NOTES ON THE IMPRIMITIVITY THEOREM

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1 Introduction

These notes were the basis of a course of (four) lectures given by the author at a Winter School on *Group Representations and Function Theory* conducted by the Indian Statistical Institute at Bangalore in October 1995. The author is particularly happy to be able to have these notes appear in a volume in honour of Professor K.R. Parthasarathy, since the author first learnt about the inducing construction and the imprimitivity theorem from his lecture notes which are referred to in the references. While there is nothing in these notes which is not already to be found in the lecture notes referred to above, it is hoped that these notes might serve some purpose as they restrict themselves to just these notions and a prospective reader who does not want to necessarily read about the mathematical foundations of quantum mechanics might be able to get directly to the imprimitivity theorem.

The importance of induced representations was recognised and emphasised by George Mackey, who first proved the imprimitivity theorem and used that to analyse the representation theory of some important classes of groups (which include the Heisenberg group and semi-direct products where the normal summand is abelian). Broadly speaking, the philosophy is that the inducing construction enables one, in favourable cases, to 'reduce' the problem of constructing irreducible representations of a group to those of 'smaller subgroups whose representation theory is known'.

These notes are organised as follows: after an initial section on measure-theoretic preliminaries, we prove the Hahn-Hellinger theorem which classifies separable *-representations of the commutative algebra $C_0(X)$ (where X denotes a locally compact space satisfying the second axiom of countability, and $C_0(X)$ denotes the space of continuous functions on X which vanish at infinity); the third section discusses the constructon of induced representations and proves the imprimitivity theorem (modulo some measure-theoretic arguments which are omitted here in the hope of 'greater transparency'); the final section discusses some examples.

Finally, two remarks are probably in order: (a) since the material discussed here is very old and 'standard', the author has almost surely been more than remiss in the matter of references and giving credit where it is due; the reader, who is desirous of getting bibliographic details and references, is referred to such standard treatises on the subject as [Kir] or [Var], for instance; and (b) the definition, given here, of the induced representation is neither the most commonly seen one, nor is it perhaps the aesthetically most pleasing one; but it does possess the virtue of being a reasonably 'amenable' and efficient one, and that is the reason for using it here; the reader must augment this treatment with more standard (and meaty) ones.

2 Some measure-theoretic generalities

In what follows, $(X, \mathcal{B}, \mu)$ will denote a *separable* σ-finite measure space. (To say that $(X, \mathcal{B}, \mu)$ is separable amounts to requiring that the Hilbert space $L^2(X, \mathcal{B}, \mu)$ is separable; and to say that μ is σ-finite means that there exists a sequence $\{E_n\} \subset \mathcal{B}$ such that $X = \bigcup_n E_n$ and $\mu(E_n) < \infty \ \forall \ n$.)

We begin with the proposition that governs 'change of variables' in Lebesgue integration - which we only state for non-negative functions, because that is the only case we will need. The reader should have no trouble formulating the most general case.

PROPOSITION 2.1 *Let $(X, \mathcal{B}_X, \mu)$ be a measure space, and let $(Y, \mathcal{B}_Y)$ be a 'measurable space'. Let $T : X \to Y$ be a measurable map - i.e., $E \in \mathcal{B}_Y \Rightarrow T^{-1}(E) \in \mathcal{B}_X$. Consider the measure $\mu \circ T^{-1}$ defined on $\mathcal{B}_Y$ by the obvious formula*

$$\mu \circ T^{-1}(F) \ = \ \mu(\ T^{-1}(F) \ ).$$

If $f : Y \to I\!R$ is a measurable non-negative real-valued function on Y, then

$$\int f \ d(\mu \circ T^{-1}) \ = \ \int (f \circ T) d\mu.$$

Proof: The assertion is true for indicator functions of sets by definition; hence the assertion holds for simple functions; the general case follows from the fact that any measurable non-negative function is a monotone limit of a sequence of simple functions (and from the monotone convergence theorem). $\square$

Recall that if μ, ν are σ-finite measures defined on $\mathcal{B}$, then ν is said to be *absolutely continuous* with respect to μ if every μ-null set is also a ν-null set - i.e., $E \in \mathcal{B}_X, \mu(E) = 0 \ \Rightarrow \ \nu(E) = 0$. In this case there exists a non-negative measurable function, which is uniquely determined a.e. (μ), denoted by $\frac{d\nu}{d\mu}$ and called the Radon-Nikodym derivative of ν with respect to μ, such that

$$\int f \left(\frac{d\nu}{d\mu}\right) d\mu = \int f \, d\nu$$

for all non-negative measurable functions f.

Measures μ, ν are said to be *equivalent* (or mutually absolutely continuous), written $\mu \cong \nu$, if the class of μ-null sets coincides with the class of ν-null sets; in this case, $\frac{d\nu}{d\mu} > 0$ *a.e.* (μ).

Definition 2.2 *An* **automorphism** *of the measure space* $(X, \mathcal{B}, \mu)$ *is a measurable self-map* $T : X \to X$ *such that (a) there exists another measurable mapping* $T^{-1} : X \to X$ *such that* $T \circ T^{-1} = T^{-1} \circ T = id_X$ *a.e.*(μ), *and (b)* $\mu \circ T \cong \mu$ $(\cong \mu \circ T^{-1})$. *The set of automorphisms of* $(X, \mathcal{B}, \mu)$ *will be denoted by* $Aut(X, \mathcal{B}, \mu)$.

As is customary, we identify any two maps which agree a.e.; with this convention, it must be clear that $Aut(X, \mathcal{B}, \mu)$ is naturally a group (with respect to the composition product).

We now state the useful *chain rule* for Radon-Nikodym derivatives.

Proposition 2.3 *If* $S, T \in Aut(X, \mathcal{B}, \mu)$, *then*

$$\frac{d(\mu \circ (S \circ T)^{-1})}{d\mu} = \left(\frac{d(\mu \circ T^{-1})}{d\mu} \circ S^{-1}\right) \frac{d(\mu \circ S^{-1})}{d\mu} \quad a.e. \ (\mu)$$

Proof: If f is any non-negative measurable function on X, then, by repeated use of 'change of variables', we find that

$$\begin{aligned}
\int f \left(\frac{d(\mu \circ T^{-1})}{d\mu} \circ S^{-1}\right) \frac{d(\mu \circ S^{-1})}{d\mu} \, d\mu &= \int f \left(\frac{d(\mu \circ T^{-1})}{d\mu} \circ S^{-1}\right) d(\mu \circ S^{-1}) \\
&= \int (f \circ S) \, \frac{d(\mu \circ T^{-1})}{d\mu} \, d\mu \\
&= \int (f \circ S) \, d(\mu \circ T^{-1}) \\
&= \int (f \circ S \circ T) \, d\mu
\end{aligned}$$

and the uniqueness of the Radon-Nikodym derivative completes the proof. □

Proposition 2.4 *If* $T \in Aut(X, \mathcal{B}, \mu)$*, then define*

$$U_T f \;=\; \left(\frac{d(\mu \circ T^{-1})}{d\mu}\right)^{\frac{1}{2}} \left(f \circ T^{-1}\right) \quad .$$

Then the mapping $T \mapsto U_T$ *defines a group-homomorphism from the group* $Aut(X, \mathcal{B}, \mu)$ *into the group of unitary operators on the Hilbert space* $L^2(X, \mu; \mathcal{H})$ *of measurable functions* $f : X \to \mathcal{H}$ *such that* $||f||^2 = \int ||f(x)||^2 d\mu(x) < \infty$*, where* $\mathcal{H}$ *is some auxiliary Hilbert space.*

Proof: Note first, thanks to the definition of the Radon-Nikodym derivative and the change of variables formula, that

$$||U_T f||^2 \;=\; \int ||f \circ T^{-1}||^2 d(\mu \circ T^{-1}) \;=\; ||f||^2 \ ,$$

so that U_T is an isometric operator.

Next, if $S, T \in Aut(X, \mathcal{B}, \mu)$, then

$$\begin{aligned} U_S(U_T f) &= \left(\frac{d(\mu \circ S^{-1})}{d\mu}\right)^{\frac{1}{2}} (U_T f) \circ S^{-1} \\ &= \left(\frac{d(\mu \circ S^{-1})}{d\mu}\right)^{\frac{1}{2}} \left(\frac{d(\mu \circ T^{-1})}{d\mu} \circ S^{-1}\right)^{\frac{1}{2}} f \circ (S \circ T)^{-1} \\ &= U_{S \circ T} f \end{aligned}$$

by the chain rule. Since $U_{id_X} = id_{L^2(X,\mu;\mathcal{H})}$, it follows that each U_T is necessarily unitary and the proof of the proposition is complete.

□

3 Representations of $C_0(X)$

Suppose X is a locally compact Hausdorff space, which we shall assume is 'second countable'. Let $C_0(X)$ denote the space of continuous complex-valued functions on X which 'vanish at ∞' - i.e., which are uniformly approximable by continuous functions of compact support. If $f, g \in C_0(X), \alpha \in \mathbb{C}$ let $f + g$, αf and fg be the functions defined by pointwise sum, scalar multiplication and product, respectively; also define $f^*(x) = \overline{f(x)}$, and

$||f|| = sup_x|f(x)|$. Then, these definitions of addition, scalar multiplication, product, 'adjoint' and norm, endow $C_0(X)$ with the structure of the prototypical commutative C^*-algebra .

We will be interested in **-representations* of $C_0(X)$ - by which we mean a map $C_0(X) \ni \phi \stackrel{\pi}{\mapsto} \pi(\phi) \in \mathcal{L}(\mathcal{H})$ which is a homomorphism of *-algebras (where the *-operation in the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$ is given by the familiar adjoint). More specifically, we will be interested in classifying all such *-representations up to (unitary) equivalence, where representations π, π' on Hilbert spaces $\mathcal{H}, \mathcal{H}'$ are said to be equivalent if there exists a unitary operator $W : \mathcal{H} \to \mathcal{H}'$ such that $W^*\pi'(\phi)W = \pi(\phi) \ \forall \phi \in C_0(X)$.

One example of such a *-representation is obtained thus: let μ be any positive measure on X; set $\mathcal{H} = L^2(X, \mu)$, and define $((\pi_\mu(\phi)f)\ (x) = \phi(x)f(x)\ \forall \phi \in C_0(X), f \in \mathcal{H}, x \in X$. It is a direct consequence of the Riesz Representation theorem that a representation is equivalent to a representation of this type (for some choice of measure μ) if and only if the representation is *cyclic* - meaning that there exists a ('cyclic') vector f_0 in the underlying Hilbert space $\mathcal{H}$ such that the set of vectors $\{\pi(\phi)f_0 : \phi \in C_0(X)\}$ is dense in $\mathcal{H}$. (By choosing a cyclic vector of norm one, we can even ensure that μ is a probability measure - i.e., is a Borel measure with $\mu(X) = 1$. Here and in the sequel, by a Borel measure on X, we shall mean a σ-finite measure defined on $\mathcal{B}_X$, the smallest σ-algebra containing every open set in X.)

An immediate consequence of the analysis of the preceding paragraph is the fact that if π is any representation of $C_0(X)$ on a separable Hilbert space, then there exists a (finite or countably infinite) sequence $\{\mu_n\}$ of probability measures on X such that $\pi \cong \oplus_n \pi_{\mu_n}$.

LEMMA 3.1 *Let $\pi : C_0(X) \to \mathcal{L}(\mathcal{H})$ denote any separable *-representation.*

*(a) Then there exists a probability measure μ on X and a *-representation $\tilde{\pi} : L^\infty(X, \mu) \to \mathcal{L}(\mathcal{H})$ such that:*

(i) $\tilde{\pi}$ is isometric - i.e., $||\tilde{\pi}(\phi)||_{\mathcal{L}(\mathcal{H})} = ||\phi||_{L^\infty(X,\mu)}$;

(ii) $\tilde{\pi}$ 'respects bounded convergence', meaning that if $\phi_n \to \phi$ a.e.(μ) and if $sup_n||\phi_n||_{L^\infty(X,\mu)} < \infty$, then $(\tilde{\pi}(\phi_n))\xi \to (\tilde{\pi}(\phi))\xi \ \ \forall \xi \in \mathcal{H}$; and

(iii) $\tilde{\pi}$ 'extends' π in the obvious sense.

(b) If $\pi_i : C_0(X) \to \mathcal{L}(\mathcal{H}_i)$, $i = 1, 2$, *are *-representations, and if* μ_i *is a measure associated with* π_i *as in (a) above, and if* $\pi_1 \cong \pi_2$, *then* $\mu_1 \cong \mu_2$.

Proof: (a) Let $\{\mu_n : n \in I\}$ be probability measures as in the paragraph preceding the lemma, where I is at most countable. In fact, we may clearly assume, without loss of generality, that actually $\pi = \oplus_{n\in I}\, \pi_{\mu_n}$ (so that $\mathcal{H} = \oplus_{n\in I}\, L^2(X, \mu_n)$).

Fix (strictly) positive numbers ϵ_n, $n \in I$ such that $\sum_{n\in I} \epsilon_n = 1$, and define $\mu = \sum_{n\in I} \epsilon_n \mu_n$. The definition implies that μ is a probability measure, and that any Borel set is a μ-null set if and only if it is a μ_n-null set for every $n \in I$. Define

$$(\tilde{\pi}(\phi))(\oplus_n \xi_n) = \oplus_n \phi \xi_n \quad \forall \phi \in L^\infty(X, \mu).$$

It is easy to see that this $\tilde{\pi}$ does everything it is supposed to.

(b) Suppose $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary operator such that $U\pi_1(\phi)U^* = \pi_2(\phi)\ \forall \phi \in C_0(X)$. Consider the class $\mathcal{C}$ consisting of bounded Borel functions on X such that (in the notation of (a) above) $U\tilde{\pi}_1(\phi)U^* = \tilde{\pi}_2(\phi)$. It follows from (a) that $C_0(X) \subset \mathcal{C}$ and that if $\{\phi_n\} \subset \mathcal{C}$, if $sup\{|\phi_n(x)| : x \in X, n \geq 1\} < \infty$, and if $\phi_n(x) \to \phi(x)\ \forall\, x \in X$, then $\phi \in \mathcal{C}$. This implies that $\mathcal{C}$ contains every bounded Borel function; in particular, $E \in \mathcal{B}_X \ \Rightarrow\ 1_E \in \mathcal{C}$; hence if $E \in \mathcal{B}_X$, then $\mu_1(E) > 0 \ \Leftrightarrow\ \mu_2(E) > 0$, and the proof is complete. □

Corollary 3.2 *If* μ, ν *are arbitrary* σ*-finite Borel measures on* X, *then*

$$\pi_\mu \cong \pi_\nu \ \Leftrightarrow\ \mu \cong \nu\ .$$

Proof: If $\mu \cong \nu$, it is easily checked that the equation $Uf = \left(\frac{d\mu}{d\nu}\right)^{\frac{1}{2}} f$ defines a unitary operator $U : L^2(X, \mu) \to L^2(X, \nu)$ which implements the desired unitary equivalence $\pi_\mu \cong \pi_\nu$.

Conversely, note that if we start with $\pi = \pi_\lambda$, then λ itself is a measure which 'works for π' in the sense of Lemma 3.1; and the desired implication is a consequence of Lemma 3.1 (b).

□

We begin with a few results needed in the proof of the main theorem of this section.

If μ is a σ-finite Borel measure on X, and if $\mathcal{H}_n$ denotes an n-dimensional Hilbert space, where $1 \leq n \leq \aleph_0$, define the *-representation $\pi_\mu^n : C_0(X) \to \mathcal{L}(L^2(X, \mu; \mathcal{H}_n))$ by $(\pi_\mu^n(\phi) f)(x) = \phi(x) f(x)$.

The proof of the next lemma is a routine verification which we omit.

Lemma 3.3 *Let $1 \leq n \leq \aleph_0$, let $\mathcal{H}$ be a Hilbert space of dimension n, and let μ be a Borel measure on X. Fix an orthonormal basis $\{\eta_j : j \in I\}$ for $\mathcal{H}$, where I is some index set (of cardinality n). Define the Hilbert spaces*

$$\mathcal{H}_1 = \oplus_{j \in I} L^2(X, \mu), \ \mathcal{H}_2 = L^2(X, \mu; \mathcal{H}), \ \mathcal{H}_3 = L^2(X, \mu) \otimes \mathcal{H} \ .$$

Then the equations

$$(W(\oplus_{j \in I} f_j)) \ (x) \ = \ \sum_{j \in I} f_j(x) \eta_j, \ V(\oplus_{j \in I} f_j) \ = \ \sum_{j \in I} f_j \otimes \eta_j$$

define unitary operators $W : \mathcal{H}_1 \to \mathcal{H}_2, \ V : \mathcal{H}_1 \to \mathcal{H}_3$ such that

$$W^* \pi_\mu^n(\phi) W \ = \ \oplus_{j \in I} \pi_\mu(\phi) \ = \ V^*(\pi_\mu(\phi) \otimes id_{\mathcal{H}}) V$$

for all $\phi \in C_0(X)$.

□

Lemma 3.4 *Let μ be a Borel measure on X. The following conditions on an operator $T \in \mathcal{L}(L^2(X, \mu))$ are equivalent:*

(a) $T \pi_\mu(\psi) = \pi_\mu(\psi) T \quad \forall \psi \in C_0(X)$;

(b) there exists $\phi \in L^\infty(X, \mu)$ such that $Tf = \phi f \quad \forall f \in L^2(X, \mu)$.

Proof: We only prove $(a) \Rightarrow (b)$, since the other implication is obvious. Also, since every Borel measure is equivalent to a probability measure and since the representations associated with equivalent measures are equivalent - see Corollary 3.2 - we may, without loss of generality, assume that μ is a probability measure; hence the constant function $\xi_0 = 1$ is a unit vector in $L^2(X, \mu)$. Define $\phi = T \xi_0$.

Notice now that for arbitrary $\psi \in C_0(X)$, we have:

$$T \psi \ = \ T \pi_\mu(\psi) \xi_0 \ = \ \psi \phi \ .$$

Hence, we see that

$$||\phi\psi||_2 \ \leq ||T|| \ ||\psi||_2 \ \ \forall \psi \in C_0(X) \ .$$

This last statement implies that necessarily $\phi \in L^\infty(X, \mu)$ and that $Tf = \phi f \ \forall f \in L^2(X, \mu)$.

□

PROPOSITION 3.5 *Let* $\mu, \mathcal{H}, n, \mathcal{H}_i, i = 1, 2, 3, V, W$ *be as in Lemma 3.3. For* $i = 1, 2, 3$, *define* $\pi_i : C_0(X) \to \mathcal{L}(\mathcal{H}_i)$ *by* $\pi_1(\phi) = \oplus_{j \in I} \pi_\mu(\phi)$, $\pi_2(\phi) = \pi^n_\mu(\phi)$, $\pi_3(\phi) = \pi_\mu(\phi) \otimes id_{\mathcal{H}}$. *The following conditions on an operator* $T \in \mathcal{L}(\mathcal{H}_1)$ *are equivalent:*

(a) $T\pi_1(\phi) = \pi_1(\phi)T \ \ \forall \phi \in C_0(X)$;

(b) there exist $\phi_{ij} \in L^\infty(X, \mu), i, j \in I$ *such that* T *is represented by the matrix* $((\pi_\mu(\phi_{ij}) \ ))$, *meaning that if* $T(\oplus_j f_j) = \oplus_i g_i$, *then* $g_i = \sum_j \pi_\mu(\phi_{ij}) f_j \ \ \forall i$.

(c) there exists a measurable mapping $X \ni x \to T(x) \in \mathcal{L}(\mathcal{H})$ *such that* $(WTW^* f)(x) \ = \ T(x) f(x)$ *a.e.*(μ) $\forall f \in \mathcal{H}_2$.

Proof: $(a) \Rightarrow (b)$: Let $((T_{ij}))$ denote the matrix (with $T_{ij} \in \mathcal{L}(L^2(X, \mu)) \ \forall i, j)$ which 'represents' T in the sense described in the statement of (b). Since $\pi_1(\phi)$ is 'represented' by the diagonal matrix $((\delta_{ij}\pi_\mu(\phi)))$, it is clear that the condition (a) translates into the condition that $T_{ij}\pi_\mu(\phi) = \pi_\mu(\phi)T_{ij} \ \ \forall \phi \in C_0(X)$. It follows now from Lemma 3.4 that there exist $\phi_{ij} \in L^\infty(X, \mu)$ such that $T_{ij} = \pi_\mu(\phi_{ij}) \ \forall \ i, j \in I$.

$(b) \Rightarrow (c)$: If the bounded operator T is related to the functions ϕ_{ij} as in (b), it is a pleasant exercise (of one's facility with arguments of a measure theoretic nature) to show that it is necessarily the case that the matrix $((\ \phi_{ij}(x) \ ))$ represents a bounded operator on $\mathcal{H}$ (with respect to the orthonormal basis $\{\eta_j\}$) - call it $T(x)$ - for μ-almost all $x \in X$, and that the mapping $X \in x \mapsto T(x) \in \mathcal{L}(\mathcal{H})$ is measurable. It is clear that (c) holds.

$(c) \Rightarrow (a)$: If (c) holds, it is immediate that WTW^* commutes with $\pi^n_\mu(\phi) \ \forall \phi \in C_0(X)$, whence the validity of (a) follows from Lemma 3.3.

□

LEMMA 3.6 *Let* $1 \leq n \leq \aleph_0$, $\mathcal{H} = L^2(X, \mu; \mathcal{H}_n)$, $\pi = \pi^n_\mu$ *as above. Suppose* $\{P_i : i \in I\}$ *is a collection of projections in* $\mathcal{L}(\mathcal{H})$ *- where* I *is some countable set - satisfying the following conditions:*

(i) $P_i \pi(\phi) = \pi(\phi) P_i \ \forall \ \phi \in C_0(X)$;

(ii) $P_i P_j = 0 \ \forall \ i \neq j, \ i, j \in I$;

(iii) $P_i \tilde{\pi}(1_F) \neq 0$ *whenever* $\mu(F) > 0$.

Then $|I| \leq n$.

Further, there exists a collection $\{P_i : i \in \tilde{I}\}$ *of projections which satisfies (i) - (iii) above, such that* $|\tilde{I}| = n$.

Proof: If $\{P_i : i \in I\}$ satisfies (i) and (ii) above, it follows first from Lemma 3.5 that there exist measurable maps $P_i : X \rightarrow \mathcal{L}(\mathcal{H})$ such that $(P_i f)(x) = P_i(x) f(x)$ *a.e.* A few moments' thought shows then that there exists a μ-null set N such that whenever $x \notin N$, the following conditions are satisfied:

(a) $P_i(x)$ is a non-zero projection in $\mathcal{L}(\mathcal{H}_n)$ for each $i \in I$;

(b) $P_i(x) P_j(x) = 0$ whenever $i, j \in I, \ i \neq j$;

The fact that $dim \ \mathcal{H}_n = n$ now guarantees that $|I| \leq n$, as desired.

For existence, just pick pairwise orthogonal non-zero projections $P_i^0, \ 1 \leq i \leq n$ in $\mathcal{L}(\mathcal{H}_n)$ such that $\sum_i P_i^0 = id_{\mathcal{H}_n}$, and define $(P_i f)(x) = P_i^0 f(x)$. $\square$

Notice, before we proceed further, that if ν, μ are measures such that ν is absolutely continuous with respect to μ, if $E = \{x : \frac{d\nu}{d\mu}(x) > 0\}$, and if we define the measure $\mu|_E$ by $\mu|_E(A) = \mu(A \cap E)$, then $\mu|_E$ is absolutely continuous with respect to μ, $\frac{d\mu|_E}{d\mu} = 1_E$, and $\mu|_E \cong \nu$.

THEOREM 3.7 **(The Hahn-Hellinger theorem)**

Let $\pi : C_0(X) \rightarrow \mathcal{L}(\mathcal{H})$ *be a *-representation of* $C_0(X)$ *on a separable Hilbert space.*

(a) Then, there exists a Borel measure μ *on* X*, and a sequence* $\{E_n : 1 \leq n \leq \aleph_0\}$ *of pairwise disjoint Borel sets in* X*, and Hilbert spaces* $\mathcal{H}_n, 1 \leq n \leq \aleph_0$ *with* $dim \ \mathcal{H}_n = n \ \forall n$*, such that* $\mu(X - U_n E_n) = 0$*, and*

$$\pi \ \cong \ \oplus_{1 \leq n \leq \aleph_0} \ \pi^n_{\mu|_{E_n}} \ .$$

(b) Further, if π' *is another representation of* $C_0(X)$*, and if* μ' *is a measure and if* $\{E'_n : 1 \leq n \leq \aleph_o\}$ *is a sequence of pairwise disjoint Borel sets such that* $\pi' \ \cong \ \oplus_{1 \leq n \leq \aleph_0} \ \pi^n_{\mu'|_{E'_n}}$ *, and if* $\pi \cong \pi'$*, then* $\mu \cong \mu'$ *and* $\mu(E_n \Delta E'_n) = 0 \ \forall n$*, where* Δ *denotes 'symmetric difference':* $A \Delta B = (A - B) \cup (B - A)$.

Proof: (a) As in the proof of Lemma 3.1, there exists (at most countably many) probability measures ν_n such that $\pi \cong \oplus_n \pi_{\nu_n}$; let $\mu = \sum_n \epsilon_n \nu_n$ be as in the proof of Lemma 3.1. It then follows that each ν_n is absolutely continuous with respect to μ. Hence, if $A_n = \{x : \frac{d\nu_n}{d\mu}(x) > 0\}$, it follows from the preceding paragraph and Lemma 3.2 that

$$\pi \cong \oplus_n \pi_{\mu|_{A_n}} .$$

For $1 \leq l \leq k \leq \aleph_0$, define $E_k = \{x : \sum_n 1_{A_n}(x) = k\}$ and define $A_{n,k,l} = \{x \in A_n \cap E_k : \sum_{j=1}^n 1_{A_j}(x) = l\}$. (Thus, $x \in E_k$ precisely when x belongs to exactly k A_n's; while $x \in A_{n,k,l}$ precisely when $x \in A_n \cap E_k$ and $x \in A_j \cap E_k$ for precisely l j's which are $\leq k$.)

A moment's thought should convince the reader that $A_n \cap E_k = \coprod_{l=1}^k A_{n,k,l}$ for all n, k and that $\coprod_n A_{n,k,l} = E_k \; \forall \; 1 \leq l \leq k$, where the symbol $\coprod$ denotes '(pairwise) disjoint union'. It follows that

$$\begin{aligned} \pi &\cong \oplus_n \; \pi_{\nu_n} \\ &\cong \oplus_n \; \pi_{\mu|_{A_n}} \\ &\cong \oplus_{n,k} \; \pi_{\mu|_{A_n \cap E_k}} \\ &\cong \oplus_{n,k,l} \; \pi_{\mu|_{A_{n,k,l}}} \\ &\cong \oplus_{k,l} \; (\oplus_n \pi_{\mu|_{A_{n,k,l}}}) \\ &\cong \oplus_{k,l} \; \pi_{\mu|_{E_k}} \\ &\cong \oplus_k \; \pi^k_{\mu|_{E_k}} \end{aligned}$$

thereby completing the proof of the first half (the existence part) of the theorem.

(b) In view of Lemma 3.1, we may assume that $\mu = \mu'$. Thus we have to prove that if μ is a Borel measure on X, and if $\{E_n^i : 1 \leq n \leq \aleph_0\}, i = 1, 2$ are two sequences of pairwise disjoint Borel subsets of X such that (i) $\mu(X - \cup_n E_n^i) = 0$, for $i = 1, 2$, and (ii) $\oplus_n \; \pi^n_{\mu|_{E_n^1}} \cong \oplus_n \; \pi^n_{\mu|_{E_n^2}}$, then $\mu(E_n^1 \Delta E_n^2) = 0 \; \forall \; n$.

To see this, let us write $\pi^i = \oplus_n \; \pi^n_{\mu|_{E_n^i}}, \mathcal{H}^i = L^2(E_n^i, \mu; \mathcal{H}_n)$, for $i = 1, 2$ and suppose $U : \mathcal{H}^1 \to \mathcal{H}^2$ is a unitary operator such that $U\pi^1(\phi) = \pi^2(\phi)U$ for all $\phi \in C_0(X)$. In the ($\pi \mapsto \tilde{\pi}$) notation of Lemma 3.1, it follows that

$$U\tilde{\pi}^1(\phi) = \tilde{\pi}^2(\phi)U \; \forall \phi \in L^\infty(X, \mu).$$

In particular,

$$U\tilde{\pi}^1(1_{E_n^1})U^* \ = \ \tilde{\pi}^2(1_{E_n^1}) \ , 1 \leq n \leq \aleph_0. \tag{3.1}$$

Notice, now, that Lemma 3.6 has the following consequence: fix $i \in \{1,2\}$, and suppose $A \in \mathcal{B}_X$, $\mu(A) > 0$, and suppose there exist projections $P_j \in \mathcal{L}(\mathcal{H})$, $j \in I$ satisfying the following properties:

(i) $P_j = P_j\tilde{\pi}^i(1_A), \ \forall \ j \in I$;

(ii) $P_jP_k = 0$ for distinct $j,k \in I$;

(iii) $F \in \mathcal{B}_X, F \subset A, \ \mu(F) > 0 \ \Rightarrow \ P_j\tilde{\pi}^i(F) \ \neq 0 \ \forall j \in I$.

It follows then from Lemma 3.6 that $\mu(A \cap E_j^i) = 0 \ \forall \ 1 \leq j < |I|$.

Now, fix $1 \leq n \leq \aleph_0$, and note, by Lemma 3.6, that there exists a set I_n of cardinality n and projections $P_j, \ j \in I_n$ satisfying (i) - (iii) of the preceding paragraph with $A = E_n^1, i = 1$. Deduce from equation 3.1 that the projections $UP_jU^*, \ j \in I_n$ satisfy (i)-(iii) with $i = 2, A = E_n^1$; conclude from the preceding paragraph that $\mu(E_n^1 \cap E_j^2) = 0 \ \forall \ 1 \leq j < n$.

Since the roles of 1 and 2 are interchangeable, we may conclude that $\mu(E_n^1 \cap E_m^2) = 0$ if $m < n$, and the proof is complete.

□

In the notation of Theorem 3.7 and Lemma 3.1, if we define $P(E) = \ \tilde{\pi}(1_E)$, then it is easy to see that the assignment $E \mapsto P(E)$ defines a *projection-valued measure* in the sense that (a) each $P(E)$ is a (self-adjoint) projection, and (b) this assignment is 'countably additive' meaning that if if $\{E_n\}$ is a sequence of pairwise disjoint Borel sets in X, then $P(\coprod_n E_n)\xi = \sum_n P(E_n)\xi$ for all $\xi \in \mathcal{H}$. Such a projection-valued measure is also sometimes referred to as a **spectral measure** on X.

Conversely, it is equally clear that if $E \mapsto P(E)$ is a spectral measure, then there exists a unique representation π of $C_0(X)$ such that $P(E) = \ \tilde{\pi}(1_E)$.

Thus, Theorem 3.7 can also be interpreted as a 'classification of unitary equivalence classes of spectral measures on X'.

4 The imprimitivity theorem

In the sequel, we shall assume that G is a locally compact second countable group and that H is a closed subgroup of G. We shall write $G/H \ = \ \{gH :$

$g \in G\}$ for the set of left-cosets of H. Also, if $g \in G$, we shall denote by $g \mapsto L_g$ the action of G on G/H; thus, $L_g(g'H) = gg'H$.

The Borel σ-algebra of the homogeneous space G/H is defined by $\mathcal{B}_{G/H} = \{E \subseteq G/H : p^{-1}(E) \in \mathcal{B}_G\}$, where $p : G \to G/H$ is the natural quotient map. One fact that we shall need is the following:

FACT 1: There exists a measurable map $s : G/H \to G$ such that $p \circ s = id_{G/H}$.

(A map s as above is sometimes called a *measurable cross-section.*)

A measure μ defined on $\mathcal{B}_{G/H}$ is said to be *quasi-invariant* if $\mu \circ L_g \cong \mu \; \forall g$.

The second fact that we shall need is the following : (see [Mac] for details concerning both these facts)

FACT 2: There exists a σ-finite quasi-invariant measure on $\mathcal{B}_{G/H}$; further, any two such measures are mutually absolutely continuous.

Existence of a quasi-invariant measure is easy to establish, as follows: let ν be a finite measure on G which is equivalent to Haar measure, and set $\mu = \nu \circ p^{-1}$.

We are now ready to define induced representations.

PROPOSITION 4.1 *Let μ be 'the' quasi-invariant measure on G/H.. Let $h \mapsto V_h$ be a unitary representation of H on a Hilbert space $\mathcal{K}$. Let $\mathcal{H} = L^2(G/H, \mu; \mathcal{K})$. Fix a measurable cross-section s as in Fact 1, and define*

$$(U_g f)(x) = \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x)\right)^{\frac{1}{2}} V_h(f(L_g^{-1}x)) \quad ,$$

where $h \in H$ is defined by $gs(L_g^{-1}x) = s(x)h$.

Then $g \mapsto U_g$ is a unitary representation of G on $\mathcal{H}$, and is called the representation induced by the representation V of the subgroup H.

Proof: First note that

$$||U_g f||^2 = \int ||f \circ L_g^{-1}||^2 \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}\right) d\mu$$

$$
\begin{aligned}
&= \int ||f||^2 \; d\mu \\
&= ||f||^2
\end{aligned}
$$

whence each U_g is an isometric operator. In view of the measurability of the section s, it is quite easy to see that the mapping $g \mapsto U_g$ is a measurable one. Since a group homomorphism of a second countable locally compact into the group of unitary operators of a separable Hilbert space is known to be strongly continuous if and only if it is measurable, we only need to verify that $U_{g_1 g_2} = U_{g_1} U_{g_2} \; \forall \; g_1, g_2 \in G$. (This, together with the obvious identity $U_{id_{G/H}} = id_{\mathcal{H}}$, also shows that each U_g is not just isometric, but actually unitary.)

Next, if $g, g' \in G$, and if $x \in G/H$, let $h, h' \in H$ be defined by $gs(L_g^{-1}x) = s(x)h$ and $g's(L_{g'}^{-1}L_g^{-1}x) = s(L_g^{-1}x)h'$ respectively; then it follows that $s(x)hh' = gg's(L_{gg'}^{-1}x)$; hence we find that

$$
\begin{aligned}
(U_g U_{g'} f)(x) &= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x) \right)^{\frac{1}{2}} V_h(\; (U_{g'} f)(L_g^{-1}x) \;) \\
&= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x) \right)^{\frac{1}{2}} \left(\frac{d(\mu \circ L_{g'}^{-1})}{d\mu}(L_g^{-1}x) \right)^{\frac{1}{2}} V_h V_{h'} \; f(L_{gg'}^{-1}x) \\
&= \left(\frac{d(\mu \circ L_{gg'}^{-1})}{d\mu}(x) \right)^{\frac{1}{2}} V_{hh'} \; f(L_{gg'}^{-1}x) \\
&= (U_{gg'} f)(x)
\end{aligned}
$$

and the proposition follows.

□

REMARK 4.2 *Suppose* $V_i : H \to \mathcal{L}(\mathcal{H}_i)$ *are unitary representations, and let* $V = V_1 \oplus V_2$. *Let* U *(resp.,* U_1, U_2*) be the representation of* G *obtained by inducing the representation* V *(resp.,* V_1, V_2*). Then the definitions easily show that* $U \cong U_1 \oplus U_2$.

Recall that a unitary representation is said to be **irreducible** *if it is not expressible as the direct sum of two unitary representations. The preceding remarks show that if* V, H, U, G *are as in Proposition 4.1, then the irreducibility of the representation* V *is a necessary condition for the irreducibility of* U.

While the construction we have given for the induced representation seems to depend upon the choices of μ, s that we have made, it is true that the construction is, in fact, independent of these choices (up to unitary equivalence). One way to see this is via a characterisation, called the *imprimitivity theorem*, of the inducing construction.

(In the following theorem, the space G/H is given the quotent topology: thus, a set E is open in G/H if and only if $p^{-1}(E)$ is open in G. Since p is continuous and open, it is clear that this defines a locally compact topology on G/H.)

THEOREM 4.3 **(The Imprimitivity Theorem)**

(a) Let $G, H, V, \mathcal{K}, U, \mathcal{H}$ be as in Proposition 4.1. Consider the representation of $C_0(G/H)$ on $\mathcal{H}$ defined by $\pi = \pi_\mu^n$. Then,

$$U_g \pi(\phi) U_g^{-1} = \pi(\phi \circ L_g^{-1}) \ \ \forall \phi, g. \tag{4.2}$$

(b) Conversely, if $g \mapsto U_g$ is a unitary representation of G on some separable Hilbert space $\mathcal{H}$, and if there exists a representation $C_0(G/H) \ni \phi \mapsto \pi(\phi) \in \mathcal{L}(\mathcal{H})$, and if these two representations satisfy the so-called imprimitivity condition given by equation 4.2, then there exists a unitary representation $h \mapsto U_h^0$ of H on a Hilbert space $\mathcal{K}$ and a unitary operator $W : \mathcal{H} \to L^2(G/H, \mu; \mathcal{K})$ such that $W\pi(\phi)W^* = \pi_\mu^{dim \mathcal{K}}(\phi)$ for all $\phi \in C_0(G/H)$, and such that $g \mapsto WU_gW^*$ is the unitary representation of G which is induced by the representation U^0 of the closed subgroup H.

Proof: The proof of (a) is an easy verification.

(b) To start with, we may assume, thanks to (the proof of) Theorem 3.7 that there exists a sequence $\{E_n : 1 \leq n \leq \aleph_0\}$ of pairwise disjoint sets in G/H, and a Borel measure μ on G/H such that $\mathcal{H} = \oplus_{1 \leq n \leq \aleph_0} L^2(E_n, \mu; \mathcal{H}_n)$ and $\pi = \oplus_{1 \leq n \leq \aleph_0} \pi^n_{\mu_{E_n}}$ for all $\phi \in C_0(G/H)$.

It is easy to see that if the measure μ is associated with the representation π (as above and) as in Lemma 3.1, and if T is a homeomorphism of X, then the measure $\mu \circ T^{-1}$ is associated to the representation defined by $\tilde{\pi}(\phi) = \pi(\phi \circ T)$.

Since the representations $\phi \mapsto \pi(\phi)$ and $\phi \mapsto \pi(\phi \circ L_g^{-1})$ are given to be unitarily equivalent, it follows from the uniqueness half of Theorem 3.7 that $\mu \cong \mu \circ L_g$ and that $\mu(E_n \Delta L_g^{-1}(E_n) = 0 \ \forall n$. Since this is true for all g and since the G-action on G/H is transitive, it is clear that (i) there exists an n such that $\mu(E_m) = 0 \; \forall \; m \neq n$; and (ii) the measure μ is (supported in E_n and is) quasi-invariant.

Hence, we may assume, without loss of generality that $\mathcal{H} = L^2(G/H, \mu; \mathcal{K})$ and that $M = \pi_\mu^n$ where $n = dim\ \mathcal{K}$. (Here, we have replaced the given representation by a unitarily equivalent one, and used the uniqueness of the quasi-invariant measure up to equivalence.)

According to Proposition 2.4, we have another unitary representation $g \mapsto V_g$ of G on $\mathcal{H}$, defined by

$$V_g f \;=\; \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}\right)^{\frac{1}{2}} \left(f \circ L_g^{-1}\right) \quad .$$

It is then easliy seen that, for arbitrary $\phi \in C_0(G/H)$ and $g \in G$, we have

$$U_g \pi(\phi) U_g^{-1} \;=\; \pi(\phi \circ L_g^{-1}) \;=\; V_g \pi(\phi) V_g^{-1} \quad .$$

On the other hand, it follows from Lemma 3.5 that the only unitary operators on $\mathcal{H}$ which commute with $\pi_\mu^n(\phi)$ for every $\phi \in C_0(G/H)$ are those of the form

$$(Wf)(x) \;=\; W(x) f(x) \tag{4.3}$$

for some measurable map $G/H \ni x \mapsto W(x) \in \mathcal{L}(\mathcal{K})$, such that $W(x)$ is a unitary operator on $\mathcal{K}$ for μ-almost all $x \in G/H$.

Hence, if we let $W_g = V_g^* U_g$, then there exists a measurable mapping $x \mapsto W_g(x)$ from G/H into the group of unitary operators on $\mathcal{K}$, such that

$$(W_g f)(x) \;=\; W_g(x) f(x) \quad . \tag{4.4}$$

The definitions imply that

$$\begin{aligned} (U_g f)(x) &= (V_g W_g f)(x) \\ &= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x)\right)^{\frac{1}{2}} W_g(L_g^{-1}x) f(L_g^{-1}x) \quad . \end{aligned} \tag{4.5}$$

Since both U and V are unitary representations, it follows that

$$\begin{aligned} W_{gg'} &= V_{gg'}^* U_{gg'} \\ &= V_{g'}^* V_g^* U_g U_{g'} \\ &= V_{g'}^* W_g V_{g'} \; W_{g'} \quad . \end{aligned}$$

On the other hand, an easy computation reveals that we have

$$(V_{g'}^* W_g V_{g'} f)(x) \;=\; W_g(L_{g'}x) f(x) \; .$$

Thus, we find that

$$W_{gg'}(x) \;=\; W_g(L_{g'}x) W_{g'}(x) \; . \tag{4.6}$$

Strictly speaking, most of the preceding equations are valid only μ-*almost everywhere*. Thus, for instance, the precise formulation of equation 4.4 is that for each $g \in G$ and $f \in \mathcal{H}$, the equation 4.4 is valid for μ-almost all $x \in G/H$.

We shall present the rest of the proof under the (not quite justified) assumption that the equations 4.4, 4.5 and 4.6 are valid for every $g, g' \in G, f \in \mathcal{H}$ and for *every* $x \in G/H$. (For the proof to be rigorously complete, we have to go through some measure theoretic calisthenics to justify this sort of assumption. The interested reader can find a complete proof in [KRP], for instance; actually, in that reference, the slightly more general situation of a 'projective representation taking values in unitary or antiunitary operators' is considered.')

Returning to the proof, let $x_0 = eH = H$ denote the identity coset in X. It follows from equation 4.6 that

$$W_g(L_{g'}x_0) \;=\; W_{gg'}(x_0) W_{g'}(x_0)^{-1} \; . \tag{4.7}$$

Since H fixes x_0, we see (from equation 4.6) that if we define $U_h^0 = W_h(x_0)$, then $h \to U_h^0$ defines a unitary representation of H on $\mathcal{K}$.

Now fix a measurable cross-section $s : G/H \to G$ as in FACT 1, and note that any $g \in G$ admits a unique factorisation of the form $g = s(x)h$ for some (uniquely determined) $x \in G/H, h \in H$. (In fact, we must have $x = gH, \; h = s(gH)^{-1}g$.)

We may now deduce from equation 4.7 and equation 4.6 that if $g \in G, x \in G/H$, then

$$\begin{aligned} W_g(x) &= W_g(L_{s(x)}x_0) \\ &= W_{gs(x)}(x_0) W_{s(x)}(x_0)^{-1} \\ &= W_{s(L_g x)}(x_0) W_h(x_0) W_{s(x)}(x_0)^{-1} \; , \end{aligned}$$

where $h \in H$ is defined by the equation $gs(x) \;=\; s(L_g x)h$.

Hence, if we write $B_x = W_x(x_0)$, then we find that

$$W_g(x) = B_{L_g x} U^0_h B_x^{-1} \quad ,$$

where $gs(x) = s(L_g x)h$. Substituting this into equation 4.5, we thus find that

$$\begin{aligned}(U_g f)(x) &= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x)\right)^{\frac{1}{2}} W_g(L_g^{-1}x) f(L_g^{-1}x) \\ &= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x)\right)^{\frac{1}{2}} B_x U^0_h B^{-1}_{L_g^{-1}x} f(L_g^{-1}x)\end{aligned}$$

where $gs(L_g^{-1}x) = s(x)h$.

Finally, define the (obviously unitary) operator $B \in \mathcal{L}(\mathcal{H})$ by

$$(Bf)(x) = B_x f(x)$$

and notice that

$$\begin{aligned}(B^{-1}U_g Bf)(x) &= B_x^{-1}(U_g Bf)(x) \\ &= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x)\right)^{\frac{1}{2}} U^0_h B^{-1}_{L_g^{-1}x}(Bf)(L_g^{-1}x) \\ &= \left(\frac{d(\mu \circ L_g^{-1})}{d\mu}(x)\right)^{\frac{1}{2}} U^0_h f(L_g^{-1}x)\end{aligned}$$

where $gs(L_g^{-1}x) = s(x)h$.

In other words, we see that $g \mapsto B^{-1}U_g B$ is indeed induced by the representation U^0 of H, and the proof of the theorem is complete.

□

5 Semi-direct products

This section is devoted to a discussion of semi-direct products, and how their representations are related to induced representations. All groups in this section will be locally compact second contable groups.

Recall that an *action* of a group K on a group H is a group homomorphism $k \mapsto \alpha_k$ from K into the group Aut H of automorphisms of H, such that the map $H \times K \ni (h,k) \mapsto \alpha_k(h) \in H$ is continuous. Given such an action, the *semi-direct product* $H \times_\alpha K$ is the group G defined as follows: G is the Cartesian product $H \times K$ as a topological space; and the group product is defined by $(h_1,k_1)(h_2,k_2) = (h_1\alpha_{k_1}(h_2), k_1k_2)$. It is easily verified that $\{1_H\} \times K$ is a closed subgroup of G, and that $H \times \{1_K\}$ is a closed normal subgroup of G; in fact, we have: $(1_H,k)(h,1_K)(1_H,k)^{-1} = (\alpha_k(h), 1_H)$.

In practice, it is customary to think of H and K as subgroups of the semi-direct product via the natural identifications.

Here are some examples of semi-direct products:

(a) The group $\mathbb{Z}_2$ acts on any abelian group A in such a way that the non-identity element acts on A as the (necessarily involutory) automorphism $a \mapsto a^{-1}$. If we denote this action by i, it is easy to see, for instance, that $\mathbb{Z}_n \times_i \mathbb{Z}_2$ is isomorphic to the *dihedral group* D_{2n}; in particular, $\mathbb{Z}_3 \times_i \mathbb{Z}_2 \cong S_3$.

(b) The group $GL_n(\mathbb{R})$ (of non-singular $n \times n$ real matrices acts on $\mathbb{R}^n$ via the action $\alpha_T(v) = Tv$, where we think of $\mathbb{R}^n$ as a set of column vectors. Then the semi-direct product $\mathbb{R}^n \times_\alpha GL_n(\mathbb{R})$ can be identified with the subgroup of $GL_{n+1}(\mathbb{R})$ consisting of matrices with block-form $\begin{pmatrix} T & v \\ 0 & 1 \end{pmatrix}$.

Our goal in this section is to analyse irreducible unitary representations of a semi-direct product $A \times_\alpha K$, where the normal subgroup A is abelian. Before doing so, it will be prudent to gather some facts about locally compact abelian groups.

If A is a locally compact abelian group, a *character* on A is, by definition, a continuous homomorphism $\chi : A \to \mathbb{T}$ $(= \{z \in \mathbb{C} : |z| = 1\})$. The set $\hat{A}$ of characters on A is an abelian group with respect to point-wise product: $(\chi_1 \cdot \chi_2)(a) = \chi_1(a)\chi_2(a)$. In fact, $\hat{A}$ becomes a locally compact group with respect to the topology of uniform convergence on compact sets. The content of the celebrated Pontrjagin duality theorem is that the character group $\hat{\hat{A}}$ of $\hat{A}$ can naturally be identified (as a topological group) with A.

The equation $\hat{f}(\chi) = \int_A f(a)\overline{\chi(a)}dm_A(a)$, where we write m_A for the Haar measure on A, is known to define a linear injective mapping $L^1(A, m_A) \ni f \mapsto \hat{f} \in C_0(\hat{A})$ onto a dense subalgebra - call it $\mathcal{F}(\hat{A})$ - of $C_0(\hat{A})$, which is contractive in the sense that $||\hat{f}||_{C_0(\hat{A})} \leq ||f||_{L^1(A,m_A)}$. The mapping $f \mapsto \hat{f}$ is called the Fourier transform, because of the 'classical examples' described in the next paragraph.

For example, with $A = I\!R$ (resp., $\mathbb{Z}$), it is true that $\chi \in \hat{I\!R}$ (resp., $\chi \in \hat{\mathbb{T}}$) if and only if there exists $t \in I\!R$ (resp., $z \in \mathbb{T}$) such that $\chi = \chi_t$ (resp., $\chi = \chi_z$), where $\chi_t(s) = exp(\sqrt{-1}st)$ (resp., $\chi_z(n) = z^n$). Thus, $\hat{R} \cong I\!R$ (resp., $\hat{\mathbb{Z}} \cong \mathbb{T}$).

It is a fact that if $\pi : A \to \mathcal{L}(\mathcal{H})$ is a unitary representation of a locally compact abelian group, then there exists a unique *-representation $\hat{\pi} : C_0(\hat{A}) \to \mathcal{L}(\mathcal{H})$ such that $\hat{\pi}(\hat{f}) = \int_A f(a)\pi(a)dm_A(a) \ \forall\ f \in L^1(A, m_A)$. It follows that, with the above notation and the notation of Lemma 3.1, that $\pi(a) = \tilde{\hat{\pi}}(\hat{a})$, where $\hat{a}$ is the function on $\hat{A}$ defined by $\hat{a}(\chi) = \overline{\chi(a)}$. Thus, Theorem 3.7 yields, via the preceding discussion, a classification, up to equivalence, of the (separable) unitary representations of a locally compact abelian group.

Suppose now that $G = A \times_\alpha K$ is a semi-direct product, where A is abellian. Then the action $\alpha : K \to Aut\ A$ induces an action $\hat{\alpha} : K \to Aut\ \hat{A}$ thus: $\hat{\alpha}_k(\chi) = \chi \circ \alpha_k^{-1}$. Suppose now that $\pi : G \to \mathcal{L}(\mathcal{H})$ is a unitary representation. By restricting the representation to A, we obtain, as above, a 'spectral measure' $\mathcal{B}_{\hat{A}} \ni E \mapsto P(E)$ (taking values in projection operators on $\mathcal{H}$). It is clear that if $E \in \mathcal{B}_{\hat{A}}$, the range of $P(E)$ is an invariant subspace for each $\pi(a), a \in A$. Also, it is not hard to show that the range of $P(E)$ is an invariant subspace for each $\pi(k), k \in K$ if and only if $\mu(E\Delta\hat{\alpha}_k(E)) = 0\ \forall k \in K$, where μ is a measure associated with the representation of $C_0(\hat{A})$ obtained as above from the restriction to A of π. It follows that if the representation π of G is irreducible, then the measure μ (as above) is *ergodic* for the action $\hat{\alpha}$ of K on $\hat{A}$ - meaning that the only Borel sets $E \in \mathcal{B}_{\hat{A}}$ which satisfy $\mu(E\Delta\hat{\alpha}_k(E)) = 0\ \forall k \in K$ are the ones for which either $\mu(E) = 0$ or $\mu(\hat{A} - E) = 0$.

On the other hand, it is known - and not hard to prove - that if a locally compact second countable group K acts as automorphisms of a *standard measure space* $(X, \mathcal{B}_X, \mu)$ - i.e., μ is a Borel measure defined on the Borel σ-algebra $\mathcal{B}_X$ of a complete separable metric space X - and if the action is ergodic (in the above sense) and *regular* - meaning that there exists a Borel set $C \in \mathcal{B}_X$ such that C meets μ- almost every G-orbit in exactly one point - then the measure is concentrated on one orbit - meaning that there exists $x_0 \in X$ such that $\mu(X - G \cdot x_0) = 0$.

(The assumption of regularity may be reformulated thus: if X/G denotes the set of G-orbits, and if $\mathcal{B} = \{E \subset X/G : p^{-1}(E) \in \mathcal{B}_X\}$ denotes the natural 'quotient σ-algebra' on X/G, (where $p(x) = G \cdot x$), then regularity of the action is equivalent to the existence of a 'measurable cross-section', i.e., a measurable map $s : X/G \to X$ such that $p \circ s = id_{X/G}$ $a.e.(\mu \circ p^{-1}.)$

Also note that if $\chi \in \hat{A}$, then the orbit $K \cdot \chi$ may be identified with the coset space K/K_χ, where $K_\chi = \{k \in K : \hat{\alpha}_k(\chi) = \chi\}$. Further, it should be clear that if we define $H_\chi = A \times_{\alpha|_{K_\chi}} K_\chi$, then we have a natural identification $G/H_\chi \cong K/K_\chi$. Putting all the above analysis together, and invoking the imprimitivity theorem, it is not hard now to write down a complete proof of part (b) of the following result. (The proof of part (a) is not very difficult, either.)

THEOREM 5.1 *Let* $G = A \times_\alpha K$ *be a semi-direct product, with* A *abelian. Let* $\hat{\alpha} : K \to Aut\ \hat{A}$ *be as above. Assume that the action* $\hat{\alpha}$ *of* K *on* $\hat{A}$ *is regular in the sense described above (of 'admitting a Borel cross section').*

(a) If $\chi \in \hat{A}$, *if* K_χ, H_χ *are as above, and if* $\theta : K_\chi \to \mathcal{L}(\mathcal{H}^0)$ *is an irreducible representation of* K_χ, *then the equation* $\pi^0_{\chi,\theta}(a, k_\chi) = \chi(a)\theta(k_\chi)$ *defines an irreducible representation* $\pi^0_{\chi,\theta} : H_\chi \to \mathcal{L}(\mathcal{H}^0)$; *further, the result* $\pi_{\chi,\theta}$ *of inducing the representation* $\pi^0_{\chi,\theta}$ *up to* G *is an irreducible representation of* G.

(b) Conversely, if π *is an irreducible representation of* G, *then there exists a* $\chi \in \hat{A}$ *and an irreducible representation* θ *of* K_χ *such that* $\pi \cong \pi_{\chi,\theta}$ *(as in (a) above).*

□

In order to ensure that (s)he has really understood what the preceding theorem (as well as the imprimitivity theorem) says, the reader will do well to try and apply this theorem to the examples (a) and (b) of semi-direct products listed at the start of this section, to list out all possible irreducible representations of those groups (up to equivalence, that is).

(After these notes were prepared, the author learnt that Professor K.R. Parthasarathy has himself written an expository article on the imprimitivity theorem, and the reader is urged to also look at [KRP1].)

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