SOME REFERENCES

Free probability was introduced by Voiculescu in [V1]. However, the reduced free product of C*–algebras was also considered by Avitzour in [A], (where simplicity was proved under certain hypotheses).

An early general reference for many basic topics (free convolution, reduced free product, the free analogue of the Gaussian functor, etc.) and some more advanced topics is the book [VDN]. See also the book by Nica and Speicher [NS] for more combinatorial aspects.

For Eric Ricard's argument that free product of faithful states (or conditional expectations) if faithful, see [I], where it is used with permission. For an example showing that the reduced free product lacks the universal property, see [DR]. For the embedding result, see [BD].

The first appearance of asymptotic freeness in random matrices was in Voiculescu's pioneering paper [V3]. This was later extended in [D1] and, vastly, in [V7], though none of these contain the nice proof that Speicher presented. Voiculescu's first applications to free group factors were in [V2]. Rădulescu's proof that the fundamental group of $L(\mathbf{F}_\infty)$ is $\mathbf{R}_+^*$ is in [R1]. The proof that $R * L(Z) \cong L(\mathbf{F}_s)$ is found in [D1]. The interpolated free group factors were found (independently) in [R2] and [D3], by slightly different constructions. The results on free products of finite dimensional, hyperfinite and other algebras, using the heuristic quantity "free dimension," are in [D2].

The results of Voiculescu involving free entropy that we discussed are contained in his papers [V4], [V5] and [V6], while Kenley Jung's reformulation of free entropy dimension is in [J1] and his computation of this for generators of hyperfinite von Neumann algebras is in [J2]. The equivalence of $\delta_0(\cdots) \geq 0$ and embeddability in R^ω is found in [BDJ].

B–valued freeness and the reduced amalgamated free product of C*–algebras was introduced by Voiculescu already in [V1], and the W*–case was well understood at this time; (but see [BD] for a careful treatment).

The result of Haagerup and Schultz on existence of hyperinvariant subspaces (discussed in the problem session) is in [HS], while the case of the quasinilpotent DT–operator is in [DH], and Tucci's operators are in [T].

REFERENCES

[A]   D. Avitzour, *Free products of C*–algebras*, Trans. Amer. Math. Soc. **271** (1982), 423–465.

[BD]   E. F. Blanchard and K. Dykema, *Embeddings of reduced free products of operator algebras*, Pacific J. Math. **199** (2001), 1–19.

[BDJ] N. Brown, K. Dykema, and K. Jung, *Free entropy dimension in amalgamated free products*, with an appendix by W. Lück, Proc. London Math. Soc. **97** (2008), 339–367.

[D1]   K. Dykema, *On certain free product factors via an extended matrix model*, J. Funct. Anal. **112** (1993), 31–60.

[D2]   _____, *Free products of hyperfinite von Neumann algebras and free dimension*, Duke Math. J. **69** (1993), 97–119.

[D3]   _____, *Interpolated free group factors*, Pacific J. Math. **163** (1994), 123–135.

[DH]   K. Dykema and U. Haagerup, *Invariant subspaces of the quasinilpotent DT–operator*, J. Funct. Anal. **209** (2004), 332-366.

[DR]   K. Dykema and M. Rørdam, *Projections in free product C^*–algebras.*, Geom. Funct. Anal. **8** (1998), 1–16; erratum **10** (2000), 975.

[I]   N. Ivanov, *On the structure of some reduced amalgamated free product C^*–algebras*, available at http://arxiv.org/abs/0705.3919.

[HS]   U. Haagerup and H. Schultz, *Invariant subspaces for operators in a general II_1–factor*, Publ. Math. Inst. Hautes Études Sci. (2009), 19-111.

[J1]   K. Jung, *A free entropy dimension lemma*, Pacific J. Math. **177** (2003), 265–271.

[J2]   _____, *The free entropy dimension of hyperfinite von Neumann algebras*, Trans. Amer. Math. Soc. **355** (2003), 5053–5089.

[NS]   A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge University Press, 2006.

[R1]   F. Rădulescu, *The fundamental group of the von Neumann algebra of a free group with infinitely many generators is $\mathbb{R}_+\backslash 0$*, J. Amer. Math. Soc. **5** (1992), 517–532.

[R2]   _____, *Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group of noninteger index*, Invent. Math. **115** (1994), 347–389.

[T]   G. Tucci, *Some quasinilpotent generators of the hyperfinite II_1 factor*, J. Funct. Anal. **254** (2008), 2969-2994.

[V1]   D. Voiculescu, *Symmetries of some reduced free product C^*–algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory (Busteni, 1983), Lecture Notes in Mathematics, vol. 1132, Springer–Verlag, 1985, pp. 556–588.

[V2]   _____, *Circular and semicircular systems and free product factors*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, 1990, pp. 45–60.

[V3]   _____, *Limit laws for random matrices and free products*, Invent. Math. **104** (1991), 201–220.

[V4]   _____, *The analogues of entropy and of Fisher's information measure in free probability theory, II*, Invent. Math. **118** (1994), 411–440.

[V5]   _____, *The analogues of entropy and of Fisher's information measure in free probability theory III: The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), 172–199.

[V6]   _____, *The analogues of entropy and of Fisher's information measure in free probability theory IV: Maximal entropy and freeness*, Free Probability Theory (Waterloo, Ontario, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., 1997, pp. 293–302.

[V7]   _____, *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, Internat. Math. Res. Notices **1998**, 41–63.

[VDN]   D. Voiculescu, K. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series, vol. 1, American Mathematical Society, Providence, RI, 1992.