II_1 factors and Ergodic Theory

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Overview

- We shall review long-standing links between ergodic theory and von Neumann algebras - from the original construction of factors[1] using the *group-measure-space construction*, to more recent use of *von Neumann dimensions* of modules over some II_1 factors for defining ℓ^2-Betti numbers of standard equivalence relations and obtaining consequent rigidity theorems.

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- Outline of lecture
  - von Neumann algebras
  - Ergodic Theory
  - Group measure space construction
  - II_1 factors
  - Standard equivalence relations
  - Orbit equivalence
  - Measurable equivalence
  - ℓ^2-Betti numbers
  - Kadison conjecture
  - strong rigidity theorems.

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Proposition: The following conditions on a subset $M \subset \mathcal{L}(\mathcal{H})$ are equivalent:

1. There exists a unitary group representation $\pi : G \to \mathcal{U}(\mathcal{H})$ sich that

$$M = \pi(G)' = \{x \in \mathcal{L}(\mathcal{H}) : x\pi(g) = \pi(g)x \ \forall g \in G\}$$

2. M is a unital *-subalgebra of $\mathcal{L}(\mathcal{H})$ satisfying

$$M = M'' = (M')'$$

Such an M is called a **von Neumann algebra**. □

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Example: $L^\infty(X, \mathcal{B}, \mu) \hookrightarrow \mathcal{L}(L^2(X, \mathcal{B}, \mu))$ via $f \cdot \xi = f\xi$. This is essentially the only abelian von Neumann algebra.

MvN equivalence and order of projections

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If $M = \pi(G)'$, then $p \in \mathcal{P}(M)$ iff *ran* p is π-stable; so $\mathcal{P}(M)$ parametrises the subrepresentations of π.

Def: For $p, q \in \mathcal{P}(M)$ say

- $$p \sim_M q \Leftrightarrow \exists u \in M \text{ such that } u^*u = p, uu^* = q$$
- $$p \prec_M q \Leftrightarrow \exists u \in M \text{ such that } u^*u = p, uu^* \leq q$$

- p is *finite* if $p \sim_M p_0 \leq p$ implies $p_0 = p$

Proposition: The following conditions on a von Neumann algebra M are equivalent:

1. $\forall p, q \in \mathcal{P}(M)$ either $p \prec_M q$ or $q \prec_M p$ (i.e., if $M = \pi(G)'$, then π is *isotypical*)
2. $Z(M) = M \cap M' = \mathbb{C}$

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Def: A factor is called *finite* if 1 is a finite projection. A finite factor which is infinite-dimensional as a $\mathbb{C}$-vector space is called a II_1 factor.

Let M be a II_1 factor. Then

- M admits a positive tracial state, i.e., there exists a linear functional $tr_M : M \to \mathbb{C}$ such that
  1. $tr_M(x^*x) \geq 0 \ \forall x \in M$
  2. $tr_M(xy) = tr_M(yx) \ \forall x, y \in M$
  3. $tr_M(1) = 1$
- The functional tr_M is uniquely determined by the above properties, and is *faithful* : i.e., $tr_M(x^*x) = 0, x \in M \Rightarrow x = 0$.
- $p \sim_M q \Leftrightarrow tr_M(p) = tr_M(q)$.
- $\{tr_M(p) : p \in \mathcal{P}(M)\} = [0, 1]$.

Def: A *module* over a II_1 factor M is a triple $(\mathcal{H}_\pi, M_\pi, \pi)$ where $\mathcal{H}_\pi$ is some Hilbert space, $M_\pi \subset \mathcal{L}(\mathcal{H}_\pi)$ is a von Neumann algebra, and $\pi : M \rightarrow M_\pi$ is an isomorphism of *-algebras.

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Proposition: M-modules are determined, up to isomorphism, by their *M-dimension*; thus, to an M-module $\mathcal{K}$ is associated a number $dim_M \mathcal{K} \in [0, \infty]$ so that

1. there exists an M-linear bounded operator mapping $\mathcal{H}_1$ isomorphically onto $\mathcal{H}_2$ iff $dim_M \mathcal{H}_1 = dim_M \mathcal{H}_2$
2. $dim_M(\oplus_{n=1}^{\infty} \mathcal{H}_n) = \sum_{n=1}^{\infty} dim_M \mathcal{H}_n$

Further, each $d \in [0, \infty]$ arises as $dim_M \mathcal{H}$ for some M-module $\mathcal{H}$.

If Γ is a countable group, let $\{\xi_\gamma : \gamma \in \Gamma\}$ denote the standard orthonormal basis of $\ell^2(\Gamma)$. Let us write λ and ρ respectively for the *left-* and *right-regular* representations $\lambda, \rho : \Gamma \to \mathcal{L}(\ell^2(\Gamma))$ defined by

$$\lambda_\gamma \xi_\kappa = \xi_{\gamma\kappa} = \rho_{\kappa^{-1}} \xi_\gamma$$

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Proposition:

1. $(L\Gamma)' = \rho(\Gamma)''$
2. the equation $tr(x) = \langle x\xi_1, \xi_1 \rangle$ defines a faithful trace on $L\Gamma$ as well as on $(L\Gamma)'$
3. $L\Gamma$ is a II_1 factor iff every conjugacy class other than $\{1\}$ in Γ is infinite, and $\Gamma \neq \{1\}$.

The setting is a triple $(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B})$ is a *standard Borel space* and μ is a (usually non-atomic) probability measure defined on $\mathcal{B}$. Our *standard probability spaces* will be assumed to be *complete* - i.e., $\mathcal{B}$ contains all μ-null sets.

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An *isomorphism* between standard probability spaces $(X_i, \mathcal{B}_i, \mu_i), i = 1, 2$ is a bimeasurable measure-preserving bijection of conull sets; i.e., it is a bijective map $T : X_1 \setminus N_1 \to X_2 \setminus N_2$, where N_i are μ_i-null sets, such that

1. $E \in \mathcal{B}_2 \Leftrightarrow T^{-1}(E) \in \mathcal{B}_1$
2. $E \in \mathcal{B}_2 \Rightarrow \mu_1(T^{-1}(E)) = \mu_2(E)$.

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Note: For each isomorphism T as above, the equation

$$\alpha_T(f) = f \circ T^{-1}$$

defines an isomorphism of von Neumann algebras:

$$\alpha_T : L^\infty(X_1, \mathcal{B}_1, \mu_1) \to L^\infty(X_2, \mathcal{B}_2, \mu_2).$$

Further, the map $T \mapsto \alpha_T$ is a homomorphism of $Aut(X, \mathcal{B}, \mu)$ into $Aut(L^\infty(X, \mathcal{B}, \mu))$.

Definition: A homomorphism $\Gamma \ni \gamma \to T_\gamma \in Aut(X, \mathcal{B}, \mu)$ is called an *action* of Γ on $(X, \mathcal{B}, \mu)$; such an action is said to be *ergodic* if it satisfies any of the following equivalent conditions:

1. $E \in \mathcal{B}, \mu(T_\gamma^{-1}(E)\Delta E) = 0 \forall \gamma \in \Gamma \Rightarrow \mu(E) = 0$ or $\mu(X \setminus E) = 0$.
2. $E, F \in \mathcal{B}, \mu(E), \mu(F) > 0 \Rightarrow \ \exists \gamma \in \Gamma$ such that $\mu(E \cap T_\gamma^{-1}(F)) > 0$
3. $f \in L^\infty(X, \mathcal{B}, \mu), f \circ T_\gamma = f$ *a.e.* $\forall \gamma \in \Gamma \Rightarrow \exists C \in \mathbb{C}$ such that $f = C$ *a.e.*
4. $f \in L^2(X, \mathcal{B}, \mu), f \circ T_\gamma = f$ *a.e.* $\forall \gamma \in \Gamma \Rightarrow \exists C \in \mathbb{C}$ such that $f = C$ *a.e.*

Group-measure space (a.k.a. crossed-product) construction

Let $A = L^\infty(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B}, \mu)$ is a standard probability space, and suppose α is an action of a countable group Γ on $(X, \mathcal{B}, \mu)$.

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The equations

$$\begin{aligned} (\pi(f)\xi)(\kappa) &= (f \circ \alpha_\kappa)\xi(\kappa) \\ (\lambda(\gamma)\xi)(\kappa) &= \xi(\gamma^{-1}\kappa) \end{aligned}$$

define, resp., a *-homomorphism of A and a unitary representation of Γ on the Hilbert space $\ell^2(\Gamma, L^2(X, \mathcal{B}, \mu))$ which satisfy the commutation relation

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Def: The crossed-product is defined to be the generated von Neumann algebra

$$A \rtimes_\alpha \Gamma = (\pi(A) \cup \lambda(\Gamma))''$$

Theorem: Let $X, \mathcal{B}, \mu)$ be a non-atomic standard probability space. Suppose $\alpha : G \to Aut(X/\mathcal{B}, \mu)$ defines a *free action* of Γ; i.e., suppose $\mu(\{x \in X : \alpha_\gamma(x) = x\}) = 0 \; \forall \gamma \neq 1 \in \Gamma$.

Then $A \times_\alpha \Gamma$ is a II_1 factor iff the action is ergodic.

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Example: Let $\Gamma = \mathbb{Z}, X = \mathbb{T}, \mathcal{B} = \mathcal{B}_\mathbb{T}$ and μ be normalised arc-length, so $\mu(X) = 1$; let the action be defined by $\alpha_n(e^{2\pi i\theta}) = e^{2\pi i(\theta + n\phi)}$, where ϕ is irrational.

More generally, we could have considered the action on a compact second countable group defined by translation of any countable dense subgroup.

Suppose Γ acts freely and ergodically on a standard probability space $(X, \mathcal{B}, \mu)$ (and preserves μ) - so $M = A \rtimes \Gamma$ is a II_1 factor.

It turns out that, as far as the factor M is concerned, the group Γ itself is not important; what matters is the relation

$$\mathcal{R} = \mathcal{R}_\Gamma = \{(x, \gamma \cdot x) : x \in X, \gamma \in \Gamma\}.$$

This equivalence relation is a standard Borel space with the Borel structure given by $\mathcal{C} = \{B \in \mathcal{B} \times \mathcal{B} : B \subset \mathcal{R}\}$, and it has countable equivalence classes. Also, there is a natural σ-finite 'counting measure' ν defined on $(\mathcal{R}, \mathcal{C})$ by

$$\begin{aligned} \nu_l(C) &= \int_X |\pi_l^{-1}(x) \cap C| d\mu(x) \\ &= \int_X |\pi_r^{-1}(y) \cap C| d\mu(y) \\ &= \nu_r(C) \end{aligned}$$

where $\pi_l : \mathcal{R} \to X$ and $\pi_r : \mathcal{R} \to X$ are the left- and right-projection defined by $\pi_l(y, z) = y = \pi_r(x, y)$.

Abstract standard equivalence relations

Feldman and Moore initiated the study of abstract standard equivalence relations $\mathcal{R}$ with countable equivalence classes, which are μ-invariant in the sense that the associated 'left- and right- counting measures' ν_l and ν_r agree. (We shall simply write ν for this 'counting' measure.)

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Such an $\mathcal{R}$ is called 'ergodic' if the only Borel subsets of X which are '$\mathcal{R}$-saturated' are μ-null or conull. They proved that any standard equivalence relation $\mathcal{R} \subset X \times X$ which is μ-invariant can be realised as an $\mathcal{R}_\Gamma$ for a necessarily ergodic and measure-preserving action of some countable group Γ, and asked if the action could always be chosen to be a free one. Later, Furman showed that this was not necessarily so.

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FM also associated a II_1 factor $L\mathcal{R}$ to such an ergodic $\mathcal{R}$, which reduces to the crossed product in the concrete example of a free ergodic action.

Def: Two (probobility measure preserving) dynamical systems $(X_i, \mathcal{B}_i, \mu_i, \Gamma_i, \alpha_i), i = 1, 2$ (or equivalently, their induced equivalence relations $\mathcal{R}_i$) are said to be *orbit equivalent* if there exists an isomorphism $T : X_1 \to X_2$ such that $T(\alpha_1(\Gamma_1)x) = \alpha_2(\Gamma_2)Tx\ \mu_1 - a.e.$ (or equivalently, $(T \times T)(\mathcal{R}_1) = \mathcal{R}_2\ mod\ \nu_2$).

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Theorem: With the foregoing notation, write $A_i = L^\infty(X_i, \mathcal{B}_i, \mu_i)$ TFAE:

1. We have an isomorphism of pairs

$$(A_1 \rtimes_{\alpha_1} \Gamma_1, A_1) \cong (A_2 \rtimes_{\alpha_2} \Gamma_2, A_2)$$

2. $\mathcal{R}_1$ and $\mathcal{R}_2$ are orbit equivalent.

Questions: When are two standard equivalence relations orbit equivalent? How much of (Γ, α) does $\mathcal{R}$ remember?

Assume henceforth that all our probability spaces are non-atomic.

Theorem: (Dye) The equivalence relations determined by any two ergodic actions of $\mathbb{Z}$ are orbit equivalent. □

A volume of work by many people, notably Dye, Connes, Feldman, Krieger, .. culminated in the following result.

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A volume of work by many people, notably Dye, Connes, Feldman, Krieger, .. culminated in the following result.

Theorem: (Ornstein-Weiss) Ergodic actions (on a standard non-atomic probabaility space) of any two infinite amenable groups produce orbit equivalent equivalence relations.

Equivalence relations determined by such actions of such groups are characterised by the following property of **hyperfiniteness**:

there exists a sequence of standard equivalence relations $\mathcal{R}_n$ on X with finite equivalence classes such that

$$\mathcal{R}_n \subset \mathcal{R}_{n+1} \forall n \text{ and } \mathcal{R} = \cup_n \mathcal{R}_n.$$

SOE and ME

For ergodic actions, the quotient space $\Gamma\backslash X$ has only a trivial Borel structure; the standard equivalence relation $\mathcal{R}$ is a good substitute. If $\mu(A) > 0$, then almost every orbit meets A, so the induced relation $\mathcal{R}_A = \mathcal{R} \cap (A \times A)$ should be an equally good candidate to describe the space of orbits in X.

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Defs.: (a) Call two equivalence relations $\mathcal{R}_i$ *stably orbit equivalent* (or simply SOE), if there exists Borel subsets $A_i \subset X_i$ of positive measure which meet almost every orbit, a constant $c > 0$, and a Borel isomorphism $f : (A_1, \mathcal{B}_{A_1}) \to (A_2, \mathcal{B}_{A_2})$ which scales measure by a factor of c, such that $(f \times f)(\mathcal{R}_{A_1}) = \mathcal{R}_{A_2}$ (mod null sets). The constant c is called the compression constant of the SOE.

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(b) On the other hand, call two countable groups $\Gamma_i, i = 1, 2$ *measurably equivalent* (or simply ME) if they admit commuting free actions on a standard (possibly σ-finite) measure space $(X, \mathcal{B}, \mu)$, which admit a *fundamental domain* F_i of finite measure; call the ratio $\frac{\mu(F_2)}{\mu(F_1)}$ the compression constant of the ME.

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Theorem: (Furman) Γ_1 is ME to Γ_2 with compression constant c if and only if Γ_1 and Γ_2 admit free actions on standard probability space such that the associated equivalence relations are *SOE* with compression constant c.

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Gaboriau then made sense, (still in terms of the von Neumann dimension of a suitable M-module of ℓ^2-chains) of ℓ^2 Betti numbers for any standard equivalence relation, and related these to the objects defined by Cheeger and Gromov.

Theorem:

1. If an equivalence relation $\mathcal{R}$ is produced by a free action of a countable group Γ, then $\beta_n(\Gamma) = \beta_n(\mathcal{R})$, where the left side is defined á la Gromov-Cheeger and the right side is defined á la Gaboriau.
2. If $\Gamma_i, i = 1, 2$ are ME with compression constant c, then $\beta_n(\Gamma_2) = c\beta_n(\Gamma_1)$; in particular, $\beta_n(\Gamma_1) = \beta_n(\Gamma_2)$ if the Γ_i admit free actions which produce orbit equivalent equivalence relations.

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The simplest example of two ME groups is a pair of lattices in a locally compact group with not necessarily compact quotients, acting by left- and right- multiplication on the ambient group.

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The simplest example of two ME groups is a pair of lattices in a locally compact group with not necessarily compact quotients, acting by left- and right- multiplication on the ambient group.

Theorem: (Gaboriau)

1. No lattice in $SP(n, 1)$ is ME to a lattice in $SP(p, 1)$ if $n \neq p$.
2. No lattice in $SU(n, 1)$ is ME to a lattice in $SU(p, 1)$ if $n \neq p$.
3. No lattice in $SO(2n, 1)$ is ME to a lattice in $SO(2p, 1)$ if $n \neq p$.

Proof: This is due to the following computations made by Borel:

$$\begin{aligned}
\beta_i(\Gamma(SP(m,1))) \neq 0 \quad &\Leftrightarrow \quad i = 2m \\
\beta_i(\Gamma(SU(m,1))) \neq 0 \quad &\Leftrightarrow \quad i = m \\
\beta_i(\Gamma(SO(2m,1)) \neq 0 \quad &\Leftrightarrow \quad i = m
\end{aligned}$$

where we write $\Gamma(G)$ to denote any lattice in G.

[2] Actually, Kadison wondered if $M_2(M) \cong M$ for any II_1 factor M.

Proof: This is due to the following computations made by Borel:

$$\begin{aligned}\beta_i(\Gamma(SP(m,1))) \neq 0 &\Leftrightarrow i = 2m\\ \beta_i(\Gamma(SU(m,1))) \neq 0 &\Leftrightarrow i = m\\ \beta_i(\Gamma(SO(2m,1)) \neq 0 &\Leftrightarrow i = m\end{aligned}$$

where we write $\Gamma(G)$ to denote any lattice in G.

Kadison's conjecture:

If M is a II_1 factor, $d \in (0,\infty)$ and $\mathcal{H}_d$ is an M-module with $dim_M(\mathcal{H}_d) = d$, then $End_M(\mathcal{H}_d) = M_d(M)$. The *fundamental group* of M is defined by

$$\mathcal{F}(M) = \{d \in (0,\infty) : M \cong M_d(M)\}$$

and Kadison's conjecture[2] (unsolved for several decades) asks if $\mathcal{F}(M)$ - which is always a mutiplicative subgroup of $\mathbb{R}^\times$ - can be trivial.

[2]Actually, Kadison wondered if $M_2(M) \cong M$ for any II_1 factor M.

Using Gaboriau's ℓ^2 Betti numbers, Popa showed the existence of many countable groups admitting free ergodic actions α which produce equivalence relations $\mathcal{R}$ such that the corresponding II_1 factor $L\mathcal{R}$ has trivial fundamental group.

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In fact $L^\infty(\mathbb{T}^2) \rtimes SL(2, \mathbb{Z})$ is an example of such a factor.

Popa's use of Gaboriau's work

Using Gaboriau's ℓ^2 Betti numbers, Popa showed the existence of many countable groups admitting free ergodic actions α which produce equivalence relations $\mathcal{R}$ such that the corresponding II_1 factor $L\mathcal{R}$ has trivial fundamental group.

In fact $L^\infty(\mathbb{T}^2) \rtimes SL(2,\mathbb{Z})$ is an example of such a factor.

Further, Popa and Gaboriau have shown that the free group $\mathbb{F}_n, 2 \leq n < \infty$ admits uncountably many free ergodic actions α_i such that

- The relations $\mathcal{R}_{\alpha_i}$ are pairwise non-SOE; and
- $\mathcal{F}(L\mathcal{R}_{\alpha_i}) = \{1\} \; \forall i$.

Popa has gone on to prove several stunning *strong rigidity theorems*. Rather than state his results too precisely, which would entail a fair bit of preparation, we shall merely content ourselves by conveying a flavour of one of his theorems:

[3] Bernoulli actions, for instance.

[4] 'Relative Kazhdan property (T)' features in a description of the permssible kind of groups.

Popa has gone on to prove several stunning *strong rigidity theorems*. Rather than state his results too precisely, which would entail a fair bit of preparation, we shall merely content ourselves by conveying a flavour of one of his theorems:

Certain kinds of free ergodic actions[3] *of certain kinds of groups* [4] *G are such that if the resulting equivalence relation $\mathcal{R}$ has the property that $\mathcal{R}_Y$ is isomorphic to $\mathcal{R}_\Gamma$ for some Borel subset Y and some free ergodic action of some countable group Γ, then Y must have full measure, and the actions of Γ and G must be conjugate through a group isomorphism.*

With $\Gamma, \mathcal{R}$ as above, if Y is a Borel set with $0 < \mu(Y) < 1$, it is seen that the relation $\mathcal{R}_Y$ cannot be obtained as the equivalence relation produced from a free ergodic action of any countable group. (We thus recover Furman''s result.)

[3] Bernoulli actions, for instance.

[4] 'Relative Kazhdan property (T)' features in a description of the permssible kind of groups.

References:

[A] Atiyah, M., *Elliptic operators, discrete groups and von Neumann algebras*, Colloque 'Analyse et Topologie' en l'Honneur de Henri Cartan, (Orsay, 1974), pp. 43-72, Asterisque, No. 32-33. Paris: Soc. Math. France, 1976.

[Bor], Borel, A., *The* L^2*-cohomology of negatively curved Riemannian symmetric spaces,* Ann. Acad. Sci. Fenn. Ser. A I Math., **10**, (1985), 95-105.

[CG] Cheeger, J., and Gromov, M., L_2*-cohomology and group cohomology*, Topology, **25**(2), (1986), 189-215.

[CFW], Connes, A., Feldman, J. and Weiss, B., *An amenable equivalence relation is generated by a single transformation,* Ergodic Theory Dynamical Systems, **1**(4), 431-450, (1981), 431-450.

[D] Dye, H., *On groups of measure preserving transformations, I,* Amer. J. Math., **81**, (1959), 119-159.

[Fu] Furman, A, *Orbit equivalence rigidity*, Ann. of Math. (2), **150**(3), (1999), 1083-1108.

[FM] Feldman, J. and Moore, C., *Ergodic equivalence relations, cohomology, and von Neumann algebras, I,II* Trans. Amer. Math. Soc., **234**(2), (1977), 289-324.

[G] Gaboriau, D., *Invariants ℓ^2 de relations d'equivalence et de groupes,* Publ. Math. I.H.E.S. **95**, (2002), 93-150.

[GP] Gaboriau, D., and Popa, S., *An uncountable family of nonorbit equivalent actions of $\mathbb{F}_n$.* J. Amer. Math. Soc. 18 (2005), no. 3, 547–559 (electronic).

[Kri] Krieger, W., *On non-singular transformations of a measure space, I,II,* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **11**, (1969), 83-97.

[OW] Ornstein, D., and Weiss, B., *Ergodic Theory of amenable group actions, I. The Rohlin lemma,* Bull. Amer. Math. Soc. (New Ser.) **2**(1), (1980), 161-164.

[V] Vaes, S., *Rigidity results for Bernouilli actions and their von Neumann algebras, (after Sorin Popa),* Asterisque 311 (2007), 237-294, also arXiv: math. OA/0603434 v2.

[vN] von Neumann, J. (1961). *Collected works vol. III. Rings of Operators.* Pergamon Press.