The Temperley-Lieb algebra

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These algebras appeared first in [TL] in the context of statistical mechanics. See appendix IIb of [GHJ] for an account of this. Our interest stems from their manner of appearance in [Jon], where the following striking result was proved:

Theorem (Jones): The index $[M : N]$ of a subfactor satisfies

$$[M : N] \in [4, \infty] \cup \{4cos^2(\frac{\pi}{n}) : n = 3, 4, 5, \cdots\}$$

References:

1. ([GHJ]) F. M. Goodman, P. de la harpe and V. F.R.Jones : "Coxeter Graphs and Towers of Algebra", MSRI Publ., 14, Springer, New York, 1989.

2. ([Jon]) V.F.R. Jones: "Index for subfactors", Inventiones Math. 72(1983)1-25.

3. ([JS]) V.Jones and V.S.Sunder : “Introduction to Subfactors”, LMS lecture note series, 234(1997).

4. ([JR]) V. F.R. Jones and Sarah A.Reznikoff: “Hilbert space representations of the annular Temperley-Lieb algebra”, Pacific J. of Math., vol. 228, No. 2,(2006),219-250.

5. ([TL]) H.N.V. Temperley and E.H. Lieb: “Relations between the ‘percolation’ and the ‘coloring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem”, Proc. Roy. Soc. (London) ser A **322** (1971) 251-280.

6. ([Wen]) H.Wenzl: “On sequences of projections”, C.R.Math. Rep. Acad. Sci. Canada 9 (1987) 5-9.

For a scalar $0 \neq \tau \in \mathbb{C}$, and positive integer n, the **Temperley-Lieb algebra** $TL_n(\tau)$ is the universal unital $\mathbb{C}$-algebra generated by $1, e_1, \cdots, e_{n-1}$ and satisfying the relations

$$\begin{array}{rcll} e_i^2 & = & e_i & \forall i \\ e_i e_j & = & e_j e_i & \text{if } |i-j| \geq 2 \\ e_i e_j e_i & = & \tau e_i & \text{if } |i-j| = 1 \end{array}$$

We shall first establish finite-dimensionality of $TL_n(\tau)$. (In fact, the dimension is the n-th Catalan number.) To this end, we begin with the following result from [Jon]:

Lemma 1:

If w is a 'word' in $e_1, \cdots, e_{n-1}$, and if k is the largest integer for which e_k occurs in w, then there exist words u, v in $e_1, \cdots, e_{k-1}$ such that $w = \tau^m u e_k v$.

Proof: We prove this by induction on k. The assertion being trivial when $k = 1$, we assume the result for all smaller values of k and establish the validity of the inductive step.

For this, we proceed by induction on the number M of times e_k occurs in the word w. The result to be proved is trivially true for $M = 1$, so we shall reduce the validity of the case when $M \geq 2$ to that of the case $M-1$. If $M \geq 2$, then $w = w_1 e_k w_2 e_k w_3$, with the w_i's being words in $e_1, \cdots, e_{k-1}$. If w_2 does not feature any e_{k-1}, then it commutes with e_k and we have $w = w_1 e_k w_2 w_3$ is an expression where e_k occurs only $(M-1)$ times. If w_2 does feature e_{k-1}, then by the (first) induction hypothesis, we may write $w_2 = \tau^r u_2 e_{k-1} v_2$, where u_2, v_2 are words in $e_1, \cdots, e_{k-2}$ and consequently commute with e_k, so that we find that, indeed

$$\begin{aligned} w &= w_1 e_k w_2 e_k w_3 \\ &= \tau^r w_1 u_2 e_k e_{k-1} e_k v_2 w_3 \\ &= \tau^{r+1} w_1 u_2 e_k v_2 w_3 \end{aligned}$$

is an expression of w featuring only $(M-1)$ e_k's, and by induction, the proof of the lemma is complete.

Proposition 1.

(a) $TL_n(\tau)$ is spanned by elements of the form

$$\begin{aligned} w = \tau^m & (e_{i_1} e_{i_1-1} \cdots e_{j_1}) \\ & (e_{i_2} e_{i_2-1} \cdots e_{j_2}) \cdots \\ & (e_{i_p} e_{i_p-1} \cdots e_{j_p}) \end{aligned} \tag{1}$$

for some integers $m, p, i_1, j_1, \cdots, i_p, j_p$ satisfying $m, p \geq 0$, and

$$\begin{aligned} 1 \leq i_1 < i_2 < \cdots < i_p < n, \\ 1 \leq j_1 < j_2 < \cdots < j_p < n, \\ j_1 \leq i_1, j_2 \leq i_2, \cdots, j_p \leq i_p \end{aligned} \tag{2}$$

(b) In particular,

$$\dim\, TL_n(\tau) \leq \frac{1}{n+1} \binom{2n}{n}.$$

Proof: (a) The proof is by induction on n.

It clearly suffices to prove that every word in the e_i's satisfies an equation of the form displayed in (a). This assertion is obvious for $n = 1$. So suppose this assertion is true for all values smaller than n.

If, now, w is a word in $e_1, \cdots, e_{n-1}$, the desired assertion holds if the w does not feature e_{n-1}. If w features e_{n-1}, then, the lemma alows us to assume that in fact, $w = ue_{n-1}v$, where u, v are words in $e_1, \cdots, e_{n-2}$. If we now 'push the e_{n-1} as far to the right as possible, we see, after appropriately utilising the induction hypothesis, that we may write w as in equation (1), where $i_p = n-1$ and all the inequalities in (2) are satisfied, except possibily $j_{p-1} < j_p$. Among all possible expressions of w, we claim that the one of shortest length will indeed satisfy $j_{p-1} < j_p$. This is because, if $j_{p-1} \geq j_p$, it can be seen that

$$\begin{aligned}
&(e_{i_{p-1}} \cdots e_{j_{p-1}})(e_{n-1}e_{n-2} \cdots e_{j_{p-1}+1}e_{j_{p-1}} \cdots e_{j_p}) \\
&= (e_{i_{p-1}} \cdots e_{j_{p-1}+1}) \\
&\quad (e_{n-1}e_{n-2} \cdots e_{j_{p-1}+2}e_{j_{p-1}}e_{j_{p-1}+1}e_{j_{p-1}} \cdots e_{j_p}) \\
&= \tau(e_{i_{p-1}} \cdots e_{j_{p-1}+1}) \\
&\quad (e_{n-1}e_{n-2} \cdots e_{j_{p-1}+2})(e_{j_{p-1}} \cdots e_{j_p}) \\
&= \tau(e_{i_{p-1}} \cdots e_{j_p})(e_{n-1}e_{n-2} \cdots e_{j_{p-1}+2}) \ ,
\end{aligned}$$

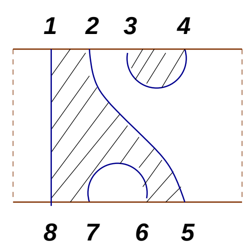
would be another such expression with smaller length. Thus, the proof of (a) is complete.

(b) We only need to show, in view of (a) above, that if $\mathcal{S}_n$ is the set of tuples $(i_1, j_1, \cdots, i_p, j_p)$ which satisfy condition (2), then $|\mathcal{S}_n|$ is at most the Catalan number $\frac{1}{n+1}\left(\begin{array}{c} 2n \\ n \end{array}\right)$. The proof is completed by observing that the assignment $(\mathbf{i}, \mathbf{j}) \mapsto \gamma_{(\mathbf{i},\mathbf{j})}$ sets up a bijection between $\mathcal{S}_n$ and the set denoted by $P_g((0,0),(n,n))$ in the proof of Proposition 2, where $\gamma_{(\mathbf{i},\mathbf{j})}$ is the path

$$(0,0) \to (i_1,0) \to (i_1,j_1) \to (i_2,j_1) \\ \to \cdots (i_p,j_p) \to (n,j_p) \to (n,n)$$

$\square$

A **Kauffman diagram** is an isotopy class of a planar (i.e., non-crossing) arrangement of n curves in a box with their ends tied to $2n$ marked points on the boundary, with n points on each horizontal bounding edge, and usually thought of as being numbered clockwise, starting from the top left corner; an example, with $n = 4$ is illustrated below:



The collection of such diagrams will be denoted by $\mathcal{K}_n$.

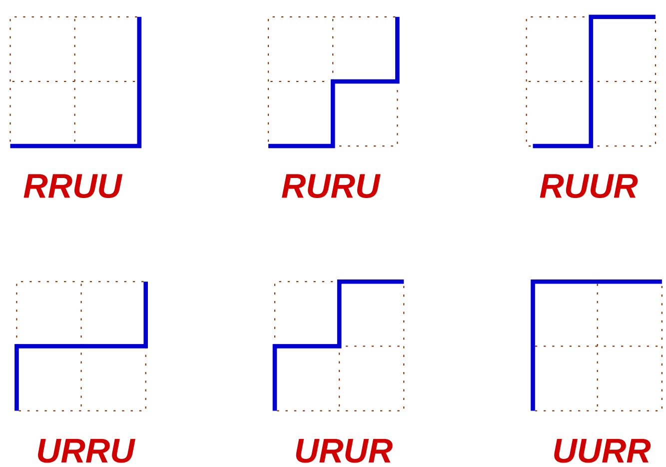
Proposition 2:

$$|\mathcal{K}_n| = \frac{1}{n+1}\binom{2n}{n}$$

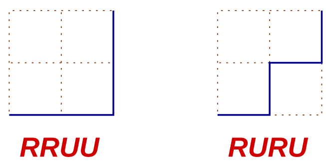
Proof: For $x, y \in \mathbb{R}^2$ such that $x_i \leq y_i$ for $i = 1, 2$, let $P(x, y)$ denote the collection of all 'walks' γ from x to y, in which each step is of unit length, and is to the right (R) or up (U). It is clear that

$$|P(x,y)| = \binom{y_1 - x_1 + y_2 - x_2}{y_1 - x_1} .$$

We will primarily be interested in $P((0,0),(n,n))$. For instance, we see that $P((0,0),(2,2))$ is as follows:



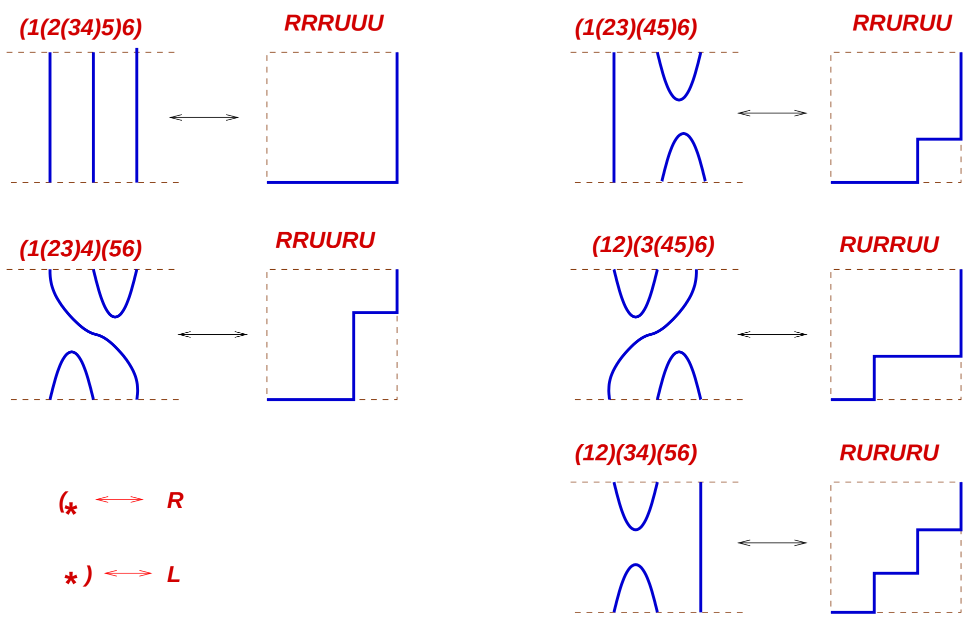
Let $P_g((0,0),(n,n))$ consist of those paths which do not cross the main diagonal (- i.e., every initial segment has at least as many R's as U's.) Thus, $P((0,0),(2,2))$ is as follows:



It is an easy exercise to verify that

$$|\mathcal{K}_n| = |P_g((0,0),(n,n))|.$$

The bijection is illustrated below, for $n = 3$:



We need to show that

$$|P_g((0,0),(n,n))| = \frac{1}{n+1}\binom{2n}{n}.$$

Note - by a shift - that $|P_g((0,0),(n,n))| = |P_g((0,1),(n+1,n))|$, and that the right side counts the ('good') paths in $P((1,0),(n+1,n))$ which do not meet the main diagonal. Consider the set $P_b((1,0),(n+1,n))$ of ('bad') paths which do cross the main diagonal. The point is that any path in $P_b((1,0),(n+1,n))$ is of the form $\gamma = \gamma_1 \circ \gamma_2 \in P_b((1,0),(n+1,n))$, where $\gamma_1 \in P((0,1),(j,j)), \gamma_2 \in P((j,j),(n+1,n))$, and (j,j) is the 'first point' where γ touches the main diagonal. Define $\widetilde{\gamma} = \gamma_1' \circ \gamma_2$, where γ_1' is the reflection of γ_1 about the main diagonal. This yields a bijection

$$P_b((1,0),(n+1,n)) \ni \gamma \leftrightarrow \widetilde{\gamma} \in P((0,1),(n+1,n))$$

Hence

$$
\begin{aligned}
&|P_g((1,0),(n+1,n))| \\
&\quad = \; |P((1,0),(n+1,n))| - |P_b((1,0),(n+1,n))| \\
&\quad = \; |P((1,0),(n+1,n))| - |P((0,1),(n+1,n))| \\
&\quad = \; \binom{2n}{n} - \binom{2n}{n+1} \\
&\quad = \; \frac{1}{n+1}\binom{2n}{n} \;,
\end{aligned}
$$

thereby completing the proof of Proposition 2.

$\square$

The diagram algebras $D_n(\beta)$

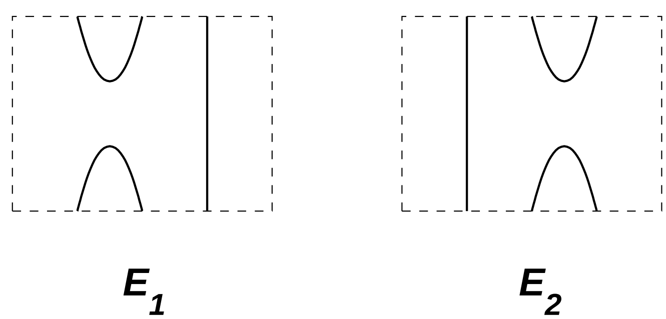
We now pass to another one-parameter family $D(\beta) = \{D_n(\beta)\}$ of towers of algebras, which depends on a complex parameter β, which we will usually take to be positive.

By definition, $D_n(\beta)$ is the $\mathbb{C}$-algebra with basis given by $\mathcal{K}_n$, and multiplication defined (on the basis) by the rule

$$ST = \beta^{\lambda(S,T)}U$$

where (1) U is the diagram obtained by concatenation - i.e., identifying the point marked $(2n - j + 1)$ for S with the point marked j for T, for $1 \leq j \leq n$ - and erasing any 'internal loops' formed in the process, and (2) $\lambda(S,T)$ is the number of 'internal loops' so erased. For example, if S and T are the elements of $\mathcal{K}_n$ represented by (1(23)4)(56) and (12)(3(45)6) respectively, then $U =$ (1(23)(45)6) and $\lambda(S,T) =$ 1 (so $ST = \beta U$), while $TS = \beta V$, where $V =$ (12)(34)(56).

Consider the elements $E_i, 1 \leq i < n$ of $\mathcal{K}_n$ defined by requiring that the pairs of vertices joined by the strings of E_i are $(i, i+1), (2n - i + 1, 2n - i)$ and $\{(j, 2n - j + 1) : j \neq i, 1 \leq j \leq n\}$; pictorially, this means that except for two strings, all strings come straight down, and the two exceptions join $(i, i+1)$ and $(2n - i + 1, 2n - i)$. We illustrate below the two E_i's, when $n = 3$:



It is not hard to see that

$$\begin{array}{rcll} E_i^2 & = & \beta E_i & \forall i \\ E_i E_j & = & E_j E_i & \text{if } |i-j| \geq 2 \\ E_i E_j E_i & = & E_i & \text{if } |i-j| = 1 \end{array}$$

It follows that if we define $e_i^D = \beta^{-1}E_i$, then the e_i^D's satisfy the Temperley-Lieb relations (with $\tau = \beta^{-2}$) , and consequently, there exists a unique unital algebra homomorphism

$$\phi^{(n)} : TL_n(\beta^{-2}) \to D_n(\beta)$$

such that $\phi^{(n)}(e_i^T) = e_i^D$ for $1 \leq i < n$, where we write $\{e_i^T : 1 \leq i < n\}$ for the generators of TL_n in order to distinguish them from the e_i^D's. We shall henceforth assume that τ and β are related by $\tau = \beta^{-2}$, and simply write TL_n and D_n without specifying the parameter, if no confusion is likely to result.

Proposition 3:

(a) D_n is generated, as a unital algebra, by $\{E_i : 1 \leq i < n\}$; and consequently,

(b) $\phi^{(n)} : TL_n(\beta^{-2}) \to D_n(\beta)$ is an isomorphism of unital algebras.

Proof: (a) This is proved by induction on n.

The proposition is clear when $n \leq 2$. For the inductive step, note that if an $S \in \mathcal{K}_n$ has a string 'coming straight down', it will follow from the induction hypotesis that S is in the unital algebra generated by the E_i's.

The general case is then reduced to two sub-cases, according as whether or not S has a *through-string*. Both sub-cases fall to the 'argument of the wiggle'.

(b) It follows from (a) that $\phi^{(n)}$ is surjective, and so

$$\dim D_n \leq \dim TL_n \; ; \tag{3}$$

On the one hand, we have (by Proposition 2)

$$\frac{1}{n+1}\binom{2n}{n} = |\mathcal{K}_n| = \dim D_n \tag{4}$$

while on the other, we also know (by Proposition 1(b)) that

$$\frac{1}{n+1}\binom{2n}{n} \geq \text{dim } TL_n \geq \text{dim } \phi^{(n)}(TL_n) \ . \tag{5}$$

Hence the three inequalities in (5) must all be inequalities, and $\phi^{(n)}$ must be an isomorphism.

$\square$

In view of the above proposition, we shall henceforth identify the Temperley-Lieb algebra with the diagram algebra.