The standard invariant of a subfactor

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Basic construction tower for finite-dimensional C^*-algebras:

Definition: Suppose $N \subset M$ is a unital inclusion of finite-dimensional C^*-algebras. A trace 'tr' is called a **Markov trace with modulus τ** for the inclusion $N \subset M$ iff it can be extended to a trace 'Tr' on $\pi_r(N)'$ with the property that

$$Tr(e_N x) = \tau \ tr(x) \ \forall x \in M.$$

Recall that $\pi_r(N)' = \langle M, e_N \rangle$ is the *-algebra generated by M and e_N in $B(L^2(M, tr))$; it is linearly spanned by $M \cup \{x e_N y : x, y \in M\}$, and consequently, the extension Tr of 'tr' to $\pi_r(N)'$ is uniquely determined by the modulus τ condition. It is existence which requires some work.

Begin by recalling that if Λ denotes the inclusion matrix for $N \subset M$, then the inclusion matrix for $M \subset \langle M, e_N \rangle$ is identifiable with Λ^t - with respect to a certain natural identification of $\mathcal{P}_Z(N)$ with $\mathcal{P}_Z(\langle M, e_N \rangle)$.

Proposition (PF): (a) If ϕ and ψ are traces on M and N respectively, then

$$\psi = \phi|_N \Rightarrow \Lambda t_\phi = t_\psi,$$

where we think of t_ϕ and t_ψ as column vectors.

(b) Let 'tr' be a positive faithful trace on M; write $t = t_{tr}$ and $s = t_{tr|_N}$ $(= \Lambda t)$. Then, 'tr' is a Markov trace of modulus τ iff $\Lambda^t \Lambda t = \tau^{-1} t$ iff $\Lambda \Lambda^t s = \tau^{-1} s$; ie., t and s are *the* 'Perron-Frobenius eigenvectors' of $\Lambda^t \Lambda$ and of $\Lambda \Lambda^t$ respectively, and τ^{-1} is the 'Perron-Frobenius eigenvalue' of both these matrices.

Suppose $M_0 = N \subset M = M_1$ is a 'connected inclusion' of finite-dimensional C^*-algebras (meaning their Bratteli diagram is a connected graph). Let τ^{-1} be the Perron-Frobenius eigenvalue of $\Lambda^t\Lambda$ and t be the unique associated Perron-Frobenius eigenvector satisfying the normalisation that $tr(1) = 1$ where $t_{tr} = t$. Then, the previous Proposition guarantees that:

(i) 'tr' is a Markov trace of modulus τ for $M_0 \subset M_1$;

(ii) there is a unique extension of 'tr' to a trace 'Tr' on $M_2 = \langle M_1, e_1 \rangle$ (where $e_1 = e_{M_0}$) with the property that $tr(x_1 e_1) = \tau tr(x_1)\ \forall x_1 \in M_1$;

(iii) 'Tr' is a Markov trace of modulus τ for $M_1 \subset M_2$; and

(iv) we may repeat the process *ad infinitum* to obtain the tower

$$M_0 \subset M_1 \subset^{e_1} M_2 \subset^{e_2} M_3 \cdots$$

where e_n is the *Jones projection implementing the 'tr'-preserving conditional expectation* $E_{M_{n-1}}$ *of* M_n *onto* M_{n-1} and $M_n \subset M_{n+1}$ is the basic construction for $M_{n-1} \subset M_n$ (so that $M_{n+1} = \langle M_n, e_n \rangle$ is the *-algebra generated by $M_n \cup \{e_n\}$).

It is a consequence of the baic construction that the e_n's satisfy the relations:

$$\begin{array}{rcll} e_i^2 & = & e_i & \forall i \\ e_i e_j & = & e_j e_i & \text{if } |i-j| \geq 2 \\ e_i e_j e_i & = & \tau e_i & \text{if } |i-j| = 1 \end{array}$$

But for the foregoing analysis to work as outlined, τ^{-1} must be the largest eigenvalue of $\Lambda^t\Lambda$ for some non-negative integer valued rectangular matrix Λ which describes the adjacency relations in a connected bipartite graph Γ. See [GHJ] for the following classical result:

Theorem:(Kronecker) For Λ, Γ as above, we must have

$$\|\Lambda\| \in [2, \infty] \cup \{2cos(\frac{\pi}{n}) : n = 3, 4, 5, \cdots\}$$

Further if $\|\Lambda\| < 2$, then Γ must be a Coxeter graph of the following type: A_n, D_n, E_6, E_7, E_8, and $\|\Lambda\| = 2cos(\frac{\pi}{h})$, where h is the 'Coxeter number' of Γ. ($h = l + 1$ for A_l, $2l - 2$ for D_l, and 12, 18, 30 for E_6, E_7, E_8.)

To be able to 'handle' the continuous range of τ's, we need II_1 *factors*.

The symbols M, N, M_i will always denote II_1 factors.

Proposition 1:

(a) If $[M : N] < \infty$, then $N' \cap M$ is finite-dimensional; in fact, $dim(N' \cap M) \leq [M : N]$; and

$$[M : N] < 4 \Rightarrow N' \cap M = \mathbb{C}.$$

(b) If $M_i \subset M_j \subset M_k$ and $[M_j : M_i] < \infty$ and $[M_k : M_j] < \infty$, then

$$[M_k : M_i] = [M_k : M_j][M_j : M_i](< \infty).$$

Corollary: If

$$M_0 \subset M_1 \subset^{e_1} M_2 \subset^{e_2} M_3 \cdots$$

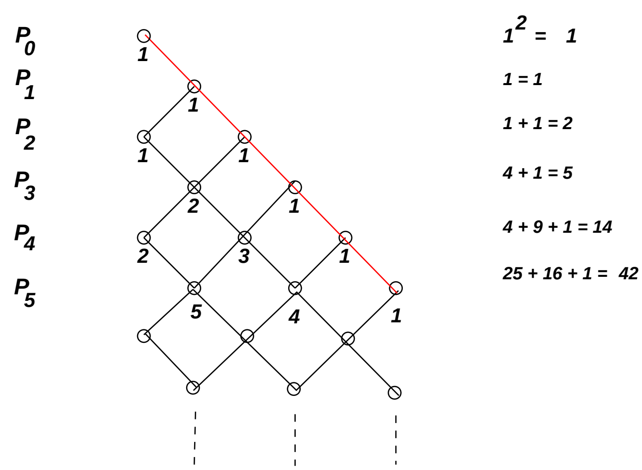
is the tower of the basic construction associated with a finite index subfactor $M_0 \subset M_1$, the following is a grid of finite-dimensional C^*-algebras:

$$\begin{array}{ccccccccc} \mathbb{C} & = & M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & \cdots \\ & & & & \cup & & \cup & & \cdots \\ & & \mathbb{C} & = & M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & \cdots \end{array}$$

Further, this comes equipped with a consistent trace (which, on $M_i' \cap M_j$ is the restriction of tr_{M_j}). This grid, with this trace, is called **the standard invariant** of $M_0 \subset M_1$.

This turns out to be a complete invariant for a 'good class' of subfactors - the so-called **extremal** ones.

To better understand this standard invariant, start by observing that the tower in the first row of the grid is described by the total Bratteli diagram obtained by glueing the several individual Bratteli diagrams together. We illustrate varous features of this tower in an example:



Here, we have written $P_k = M_0' \cap M_k$. This diagram illustrates the following features present in the corresponding diagram of relative commutants for every subfactor:

(a) The part of the diagram between the nth and $(n+1)$-th floors consists of two parts: (i) a (horizontal) mirror-reflection of the part of the diagram between the $(n-1)$-th and nth floors, and (ii) a 'new part'. In fact, new verices, if any, on the $(n+1)$-th floor are connected *only* to new vertices on the n-th floor.

(b) The (red) graph comprising all the 'new parts' is called the **principal graph** Γ of the subfactor $M_0 \subset M_1$. (It follows from (a) that the Bratteli diagram for the entire tower $\{M_0' \cap M_k : k \geq 0\}$ is determined by the principal graph.)

(c) In fact, the Bratteli diagram for the entire tower $\{M_1' \cap M_k : k \geq 0\}$ is recovered in the same fashion from the so-called **dual principal graph** $\widetilde{\Gamma}$, which is just the principal graph of $M_1 \subset M_2$.

(d) In the exhibited example, the principal graph is the finite graph A_6, and the dual principal graph turns out to be the same. It is fact that Γ is finite iff $\widetilde{\Gamma}$ is finite, in which case the subfactor is said to have **finite depth**.

(e) In addition to the two principal graphs, which only describe the two towers of relative commutants, one also needs to encode the data of how one tower is embedded into the next. This has been done in at least three ways: in a **paragroup** (Ocneanu), a λ-**lattice** (Popa), or in a **planar algebra** (Jones). (We shall elaborate later on the last.) Any one of these notions is equivalent to the ‘standard invariant, and is a complete invariant, provided the subfactor is **extremal**. (Finite depth subfactors are known to be extremal, and thus determined by their standard invariant.)

The richness of the theory of subfactors may be surmised from the following facts:

(a) To every finite group G is associated a canonical subfactor $R^G \subset R$ such that

$$(R^{G_1} \subset R) \cong (R^{G_2} \subset R) \Leftrightarrow G_1 \cong G_2$$

(b) More generally, to every finite-dimensional Hopf C^*-algebra H is associated a canonical subfactor $R^H \subset R$ such that

$$(R^{H_1} \subset R) \cong (R^{H_2} \subset R) \Leftrightarrow H_1 \cong H_2$$

(c) In fact, subfactors as in (b) are characterise by the property that they have 'depth 2'; the principal graph of $R^H \subset R$ is the bipartite graph with even vertices indexed by $\widehat{H}$ (the set of irreducible *-algebra representations of H), with one odd vertex, and with the degree of the odd vertex indexed by $\pi \in \widehat{H}$ being given by the degree d_π of the representation π.

Planar algebras (PAs):

A planar algebra is a collection $\{P_n : n \geq 0\}$ of $\mathbb{C}$-vector spaces which admits an action by the *coloured operad of planar tangles*. Here is an example of a planar tangle:

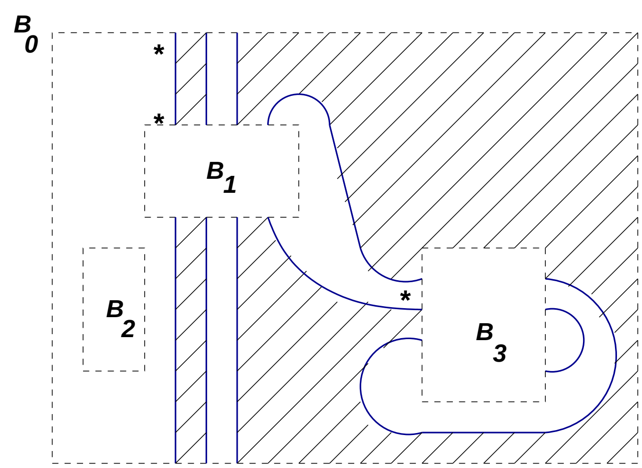


Figure 1: Tangle T

A planar tangle T has the following features:

(a) its boundary consists of an external box (labelled B_0), and some number b (which is 3 in this example, and can, in general, even be 0) of internal boxes (labelled $B_1, \cdots B_b$).

(b) each box B_i has an even number $2k_i$ of marked points, and is said to be of *colour* k_i. In this example,

$$k_0 = 3, k_1 = 4, k_2 = 0, k_3 = 3.$$

(c) There are a number of non-crossing 'strings' which are either closed curves or have their two ends on a marked point of one of the boxes, in such a way that every marked point is the end-point of some string.

(d) The entire configuration comes with a checker-board shading.

(e) One special marked point on each box of non-zero colour is labelled with a '*' in such a way that as one travels outward (resp., in-ward) from the *-point of an internal (resp., the external) box, the black region is to the right.

The one thing one can do with tangles is *composition*, when that makes sense: thus, if S and T are tangles, such that the external box of S has the same colour as the i-th internal box of T, then we may form a new tangle $T \circ_i S$ by 'glueing S into the i-th internal box of T in such a way that the *-points and the strings at the common boundary are aligned.

A tangle T with boxes coloured $k_0, \cdots, k_b$ is required to induce a linear map

$$(Z_T^P =) Z_T : \otimes_{i=1}^b P_{k_i} \to P_{k_0}$$

and these maps are to satisfy some natural compatibility requirements, the most important being compatibility with composition of tangles:

Rather than going through all the requirements of a planar algebra, let us look at one of the most elementary examples, the *Temperley-Lieb planar algebra*. Fix $0 < \tau < 1/4$, and let $P_0 = \mathbb{C}$, and $P_n = TL_n(\tau)$, the $\mathbb{C}$-vector space with basis $\mathcal{K}_n$, the set of Kauffman diagrams. We define the action of a tangle on 'basis vectors': thus, for example, if T denotes the tangle of Figure 1, and if $S_0 \in \mathcal{K}_3, S_1 \in \mathcal{K}_4$ and $S_3 \in \mathcal{K}_3$ are the Kauffman diagrams shown in Figure 2, and $1 \in \mathbb{C} = TL_0(\tau)$, then

$$Z_T(S_1 \otimes 1 \otimes S_3) = \beta^2 S_0,$$

where $\beta = \tau^{-2}$ (since each loop counts for a multiplicative factor of β).

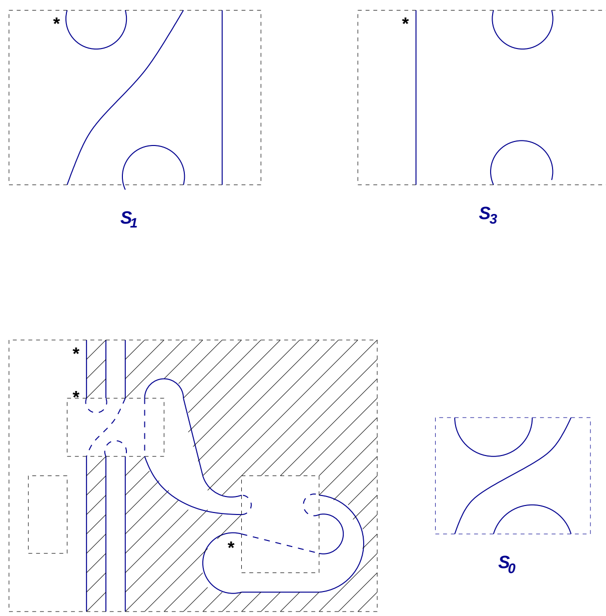


Figure 2: tangle action in TL_n

By a homomorphism π between planar algebras $P = \{P_k : k \geq 0\}$ and $Q = \{Q_k : k \geq 0\}$, one understands a collection of $\mathbb{C}$-linear maps $\pi_k : P_k \to Q_k$ which are 'equivariant' with respect to the tangle actions: thus, if T is a k_0-tangle with internal boxes of colours $k_1, \cdots, k_b$, then we must have

$$\pi_{k_0} \circ Z_T^P = Z_T^Q \circ (\otimes_{i=1}^b \pi_{k_i})$$

The generators-and-relations approach to planar algebras:

For any 'graded set' $L = \coprod_{n \geq 0} L_n$ - where some L_n's may be empty, define an L-labelled tangle T to be a tangle equipped with a labelling of each internal box of colour k by an element of L_k. (In particular, if $L_k = \emptyset$ for some k, then an L-labelled tangle cannot have an internal k-box.)

The universal PA on label set L:

Let $\mathcal{P}_k(L)$ be a $\mathbb{C}$-vector space with basis indexed by the set of all L-labelled k-tangles (= tangles with external box of colour k). It is not hard to see that $\mathcal{P}(L) = \{\mathcal{P}_k(L) : k \geq 0\}$ has a natural struture of a planar algebra; this is the *universal planar algebra on label set L* in the sense that: given set functions $f_k : L_k \to P_k$, for some planar algebra P, there is a unique planar algebra homomorphism $\pi : \mathcal{P}(L) \to P$ such that 'π_k extends f_k' for each k.

Definition: A **planar ideal** $\mathcal{I}$ of a PA P is a collection $\mathcal{I} = \{I_k : k \geq 0\}$ of subspaces of $P = \{P_k : k \geq 0\}$ such that $Z_T(\otimes_{i=1}^b x_i) \in I_{k_0}$ whenever T is a tangle and $x_i \in P_{k_i}$ $\forall$ $1 \leq i \leq b$ provided $x_j \in I_{k_j}$ for at least one j.

It is easily shown that $\mathcal{I}$ is a planar ideal in P iff there is a PA homomorphism $\pi : P \to Q$ (for some PA Q) such that $I_j = ker(\pi_j)$ $\forall j$. (This Q may be chosen as $P/|CI = \{P_k/I_k : k \geq 0\}$ with its natural PA structure.)

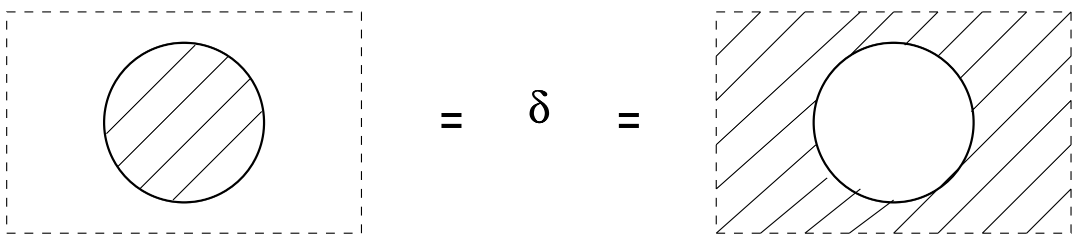
It is another routine matter to verify that given any 'subset' $\mathcal{R} = \{R_k : k \geq 0\}$ of a PA P, there exists a smallest planar ideal $\mathcal{I}(\mathcal{R})$ of P with the property that $R_k \subset I_k \; \forall k$.

Finally, given a label set $L = \coprod_k L_k$ and a 'subset' $\mathcal{R} = \{R_k : k \geq 0\}$ of the PA $\mathcal{P}(L)$, define $P(\langle L, \mathcal{R} \rangle) = \mathcal{P}(l)/\mathcal{I}(\mathcal{R})$. This is the PA with *presentation given by label set L and relations $\mathcal{R}$*.

We shall conclude with some examples of presentations of planar algebras:

Temperley-Lieb Planar algebra, for $\tau < 1/4$:

This has label set $L = \emptyset$, and the two relations listed below. (Taking a cue from group theory, we think of relations as equations; thus, we say $X = 0$ is a relation if $X \in \mathcal{R}$.)

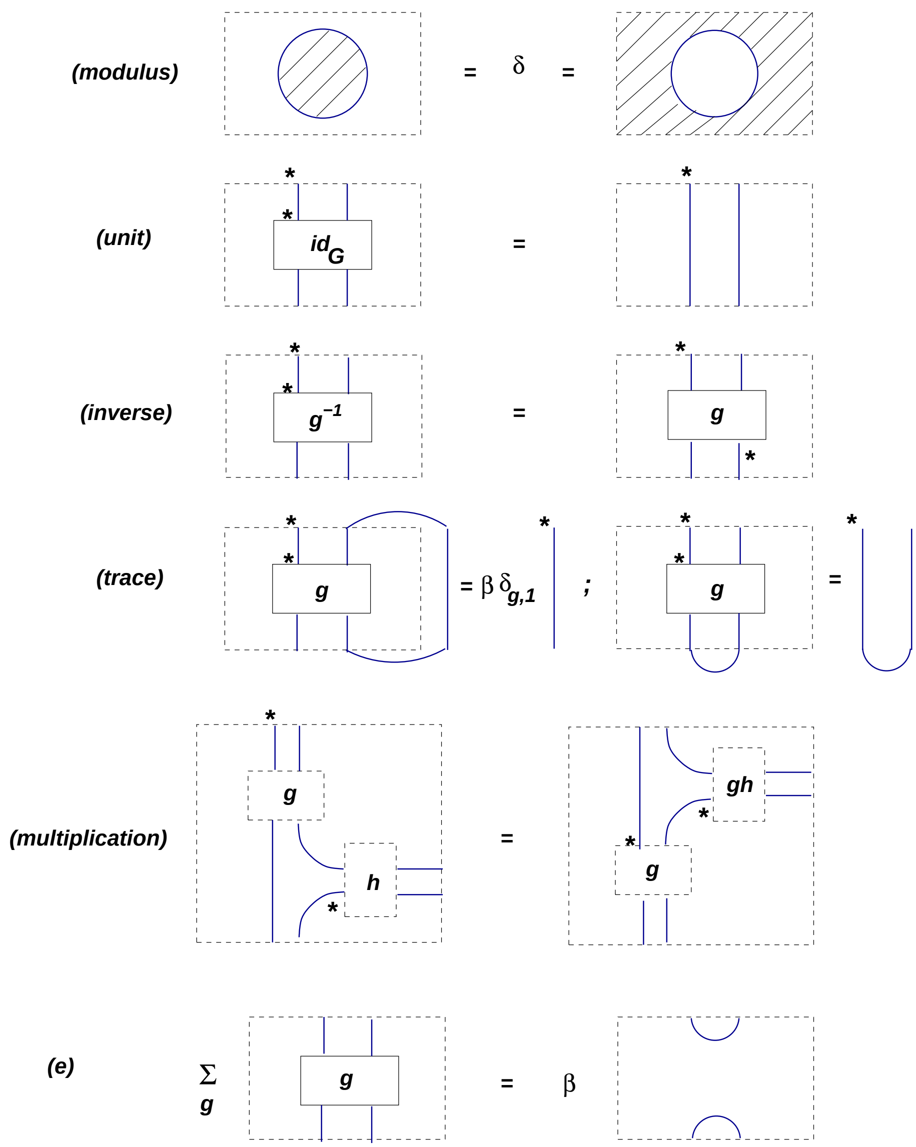


The PA for $R^G \subset R$:

For a finite group G, the label set is taken as

$$L_k = \begin{cases} G & \text{if } k = 2 \\ \emptyset & otherwise \end{cases}$$

and the relations are as follows (where we write $\beta = \sqrt{|G|}$ and use δ for the 'Kronecker delta'):


(modulus)
= δ =
(unit)
id_G
=
(inverse)
g^{-1}
= g
(trace)
g = β $\delta_{g,1}$; g =
(multiplication)
g h = gh g
(e)
$\sum_g$ g = β