\mathbb{N}} \mathrm{vN}(x_k | k \geq n)$.

Foundational result on distributional symmetries and invariance principles in free probability

Replacing permutation groups by Wang's quantum permutation groups ...

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables $x_1, x_2, \ldots$ in a W*-algebraic probability space $(\mathcal{A}, \varphi)$:

(a) the sequence is **quantum exchangeable**

(b) the sequence is identically distributed and **freely independent with amalgamation over** $\mathcal{T}$

(c) the sequence canonically embeds into $\bigstar_{\mathcal{T}}^{\mathbb{N}} \mathrm{vN}(x_1, \mathcal{T})$, a von Neumann algebraic **amalgamated free product over** $\mathcal{T}$

Here $\mathcal{T}$ denotes the **tail algebra** $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathrm{vN}(x_k | k \geq n)$.

Foundational result on distributional symmetries and invariance principles in free probability

Replacing permutation groups by Wang's quantum permutation groups . . .

Theorem (K. & Speicher 2008)

The following are equivalent for an infinite sequence of random variables $x_1, x_2, \ldots$ in a W*-algebraic probability space $(\mathcal{A}, \varphi)$:

(a) the sequence is **quantum exchangeable**

(b) the sequence is identically distributed and **freely independent with amalgamation over** $\mathcal{T}$

(c) the sequence canonically embeds into $\bigstar_{\mathcal{T}}^{\mathbb{N}} \mathrm{vN}(x_1, \mathcal{T})$, a von Neumann algebraic **amalgamated free product over** $\mathcal{T}$

Here $\mathcal{T}$ denotes the **tail algebra** $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathrm{vN}(x_k | k \geq n)$.

See talks of Curran and Speicher for more recent developments.

Foundational result on the representation theory of the infinite symmetric group $\mathbb{S}_\infty$

$\mathbb{S}_\infty$ is the inductive limit of the symmetric group $\mathbb{S}_n$ as $n \to \infty$, acting on $\{0, 1, 2, \ldots\}$. A function $\chi \colon \mathbb{S}_\infty \to \mathbb{C}$ is a **character** if it is constant on conjugacy classes, positive definite and normalized at the unity.

Foundational result on the representation theory of the infinite symmetric group $\mathbb{S}_\infty$

$\mathbb{S}_\infty$ is the inductive limit of the symmetric group $\mathbb{S}_n$ as $n \to \infty$, acting on $\{0, 1, 2, \dots\}$. A function $\chi \colon \mathbb{S}_\infty \to \mathbb{C}$ is a **character** if it is constant on conjugacy classes, positive definite and normalized at the unity.

Elementary observation

Let $\gamma_i := (0, i)$. Then the sequence $(\gamma_i)_{i \in \mathbb{N}}$ is **exchangeable**, i.e.

$$\chi\big(\gamma_{\mathbf{i}(1)}\gamma_{\mathbf{i}(2)} \cdots \gamma_{\mathbf{i}(n)}\big) = \chi\big(\gamma_{\sigma(\mathbf{i}(1))}\gamma_{\sigma(\mathbf{i}(2))} \cdots \gamma_{\sigma(\mathbf{i}(n))}\big)$$

for $\sigma \in \mathbb{S}_\infty$ with $\sigma(0) = 0$, n-tuples $\mathbf{i} \colon \{1, \dots, n\} \to \mathbb{N}$ and $n \in \mathbb{N}$.

Foundational result on the representation theory of the infinite symmetric group $\mathbb{S}_\infty$

$\mathbb{S}_\infty$ is the inductive limit of the symmetric group $\mathbb{S}_n$ as $n \to \infty$, acting on $\{0, 1, 2, \ldots\}$. A function $\chi \colon \mathbb{S}_\infty \to \mathbb{C}$ is a **character** if it is constant on conjugacy classes, positive definite and normalized at the unity.

Elementary observation

Let $\gamma_i := (0, i)$. Then the sequence $(\gamma_i)_{i \in \mathbb{N}}$ is **exchangeable**, i.e.

$$\chi\big(\gamma_{\mathbf{i}(1)}\gamma_{\mathbf{i}(2)} \cdots \gamma_{\mathbf{i}(n)}\big) = \chi\big(\gamma_{\sigma(\mathbf{i}(1))}\gamma_{\sigma(\mathbf{i}(2))} \cdots \gamma_{\sigma(\mathbf{i}(n))}\big)$$

for $\sigma \in \mathbb{S}_\infty$ with $\sigma(0) = 0$, n-tuples $\mathbf{i} \colon \{1, \ldots, n\} \to \mathbb{N}$ and $n \in \mathbb{N}$.

Task

Identify the convex combination of extremal characters of $\mathbb{S}_\infty$. In other words: **prove a noncommutative de Finetti theorem!**

Thoma's theorem as a noncommutative de Finetti theorem

Theorem (Thoma 1964)

An extremal character of the group $\mathbb{S}_\infty$ is of the form

$$\chi(\sigma) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\sigma)} .$$

Here $m_k(\sigma)$ is the number of k-cycles in the permutation σ and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

Alternative proofs

Vershik & Kerov 1981: asymptotic representation theory
Okounkov 1997: Olshanski semigroups and spectral theory
Gohm & K. 2010: operator algebraic proof (see next talk)

Towards a braided version of Thoma's theorem

The Hecke algebra $H_q(\infty)$ over $\mathbb{C}$ with parameter $q \in \mathbb{C}$ is the unital algebra with generators $g_0, g_1, \ldots$ and relations

$$
\begin{aligned}
g_n^2 &= (q-1)g_n + q; \\
g_m g_n &= g_n g_m \qquad \text{if } \mid n-m \mid \geq 2; \\
g_n g_{n+1} g_n &= g_{n+1} g_n g_{n+1}.
\end{aligned}
$$

Towards a braided version of Thoma's theorem

The Hecke algebra $H_q(\infty)$ over $\mathbb{C}$ with parameter $q \in \mathbb{C}$ is the unital algebra with generators $g_0, g_1, \ldots$ and relations

$$\begin{aligned} g_n^2 &= (q-1)g_n + q; \\ g_m g_n &= g_n g_m \qquad \text{if } \mid n-m \mid \geq 2; \\ g_n g_{n+1} g_n &= g_{n+1} g_n g_{n+1}. \end{aligned}$$

If q is a **root of unity** there exists an involution and a trace on $H_q(\infty)$ such that the g_n are unitary.

Towards a braided version of Thoma's theorem

The Hecke algebra $H_q(\infty)$ over $\mathbb{C}$ with parameter $q \in \mathbb{C}$ is the unital algebra with generators $g_0, g_1, \ldots$ and relations

$$\begin{aligned} g_n^2 &= (q-1)g_n + q; \\ g_m g_n &= g_n g_m \qquad \text{if } \mid n-m \mid \geq 2; \\ g_n g_{n+1} g_n &= g_{n+1} g_n g_{n+1}. \end{aligned}$$

If q is a **root of unity** there exists an involution and a trace on $H_q(\infty)$ such that the g_n are unitary. Now let

$$\gamma_1 := g_1, \quad \gamma_n := g_1 g_2 \cdots g_{n-1} g_n g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1}$$

Towards a braided version of Thoma's theorem

The Hecke algebra $H_q(\infty)$ over $\mathbb{C}$ with parameter $q \in \mathbb{C}$ is the unital algebra with generators $g_0, g_1, \ldots$ and relations

$$\begin{aligned} g_n^2 &= (q-1)g_n + q; \\ g_m g_n &= g_n g_m \qquad \text{if } \mid n-m \mid \geq 2; \\ g_n g_{n+1} g_n &= g_{n+1} g_n g_{n+1}. \end{aligned}$$

If q is a **root of unity** there exists an involution and a trace on $H_q(\infty)$ such that the g_n are unitary. Now let

$$\gamma_1 := g_1, \quad \gamma_n := g_1 g_2 \cdots g_{n-1} g_n g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1}$$

Replacing the role of $\mathbb{S}_\infty$ by the braid group $\mathbb{B}_\infty$ turns exchangeability into **braidability**....

Theorem (Gohm & K.)

The sequence $(\gamma_n)_{n\in\mathbb{N}}$ *is* **braidable**.

This will become more clear later . . . see talk of Gohm

Noncommutative random variables

A (noncommutative) **probability space** $(\mathcal{A}, \varphi)$ is a von Neumann algebra $\mathcal{A}$ (with separable predual) equipped with a faithful normal state φ.

A **random variable** $\iota \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is is an injective *-homomorphism from $\mathcal{A}_0$ into $\mathcal{A}$ such that $\varphi_0 = \varphi \circ \iota$ and the φ-preserving conditional expectation from $\mathcal{A}$ onto $\iota(\mathcal{A}_0)$ exists.

Given the sequence of random variables

$$(\iota_n)_{n \geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi),$$

fix some $a \in \mathcal{A}_0$. Then $x_n := \iota_n(a)$ defines the operators $x_0, x_1, x_2, \ldots$ (now random variables in the operator sense).

Noncommutative distributions

Two sequences of random variables $(\iota_n)_{n\geq 0}$ and $(\tilde{\iota}_n)_{n\geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ have the same **distribution** if

$$\varphi\big(\iota_{\mathbf{i}(1)}(a_1)\iota_{\mathbf{i}(2)}(a_2)\cdots\iota_{\mathbf{i}(n)}(a_n)\big) = \varphi\big(\tilde{\iota}_{\mathbf{i}(1)}(a_1)\,\tilde{\iota}_{\mathbf{i}(2)}(a_2)\cdots\tilde{\iota}_{\mathbf{i}(n)}(a_n)\big)$$

for all n-tuples $\mathbf{i} \colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$, $(a_1, \ldots, a_n) \in \mathcal{A}_0^n$ and $n \in \mathbb{N}$.

Noncommutative distributions

Two sequences of random variables $(\iota_n)_{n\geq 0}$ and $(\tilde{\iota}_n)_{n\geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ have the same **distribution** if

$$\varphi\big(\iota_{\mathbf{i}(1)}(a_1)\iota_{\mathbf{i}(2)}(a_2)\cdots\iota_{\mathbf{i}(n)}(a_n)\big) = \varphi\big(\tilde{\iota}_{\mathbf{i}(1)}(a_1)\,\tilde{\iota}_{\mathbf{i}(2)}(a_2)\cdots\tilde{\iota}_{\mathbf{i}(n)}(a_n)\big)$$

for all n-tuples $\mathbf{i} \colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$, $(a_1, \ldots, a_n) \in \mathcal{A}_0^n$ and $n \in \mathbb{N}$.

Notation $$(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\tilde{\iota}_0, \tilde{\iota}_1, \tilde{\iota}_2, \ldots)$$

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n\geq 0}$ is

- **exchangeable** if $(\iota_0, \iota_1, \iota_2, \dots) \stackrel{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \dots)$ for any (finite) permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n \geq 0}$ is

- **exchangeable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \ldots)$ for any (finite) permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.
- **spreadable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \ldots)$ for any subsequence $(n_0, n_1, n_2, \ldots)$ of $(0, 1, 2, \ldots)$.

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n\geq 0}$ is

- **exchangeable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \ldots)$ for any (finite) permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.
- **spreadable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \ldots)$ for any subsequence $(n_0, n_1, n_2, \ldots)$ of $(0, 1, 2, \ldots)$.
- **stationary** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_k, \iota_{k+1}, \iota_{k+2}, \ldots)$ for all $k \in \mathbb{N}$.

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n\geq 0}$ is

- **exchangeable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \ldots)$ for any (finite) permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.
- **spreadable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \ldots)$ for any subsequence $(n_0, n_1, n_2, \ldots)$ of $(0, 1, 2, \ldots)$.
- **stationary** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_k, \iota_{k+1}, \iota_{k+2}, \ldots)$ for all $k \in \mathbb{N}$.

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A sequence $(x_n)_{n\geq 0}$ is

- **exchangeable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \ldots)$ for any (finite) permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.
- **spreadable** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \ldots)$ for any subsequence $(n_0, n_1, n_2, \ldots)$ of $(0, 1, 2, \ldots)$.
- **stationary** if $(\iota_0, \iota_1, \iota_2, \ldots) \stackrel{\text{distr}}{=} (\iota_k, \iota_{k+1}, \iota_{k+2}, \ldots)$ for all $k \in \mathbb{N}$.

Lemma (Hierarchy of distributional symmetries)

exchangeability $\Rightarrow$ spreadability $\Rightarrow$ stationarity $\Rightarrow$ identical distr.

Noncommutative conditional independence

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ with φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ $(i = 0, 1, 2)$.

Noncommutative conditional independence

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ with φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ $(i = 0, 1, 2)$.

Definition

$\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{A}_0$-**independent** if

$$E_0(xy) = E_0(x)E_0(y) \qquad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

Noncommutative conditional independence

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ with φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ $(i = 0, 1, 2)$.

Definition

$\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{A}_0$-**independent** if

$$E_0(xy) = E_0(x)E_0(y) \qquad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

Remarks

- $\mathcal{A}_0 \subset \mathcal{A}_0 \vee \mathcal{A}_1, \mathcal{A}_0 \vee \mathcal{A}_2 \subset \mathcal{A}$ is a **commuting square**
- $\mathcal{A}_0 \simeq \mathbb{C}$: Kümmerer's notion of n.c. independence
- $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$: cond. independence w.r.t. sub-σ-algebra

Noncommutative conditional independence

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ with φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ $(i = 0, 1, 2)$.

Definition

$\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{A}_0$-**independent** if

$$E_0(xy) = E_0(x)E_0(y) \qquad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

Remarks

- $\mathcal{A}_0 \subset \mathcal{A}_0 \vee \mathcal{A}_1, \mathcal{A}_0 \vee \mathcal{A}_2 \subset \mathcal{A}$ is a **commuting square**
- $\mathcal{A}_0 \simeq \mathbb{C}$: Kümmerer's notion of n.c. independence
- $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$: cond. independence w.r.t. sub-σ-algebra
- Many different forms of noncommutative independence!

Noncommutative conditional independence

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ with φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ $(i = 0, 1, 2)$.

Definition

$\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathcal{A}_0$-**independent** if

$$E_0(xy) = E_0(x)E_0(y) \qquad (x \in \mathcal{A}_0 \vee \mathcal{A}_1, y \in \mathcal{A}_0 \vee \mathcal{A}_2)$$

Remarks

- $\mathcal{A}_0 \subset \mathcal{A}_0 \vee \mathcal{A}_1, \mathcal{A}_0 \vee \mathcal{A}_2 \subset \mathcal{A}$ is a **commuting square**
- $\mathcal{A}_0 \simeq \mathbb{C}$: Kümmerer's notion of n.c. independence
- $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$: cond. independence w.r.t. sub-σ-algebra
- Many different forms of noncommutative independence!
- $\mathbb{C}$-independence & Speicher's universality rules
  $\rightsquigarrow$ tensor independence or free independence

Conditional independence of sequences

A sequence of random variables $(\iota_n)_{n\in\mathbb{N}_0}\colon (\mathcal{A}_0,\varphi_0)\to(\mathcal{A},\varphi)$ is **(full)** $\mathcal{B}$**-independent** if

$$\bigvee\{\iota_i(\mathcal{A}_0)\,|\,i\in I\}\vee\mathcal{B}\quad\text{and}\quad\bigvee\{\iota_j(\mathcal{A}_0)\,|\,j\in J\}\vee\mathcal{B}$$

are $\mathcal{B}$-independent whenever $I\cap J=\emptyset$ with $I,J\subset\mathbb{N}_0$.

Conditional independence of sequences

A sequence of random variables $(\iota_n)_{n\in\mathbb{N}_0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is **(full) $\mathcal{B}$-independent** if

$$\bigvee\{\iota_i(\mathcal{A}_0) \,|\, i \in I\} \vee \mathcal{B} \quad \text{and} \quad \bigvee\{\iota_j(\mathcal{A}_0) \,|\, j \in J\} \vee \mathcal{B}$$

are $\mathcal{B}$-independent whenever $I \cap J = \emptyset$ with $I, J \subset \mathbb{N}_0$.

Remark

Many interesting notions are possible for sequences:

1. conditional top-order independence
2. conditional order independence
3. **conditional full independence**
4. discrete noncommutative random measure factorizations

Noncommutative extended De Finetti theorem

Let $(\iota_n)_{n\geq 0}$ be random variables as before with tail algebra

$$\mathcal{A}^{\mathrm{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \{\iota_k(\mathcal{A}_0)\},$$

and consider:

(a) $(\iota_n)_{n\geq 0}$ is exchangeable
(c) $(\iota_n)_{n\geq 0}$ is spreadable
(d) $(\iota_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\mathrm{tail}}$-independent
(e) $(\iota_n)_{n\geq 0}$ is identically distributed and $\mathcal{A}^{\mathrm{tail}}$-independent

Noncommutative extended De Finetti theorem

Let $(\iota_n)_{n\geq 0}$ be random variables as before with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \{\iota_k(\mathcal{A}_0)\},$$

and consider:

(a) $(\iota_n)_{n\geq 0}$ is exchangeable
(c) $(\iota_n)_{n\geq 0}$ is spreadable
(d) $(\iota_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent
(e) $(\iota_n)_{n\geq 0}$ is identically distributed and $\mathcal{A}^{\text{tail}}$-independent

Theorem (K. '07-'08)

(a) $\Rightarrow$ *(c)* $\Rightarrow$ *(d)* $\Rightarrow$ *(e)*,

Noncommutative extended De Finetti theorem

Let $(\iota_n)_{n\geq 0}$ be random variables as before with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \{\iota_k(\mathcal{A}_0)\},$$

and consider:

(a) $(\iota_n)_{n\geq 0}$ is exchangeable
(c) $(\iota_n)_{n\geq 0}$ is spreadable
(d) $(\iota_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent
(e) $(\iota_n)_{n\geq 0}$ is identically distributed and $\mathcal{A}^{\text{tail}}$-independent

Theorem (K. '07-'08, Gohm & K. '08)

(a) $\Rightarrow$ *(c)* $\Rightarrow$ *(d)* $\Rightarrow$ *(e)*, *but (a)* $\not\Leftarrow$ *(c)* $\not\Leftarrow$ *(d)* $\not\Leftarrow$ *(e)*

Noncommutative extended De Finetti theorem

Let $(\iota_n)_{n\geq 0}$ be random variables as before with tail algebra

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \{\iota_k(\mathcal{A}_0)\},$$

and consider:

(a) $(\iota_n)_{n\geq 0}$ is exchangeable

(c) $(\iota_n)_{n\geq 0}$ is spreadable

(d) $(\iota_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent

(e) $(\iota_n)_{n\geq 0}$ is identically distributed and $\mathcal{A}^{\text{tail}}$-independent

Theorem (K. '07-'08, Gohm & K. '08)

(a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), but (a) $\not\Leftarrow$ (c) $\not\Leftarrow$ (d) $\not\Leftarrow$ (e)

Remark

(a) to (e) are equivalent if the operators x_n mutually commute.
[De Finetti '31, Ryll-Nardzewski '57,..., Størmer '69,... Hudson '76, ... Petz '89,..., Accardi&Lu '93, ...]

An ingredient for proving full cond. independence

Localization Preserving Mean Ergodic Theorem (K '08)

Let $(\mathcal{M}, \psi)$ be a probability space and suppose $\{\alpha_N\}_{N \in \mathbb{N}_0}$ is a family of ψ-preserving completely positive linear maps of $\mathcal{M}$ satisfying

1. $\mathcal{M}^{\alpha_N} \subset \mathcal{M}^{\alpha_{N+1}}$ for all $N \in \mathbb{N}_0$;
2. $\mathcal{M} = \bigvee_{N \in \mathbb{N}_0} \mathcal{M}^{\alpha_N}$.

Further let

$$M_N^{(n)} := \frac{1}{n} \sum_{k=0}^{n-1} \alpha_N^k \quad \text{and} \quad T_N := \overrightarrow{\prod}_{l=0}^{N} \alpha_l^{lN} M_l^{(N)}.$$

Then we have

$$\text{SOT-}\lim_{N \to \infty} T_N(x) = E_{\mathcal{M}^{\alpha_0}}(x)$$

for any $x \in \mathcal{M}$.

Discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073-1120 (2010)

Discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares
- Exchangeability is too weak to identify the structure of the underlying noncommutative probability space

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073-1120 (2010)

Discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares
- Exchangeability is too weak to identify the structure of the underlying noncommutative probability space
- All reverse implications in the noncommutative extended de Finetti theorem fail due to deep structural reasons!

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073-1120 (2010)

Discussion

- Noncommutative conditional independence emerges from distributional symmetries in terms of commuting squares
- Exchangeability is too weak to identify the structure of the underlying noncommutative probability space
- All reverse implications in the noncommutative extended de Finetti theorem fail due to deep structural reasons!
- This will become clear from braidability...

For further details see:

C. Köstler. *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073-1120 (2010)

R. Gohm & C. Köstler. *Noncommutative independence from the braid group $\mathbb{B}_\infty$.* Commun. Math. Phys. **289**, 435–482 (2009)

Algebraic Definition (Artin 1925)

$\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } \mid i - j \mid = 1 \qquad \text{(B1)}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } \mid i - j \mid > 1 \qquad \text{(B2)}$$

Figure: Artin generators σ_i (left) and σ_i^{-1} (right)

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \ldots \subset \mathbb{B}_\infty$ (inductive limit)

Braidability

Definition (Gohm & K. '08)

A sequence $(\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is **braidable** if there exists a representation $\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ satisfying:

$$\iota_n = \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1)\iota_0 \qquad \text{for all } n \geq 1;$$
$$\iota_0 = \rho(\sigma_n)\iota_0 \qquad \text{if } n \geq 2.$$

Braidability

Definition (Gohm & K. '08)

A sequence $(\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is **braidable** if there exists a representation $\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ satisfying:

$$\begin{aligned} \iota_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1)\iota_0 && \text{for all } n \geq 1; \\ \iota_0 &= \rho(\sigma_n)\iota_0 && \text{if } n \geq 2. \end{aligned}$$

Braidability extends exchangeability

- If $\rho(\sigma_n^2) = \mathrm{id}$ for all n, one has a representation of $\mathbb{S}_\infty$.

Braidability

Definition (Gohm & K. '08)

A sequence $(\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is **braidable** if there exists a representation $\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ satisfying:

$$\begin{aligned} \iota_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1)\iota_0 && \text{for all } n \geq 1; \\ \iota_0 &= \rho(\sigma_n)\iota_0 && \text{if } n \geq 2. \end{aligned}$$

Braidability extends exchangeability

- If $\rho(\sigma_n^2) = \mathrm{id}$ for all n, one has a representation of $\mathbb{S}_\infty$.
- $(\iota_n)_{n\geq 0}$ is exchangeable $\Leftrightarrow \begin{cases} (\iota_n)_{n\geq 0} \text{ is braidable and} \\ \rho(\sigma_n^2) = \mathrm{id} \text{ for all } n. \end{cases}$

Braidability implies spreadability

Consider the conditions:

(a) $(x_n)_{n\geq 0}$ is exchangeable

(b) $(x_n)_{n\geq 0}$ is **braidable**

(c) $(x_n)_{n\geq 0}$ is spreadable

(d) $(x_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent

Braidability implies spreadability

Consider the conditions:

(a) $(x_n)_{n\geq 0}$ is exchangeable

(b) $(x_n)_{n\geq 0}$ is **braidable**

(c) $(x_n)_{n\geq 0}$ is spreadable

(d) $(x_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent

Theorem (Gohm & K. '08)

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \quad \text{but } (a) \not\Leftarrow (b) \not\Leftarrow (d)$$

Braidability implies spreadability

Consider the conditions:

(a) $(x_n)_{n\geq 0}$ is exchangeable

(b) $(x_n)_{n\geq 0}$ is **braidable**

(c) $(x_n)_{n\geq 0}$ is spreadable

(d) $(x_n)_{n\geq 0}$ is stationary and $\mathcal{A}^{\text{tail}}$-independent

Theorem (Gohm & K. '08)

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \quad \text{but } (a) \not\Leftarrow (b) \not\Leftarrow (d)$$

Observation

Large and interesting class of spreadable sequences is obtained from braidability.

Remarks

There are many examples of braidability!

- subfactor inclusion with small Jones index ('Jones-Temperley-Lieb algebras and Hecke algebras')
- left regular representation of $\mathbb{B}_\infty$
- vertex models in quantum statistical physics ('Yang-Baxter equations')
- ...
- representations of the symmetric group $\mathbb{S}_\infty$

For further details see next talk by Gohm and:

R. Gohm & C. Köstler. *Noncommutative independence from the braid group $\mathbb{B}_\infty$*. Commun. Math. Phys. **289**, 435–482 (2009)

References

C. Köstler. *A noncommutative extended de Finetti theorem.* J. Funct. Anal. **258**, 1073-1120 (2010)
(electronic: `arXiv:0806.3621v1`)

R. Gohm & C. Köstler. *Noncommutative independence from the braid group $\mathbb{B}_\infty$.* Commun. Math. Phys. **289**, 435–482 (2009)
(electronic: `arXiv:0806.3691v2`)

C. Köstler & R. Speicher. *A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation.* Commun. Math. Phys. **291**(2), 473–490 (2009)
(electronic: `arXiv:0807.0677v1`)

C. Köstler. *On Lehner's 'free' noncommutative analogue of de Finetti's theorem.* 11 pages. To appear in Proc. Amer. Math. Soc.
(electronic: `arXiv:0806.3632v1`)

R. Gohm & C. Köstler. *Noncommutative independence from characters of the symmetric group $\mathbb{S}_\infty$.* 47 pages. Preprint (2010).
(electronic: `arXiv:1005.5726`)