von Neumann algebras, II_1 factors, and their subfactors

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Lecture 3 at IIT Mumbai, April 24th, 2007

Finite-dimensional C^*-algebras:

Recall:

Definition: A linear functional ‘tr’ on an algebra A is said to be

- a *trace* if tr $(xy) =$ tr (yx) forall $x, y \in A$;
- *normalised* if A is unital and tr$(1) = 1$;
- *positive* if A is a *-algebra and tr $(x^*x) \geq 0 \forall x \in A$;
- *faithful and positive* if A is a *-algebra and tr $(x^*x) > 0 \ \forall \ 0 \neq x \in A$.

For example, $M_n(\mathbb{C})$ admits a unique normalised trace (tr$(x) = \frac{1}{n}\sum_{i=1}^{n} x_{ii}$) which is automatically faithful and positive.

Proposition FDC*: The following conditions on a finite-dimensional unital *-algebra A are equivalent:

1. There exists a unital *-monomorphism $\pi : A \to M_n(\mathbb{C})$ for some n.

2. There exists a faithful positive normalised trace on A.

$\square$

For a finite-dimensional C^*-algebra M with faithful positive normalised* trace 'tr', let us write $L^2(M, tr) = \{\widehat{x} : x \in M\}$, with $\langle \widehat{x}, \widehat{y} \rangle = \text{tr}(y^* x)$, as well as $\pi_l, \pi_r : M \to B(L^2(M, tr))$ for the maps (injective unital *-homomorphism and *-antihomomorphism, repectively) defined by

$$\pi_l(x)(\widehat{y}) = \widehat{xy} = \pi_r(y)(\widehat{x}) \ .$$

We shall usually identify $x \in M$ with the operator $\pi_l(x)$ and thus think of M as a subset of $B(L^2(M, tr))$.

Fact: $\pi_l(M)' = \pi_r(M)$ and $\pi_r(M)' = \pi_l(M)$, where we define the *commutant* S' of any set S of operators on a Hilbert space H by

$$S' = \{x' \in B(H) : xx' = x'x \ \forall x \in S\})$$

*It is a fact that every finite-dimensional C^*-algebra is unital.

Write $\mathcal{P}_Z(M)$ for the set of minimal central projections of a finite-dimensional C^*-algebra. It is a fact that there is a well-defined function $m : \mathcal{P}_Z(M) \to \mathbb{N}$, such that $Mq \cong M_{m(q)}(\mathbb{C})\ \forall q \in \mathcal{P}_Z(M)$; thus the map $M \ni x \overset{\pi_q}{\mapsto} xq$ defines an irreducible representation of M; and in fact, $\{\pi_q : q \in \mathcal{P}_Z(M)\}$ is a complete list, up to unitary equivalence, of pairwise inequivalent irreducible representations of M, and

$$M = \sum_{q \in \mathcal{P}_Z(M)} Mq \cong \oplus_{q \in \mathcal{P}_Z(M)} M_{m(q)}(\mathbb{C})$$

Since every trace on the full matrix algebra $M_n(\mathbb{C})$ is a multiple of the usual trace. It follows that any trace ϕ on M is uniquely determined by the function $t_\phi : \mathcal{P}_Z(M) \to \mathbb{C}$ defined by $t_\phi(q) = \phi(q_0)$ where q_0 is a minimal projection in Mq. It is clear that ϕ is positive (resp., faithful, or normalised) iff $t_\phi(q) \geq 0\ \forall q$ (resp., $t_\phi(q) > 0\ \forall q$, or $\sum_{q \in \mathcal{P}_Z(M)} m(q) t_\phi(q) = 1$).

If $N \subset M$ is a unital C^*-subalgebra of M, the associated *inclusion matrix* Λ is the matrix with rows and columns indexed by $\mathcal{P}_Z(N)$ and $\mathcal{P}_Z(M)$ repectively, defined by setting $\Lambda_{pq} = \sqrt{\frac{dim \ \ qpMqp}{dim \ \ qpNqp}}$. Alternatively, if we write ρ_p for the irreducible representation of N corresponding to p, then Λ_{pq} is nothing but the 'multiplicity with which ρ_p occurs in the irreducible decomposition of $\pi_q|_N$'. This data is sometimes also recorded in a bipartite graph with even and odd vertices indexed by $\mathcal{P}_Z(N)$ and $\mathcal{P}_Z(M)$ repectively, with Λ_{pq} edges joining the vertices indexed by p and q; this bipartite graph is usually called the *Bratteli diagram* of the inclusion.

Writing E_N for the tr-preserving conditional expectation of M onto N, and e_N for the orthogonal projection of $L^2(M, tr)$ onto the subspace $L^2(N, tr|_N)$, we have the following result.

Propostion (bc): Suppose $N \subset M$ is a unital inclusion of finite dimensional C* algebras. Let tr be a faithful, unital, positive trace on M. Then,

(1) The C* algebra generated by M and e_N in $B(L^2(M, tr))$ is $\pi_r(N)'$.

(2) The central support of e_N in $\pi_r(N)'$ is 1.

(3) $e_N x e_N = E(x) e_N$ for $x \in M$.

(4) $N = M \cap \{e_N\}'$.

(5) If Λ is the inclusion matrix for $N \subset M$ then Λ^t is the inclusion matrix for $M \subset \pi_r(N)'$. $\square$

This *basic construction* - i.e., the passage from $N \subset M$ to $M \subset \pi_r(N)'$ extends almost verbatim from inclusions of finite-dimensional C^*-algebras to *finite-depth subfactors*!

von Neumann algebras :

Introduced in - and referred to, by them, as - *Rings of Operators* in 1936 by F.J. Murray and von Neumann, because - in their own words:

the elucidation of this subject is strongly suggested by

- *our attempts to generalise the theory of unitary group-representations, and*

- *various aspects of the quantum mechanical formalism*

Def 1: A vNa is the commutant of a unitary group representation: i.e.,

$$M = \{x \in \mathcal{L}(\mathcal{H}) : x\pi(g) = \pi(g)x \ \forall g \in G\}$$

Note that $\mathcal{L}(\mathcal{H})$ is a Banach *-algebra w.r.t. $\|x\| = \sup\{\|x\xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}$ ('operator norm') and 'Hilbert space adjoint'.

Defs: (a) $S' = \{x' \in \mathcal{L}(\mathcal{H}) : xx' = x'x \ \forall x \in S\}$, for $S \subset \mathcal{L}(\mathcal{H})$

(b) SOT on $\mathcal{L}(\mathcal{H})$: $x_n \to x \Leftrightarrow \|x_n\xi - x\xi\| \to 0 \ \forall\xi$ (i.e., $x_n\xi \to x\xi$ strongly $\forall\xi$)

(c) WOT on $\mathcal{L}(\mathcal{H})$: $x_n \to x \Leftrightarrow \langle x_n\xi - x\xi, \eta\rangle \to 0 \forall \xi, \eta$ (i.e., $x_n\xi \to x\xi$ weakly $\forall\xi$)

(Our Hilbert spaces are always assumed to be **separable**.)

von Neumann's double commutant theorem (DCT: Let M be a unital self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$. TFAE:

(i) M is SOT-closed

(ii) M is WOT-closed

(iii) $M = M'' = (M')'$ $\square$

Def 2: A vNa is an M as in DCT above.

The equivalence of definitions 1 and 2 is a consequence of the spectral theorem and the fact that any norm-closed unital *-subalgebra A of $\mathcal{L}(\mathcal{H})$ is linearly spanned by the set $\mathcal{U}(A) = \{u \in A : u^*u = uu^* = 1\}$ of its **unitary** elements.

Some consequences of DCT:

(a) A von Neumann algebra is closed under all 'canonical constructions':

for instance, if $x \to \{1_E(x) : E \in \mathcal{B}_{\mathbb{C}}\}$ is the spectral measure associated with a normal operator x, then $x \in M \Leftrightarrow 1_E(x) \in M \ \forall \ E \in \mathcal{B}_{\mathbb{C}}$.

(*Reason:* $1_E(uxu^*) = u1_E(x)u^*$ for all unitary u; so implication $\Rightarrow$ follows from

$$\begin{aligned} x \in M, u' \in \mathcal{U}(M') \ &\Rightarrow \ u'1_E(x)u'^* = 1_E(u'xu'^*) \\ &\Rightarrow \ 1_E(x) \in \Big(\mathcal{U}(M')\Big)' = M \quad) \end{aligned}$$

(b) For implication $\Leftarrow$, uniform approximability of bounded measurable functions implies (by the spectral theorem) that

$$M = [\mathcal{P}(M)] = (span \ \mathcal{P}(M))^- \quad (*),$$

where $\mathcal{P}(M) = \{p \in M : p = p^2 = p^*\}$ is the set of projections in M.

Suppose $M = \pi(G)'$ as before. Then

$$p \leftrightarrow ran\ p$$

establishes a bijection

$$\mathcal{P}(M) \leftrightarrow G\text{-stable subspaces}$$

So, for instance, eqn. (*) shows that

$$(\pi(G))'' = \mathcal{L}(\mathcal{H}) \Leftrightarrow M = \mathbb{C} \ \Leftrightarrow \ \pi \text{ is irreducible}$$

Under the correspondence, of sub-reps of π to $\mathcal{P}(M)$, (unitary) equivalence of sub-repreps of π translates to *Murray-von Neumann equivalence* on $\mathcal{P}(M)$:

$$p \sim_M q \Leftrightarrow \exists u \in M \text{ such that } u^*u = p,\ uu^* = q$$

More generally, define

$$p \preceq_M q \Leftrightarrow \exists p_0 \in \mathcal{P}(M) \text{ such that } p \sim_M p_0 \leq q$$

Proposition: TFAE:

1. Either $p \preceq_M q$ or $q \preceq_M p$, $\forall p, q \in \mathcal{P}(M)$.

2. M has trivial center: $Z(M) = M \cap M' = \mathbb{C}$

Such an M is called a **factor**. $\square$

If $M = \pi(G)'$, with G finite, then M is a factor iff π is isotypical.

In general, any vNa is a 'direct integral' of factors.

Say a projection $p \in \mathcal{P}(M)$ is **infinite rel** M if $\exists p_0 \neq p \in \mathcal{P}(M)$ such that $p \sim_M p_0 \leq p$; otherwise, call p **finite** (rel M).

Say M is finite if 1 is finite.

Murray von-Neumann classification of factors: A factor M is said to be of type:

1. I if there is a minimal non-zero projection in M.

2. II if it contains non-zero finite projections, but no minimal non-zero projection.

3. III if it contains no non-zero finite projection.

Def. 3: (Abstract Hilbert-space-free def) M is a vNa if

- M is a C^*-algebra (i.e., a Banach *-algebra satisfying $\|x * x\| = \|x\|^2 \ \forall\ x$)

- M is a dual Banach space: i.e., $\exists$ a Banach space M_* such that $M \cong M_*^*$ as a Banach space.

Example: $M = L^\infty(\Omega, \mathcal{B}, \mu)$. Can also view it as acting on $L^2(\Omega, \mathcal{B}, \mu)$ as multiplication operators. (In fact, every commutative vNa is isomorphic to an $L^\infty(\Omega, \mathcal{B}, \mu)$.)

Fact: The predual M_* of M is unique up to isometric isomorphism. (So, (by Alaoglu), $\exists$ a canonical loc. cvx. (weak-*) top. on M w.r.t. which the unit ball of M is compact. This is called the σ-**weak topology** on M.

A linear map between vNa's is called **normal** if it is continuous w.r.t. the σ-weak topologies on domain and range.

The morphisms in the category of vNa's are unital normal $*$-homomorphisms.

The algebra $\mathcal{L}(\mathcal{H})$, for any Hilbert space $\mathcal{H}$, is a vNa - with pre-dual being the space $\mathcal{L}_*(\mathcal{H})$ of trace-class operators.

Any σ-weakly closed $*$-subalgebra of a vNa is a vNa.

Gelfand-Naimark theorem: Any vNa is isomorphic to a vN-subalgebra of some $\mathcal{L}(\mathcal{H})$. (So the abstract and concrete (= tied down to Hilbert space) definitions are equivalent.)

In some sense, the most interesting factors are the so-called *type II_1 factors* (= finite type II factors).

Theorem: Let M be a factor. TFAE:

1. M is finite.

2. $\exists$ a **trace** tr_M on M - i.e., linear functional satisfying:

   - $tr_M(xy) = tr_M(yx)\ \forall x, y \in M$ (*trace*)
   - $tr_M(x^*x) \geq 0 \forall x \in M$ (*positive*)
   - $tr_M(1) = 1$ (*normaliised*)

Such a trace is automatically unique, and *faithful* - i.e., it satisfies $tr_M(x^*x) = 0 \Leftrightarrow x = 0$

For $p, q \in \mathcal{P}(M)$, M a finite factor, TFAE:

1. $p \sim_M q$

2. $tr_M p = tr_M q$

3. $\exists u \in \mathcal{U}(M)$ such that $upu^* = q$.

If $dim_{\mathbb{C}} M < \infty$, then $M \cong M_n(\mathbb{C}) = \mathcal{L}(\mathbb{C}^n)$ for a unique n.

If $dim_{\mathbb{C}} M = \infty$, then M is a II_1 factor, and in this case, $\{tr_M p : p \in \mathcal{P}(M)\} = [0, 1]$.

So II_1 factors are the arena for continuously varying dimensions; they got von Neumann looking at *continuous geometries*.

Henceforth, M will be a II_1 factor.

Def: An M-module is a separable Hilbert space $\mathcal{H}$, equipped with a vNa morphism $\pi : M \to \mathcal{L}(\mathcal{H})$. Two M-modules are isomorphic if there exists an invertible (equivalently, unitary) M-linear map between them.

Proposition: There exists a complete isomorphism invariant

$$\mathcal{H} \mapsto dim_M \mathcal{H} \in [0, \infty]$$

of M-modules such that:

1. $\mathcal{H} \cong \mathcal{K} \Leftrightarrow dim_M \mathcal{H} = dim_M \mathcal{K}$.

2, $dim_M(\oplus_n \mathcal{H}_n) = \sum_n dim_M \mathcal{H}_n$.

3. For each $d \in [0, \infty]$, $\exists$ an M-module $\mathcal{H}_1$ with $dim_M \mathcal{H}_d = d$.

The equation

$$\langle x, y\rangle = tr_M(y^*x)$$

defines an inner-product on M. Call the completion $L^2(M, tr_M)$. Then $L^2(M, tr_M)$ is an $M - M$ bimodule with left- and right- actions given by multiplication.

$\mathcal{H}_1 = L^2(M, tr_M)$.

If $0 \leq d \leq 1$, then $\mathcal{H}_d = L^2(M, tr_M).p$ where $p \in \mathcal{P}(M)$ satisfies $tr_M p = d$.

$\mathcal{H}_d$ is a finitely generated projective module if $d < \infty$.

In particular $K_0(M) \cong \mathbb{R}$.

The hyperfinite II_1 factor R: Among II_1 factors, pride of place goes to the ubiquitous hyperfinite II_1 factor R. It is characterised as the unique II_1 factor which has any of many properties, such as injectivity and approximate finite-dimensionality (= hyperfiniteness).

Thus, $\exists$ a unique II_1 factor R which contains an increasing sequence

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$$

such that $\cup_n A_n$ is σ-weakly dense in R.

Examples of II_1 factors: Let $\lambda : G \to \mathcal{U}(\mathcal{L}(\ell^2(G)))$ denote the 'left-regular representation' of a countable infinite group G, and let $LG = (\lambda(G))''$. Then LG is a II_1 factor iff every conjugacy class of G other than $\{1\}$ is infinite.

$L\Sigma_\infty \cong R$, while $L\mathbb{F}_2$ is not hyperfinite.
Big open problem: is $L\mathbb{F}_2 \cong L\mathbb{F}_3$?

The study of bimodules over II_1 factors is essentially equivalent to that of 'subfactors'. (The bimodule ${}_N\mathcal{H}_M$ corresponds to $\pi_l(N) \subset \pi_r(M)'$.)

A **subfactor** is a unital inclusion $N \subset M$ of II_1 factors. For a subfactor as above, Jones defined the index of the subfactor to be

$$[M:N] = dim_N L^2(M, tr_M)$$

and proved:

$$[M:N] \in [4,\infty] \cup \{4cos^2(\frac{\pi}{n} : n \geq 3\}$$

A subfactor N is said to be **irreducible** if $N' \cap M = \mathbb{C}$ - or equivalently, $L^2(M, tr_M)$ is irreducible as an $N - M$ bimodule.

It is known that if a subfactor $N \subset M$ has finite index, then N is hyperfinite if and only if M is. In this case, call the subfactor hyperfinite.

Very little is known about the set $\mathcal{I}_R^0$ of possible index values of irreducible hyperfinite subfactors.

Some known facts:

(a) (Jones) $\mathcal{I}_R = [4, \infty] \cup \{4cos^2(\frac{\pi}{n}) : n \geq 3\}$ and $\mathcal{I}_R^0 \supset \{4cos^2(\frac{\pi}{n} : n \geq 3\}$

(b) $\left(\frac{N+\sqrt{N^2+4}}{2}\right)^2, \left(\frac{N+\sqrt{N^2+8}}{2}\right)^2 \in \mathcal{I}_R^0 \ \forall N \geq 1$

(c) $(N + \frac{1}{N})^2$ is the limit of an increasing sequence in $\mathcal{I}_R^0$.

What is relevant for us is that if $N \subset M$ is a subfactor of finite index, then the 'basic construction' goes through exactly as in finite dimensions.

Proposition: (subf1) Let $L^2(M, tr_M)$ denote the completion of the inner-product space $V = \{\widehat{x} : x \in M\}$ (with inner-product defined by $\langle \widehat{x}, \widehat{y} \rangle = \mathrm{tr}(y^* x)$), let $L^2(N, tr_N)$ be identified with the subspace defined as the closure of $V_0 = \{\widehat{x} : x \in N\}$, and let e_N denote the orthogonal projection of $L^2(M, tr_M)$ onto $L^2(N, tr_N)$.

(1) Then there exists a map $E_N : M \to N$ satisfying, for all $x \in M, a, b \in N$:

(i) $E_N(axb) = aE_N(x)b$

(ii) $E_N(a) = a$

(iii) $tr|_N \circ E_N = tr$

(iv) $e_N x e_N = (E_N x) e_N$

(2) Further, if we write π_l and π_r for the maps defined earlier, if we identify M with $\pi_l(M)$, then,

(a) $\pi_r(M)' = \pi_l(M) = M$

(b) $\pi_r(N)' = \langle M, e_N \rangle = (M \cup \{e_N\})''$ is also a II_1 factor, and $[\langle M, e_N \rangle : M] = [M : N]$

(c) $tr_{\langle M, e_N \rangle}(xe_N) = \tau tr_M(x)$ for all $x \in M$, where we write $\tau = [M : N]^{-1}$.

(d) $N = M \cap \{e_N\}'$ $\square$

All the necessary ingredients are in place for us to build the Jones tower

$$M_0 \subset M_1 \subset^{e_1} M_2 \subset^{e_2} M_3 \cdots$$

where e_n is the *Jones projection implementing the 'tr'-preserving conditional expectation* $E_{M_{n-1}}$ *of* M_n *onto* M_{n-1} and $M_n \subset M_{n+1}$ is the basic construction for $M_{n-1} \subset M_n$ (so that $M_{n+1} = \langle M_n, e_n \rangle$).

It is easy to deduce from the preceding proposition (applied to appropriate members of the Jones tower) that

$$\begin{array}{rcl} e_i^2 & = & e_i \quad \forall i \\ e_i e_j & = & e_j e_i \quad \text{if } |i-j| \geq 2 \\ e_i e_j e_i & = & \tau e_i \quad \text{if } |i-j| = 1 \end{array}$$

where $\tau = [M : N]^{-1}$. In fact, more generally than the last equation above, it is true that:

$$\begin{array}{rcl} e_n x e_n & = & (E_{M_{n-1}} x) e_n \ \forall x \in M_n \\ tr_{M_{n+1}}(x e_n) & = & \tau tr_{M_n}(x) \ \forall x \in M_n \end{array}$$

In fact, since there is a unique normalised trace on a II_1 factor, we can unambiguously use the symbol 'tr' for the functional on $\cup_n M_n$ which restricts on M_n to tr_{M_n}.