von Neumann Algebras

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Lecture 1: Gelfand Naimark Theorems

Lecture 2: Rings of operators

Lecture 3: Crossed products and examples of factors

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Gelfand Naimark theorems

The 'G-N' theorems lead to the 'philosophy' of regarding C^*-algebras as *non-commutative analogues of topological spaces*.

- *(commutative G-N th)* A is a unital commutative C^*-algebra if and only if $A \cong C(X)$ (the algebra of continuous functions on a compact Hausdorff space).

- *(non-commutative G-N th)* A is a C^*-algebra if and only if A is isomorphic to a closed *-subalgebra of $\mathcal{L}(\mathcal{H})$ (the C^*-algebra of 'bounded operators' on Hilbert space).

A **Banach algebra** is a triple $(A, \|\cdot\|, \cdot)$, where:

- $(A, \|\cdot\|)$ is a Banach space

- $(A, \cdot)$ is a ring

- The map $A \ni x \mapsto L_x \in \mathcal{L}(A)$ defined by $L_x(y) = xy$ is a linear map and a ring-homomorphism satisfying

$$\|xy\| \leq \|x\|\|y\|$$

(or equivalently, $\|L_x\| \leq \|x\| \ \forall x \in A$).

A is **unital** if it has a multiplicative identity 1, usually assumed to satisfy $\|1\| = 1$. (We only consider such unital algebras here.)

Define $GL(A) = \{x \in A : x \text{ is invertible}\}$

Lemma: $\|x\| < 1 \Rightarrow$

- $1 - x \in GL(A)$
- $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$
- $\|(1 - x)^{-1} - 1\| \leq \|x\|(1 - \|x\|)^{-1}$

Corollary: $GL(A)$ is open, and $x \mapsto x^{-1}$ is a continous self-map of $GL(A)$.

Define the **spectrum** of an element $x \in A$ by

$$sp(x) = \{\lambda \in \mathbb{C} : x - \lambda \notin GL(A)\}$$

and its **spectral radius** by

$$r(x) = \mathrm{sup}\{|\lambda| : \lambda \in sp(x)\}$$

Theorem: The spectrum is always non-empty, and we have the **spectral radius formula**

$$r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} \ .$$

Caution: We must exercise some caution and talk about $sp_A(x)$, since if D is a unital Banach subalgebra of A and if $x \in D$, it may be the case that $sp_D(x) \neq sp_A(x)$. For example, by the maximum modulus principle, the *Disc algebra*

$$D = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is holomorphic}\}$$

imbeds isometrically as a Banach subalgebra of $A = C(\partial\mathbb{D})$, and

$$f \in D \Rightarrow sp_D(f) = f(\overline{\mathbb{D}}), sp_A(f) = f(\partial\mathbb{D}) \ .$$

But it turns out that there is no such pathology if our Banach algebras are C^*-algebras.

Assume henceforth that A is a unital **commutative Banach algebra**. Let $\mathcal{M}(A)$ denote the collection of **maximal ideals** in A. (*Conventions:* (a) $J \in \mathcal{M}(A) \Rightarrow \{0\} \neq J \neq A$, if $A \neq \mathbb{C}$, but (b) $\{0\} \in \mathcal{M}(\mathbb{C})$)

Lemma: Let $x \in A$. T.F.A.E.:

1. $x \notin GL(A)$

2. $\exists J \in \mathcal{M}(A)$ such that $x \in J$.

Proof: For (1) $\Rightarrow$ (2), note that $I = Ax$ is a proper ideal; pick $J \in \mathcal{M}(A)$ such that $I \subset J$.

Note that maximal ideals are closed (since 1 is in the exterior of any proper ideal). This implies:

Proposition: Write $\widehat{A}$ for the collection of unital homomorphisms $\phi : A \to \mathbb{C}$. Then

(a) $J \in \mathcal{M}(A) \Leftrightarrow \exists \phi \in \widehat{A}$ such that $J = ker\ \phi$.

(b) $\phi \in \widehat{A} \Rightarrow \phi(x) \in sp(x)$, and so, $|\phi(x)| \leq r(x) \leq \|x\|$, and $\widehat{A} \subset ball(A^*)$.

$\widehat{A}$ is closed and hence compact in the *weak-** topology of $ball(A^*)$.

Proposition: The **Gelfand transform** of A, which is the map $\Gamma : A \to C(\widehat{A})$ defined by

$$(\Gamma(x))(\phi) = \phi(x) \ \forall \phi \in \widehat{A}$$

is a contractive homomorphism of Banach algebras.

($\widehat{x} = \Gamma(x)$ is called the Gelfand transform of x.)

Question: When is Γ an isometric isomorphism onto $C(\widehat{A})$?

Answer: When A is a C^*-algebra!

A **C^*-algebra** is a Banach algebra A equipped with an involution - i.e., a self-map $a \ni x \mapsto x^* \in A$ satisfying

- $(\alpha x + y)^* = \bar{\alpha} x^* + y^*$

- $(xy)^* = y^* x^*$

- $(x^*)^* = x$

- which is related to the norm on A by the C^*-**identity** $\|x\|^2 = \|x^* x\|$.

The commutative G-N theorem

The Gelfand transform of a commutative Banach algebra A is an isometric surjection if and only if A has the structure of a commutative C^*-algebra.

In this case, Γ is automatically an isomorphism of C^*-algebras.

Sketch of Proof: Suppose A is a C^*-algebra and $x = x^*$ is 'self-adjoint'. For $t \in \mathbb{R}$, define $u_t = e^{itx} = \sum_{n=0}^{\infty} \frac{(itx)^n}{n!}$ and note that $u_t^* = u_{-t} = u_t^{-1}$. So, by the C^*-identity,

$$\|u_t\|^2 = \|u_t^* u_t\| = 1.$$

Hence

$$\phi \in \widehat{A} \Rightarrow 1 \geq |\phi(u_t)| = |e^{it\phi(x)}| \ .$$

Since $t \in \mathbb{R}$ is arbitrary, deduce that $\phi(x) \in \mathbb{R}$.

Also, for self-adjoint x, note that

$$\|x\| = \|x^* x\|^{\frac{1}{2}} = \|x^2\|^{\frac{1}{2}}$$

so

$$\|x\| = \|x^2\|^{\frac{1}{2}} = \cdots = \lim_{n \to \infty} \|x^{2^n}\|^{\frac{1}{2^n}} = r(x) = \|\Gamma(x)\|$$

For a (possibly non-commutative) unital C^*-algebra A, and $x \in A$, let $C^*(x)$ be the C^*-subalgebra of A generated by the set $\{1, x\}$.

Proposition: (a) T.F.A.E.:

(1) $C^*(x)$ is commutative

(2) $x^*x = xx^*$ (such x's are called **normal**).

(b) If x is normal, there exists a unique unital C^*-algebra isomorphism $\gamma_x : C(sp(x)) \to C^*(x)$ such that $\gamma_x(id_{sp(x)}) = x$.

It is customary to write $\gamma_x(f) = f(x)$ and call γ_x the *continuous functional calculus for x*.

Sub-classes of normal elements:

Proposition: An element satisfies the algebraic condition in the second column of the table below if and only if it is normal and its spectrum is contained in the set listed in the third column.

Name	Alg. def.	$sp(x) \subset ?$
self-adjoint	$x = x^*$	$\mathbb{R}$
unitary	$x^*x = xx^* = 1$	$\mathbb{T}$
projection	$x^2 = x^* = x$	$\{0, 1\}$

Study of general C^*-algebras is facilitated by applying the commutative theory to normal elements of these types. Normal elements can be dealt with the same facility as functions. Here is a sample of such results:

- (*Cartesian decomposition*) Every element $z \in A$ admits a unique deomposition $z = x + iy$, with x, y self-adjoint; in fact, $x = \frac{z+z^*}{2}, y = \frac{z-z^*}{2i}$

- Every self-adjoint element $x \in A$ admits a unique decomposition $x = x^+ - x^-$, where $x^\pm$ are positive (in the sense of the next theorem) and satisfy $x^+x^- = 0$; in fact, $x^\pm = f^\pm(x)$, where $f^\pm \in C(\mathbb{R})$ are defined by $f^\pm(t) = \frac{|t|\pm t}{2}$

The most important notion in the theory involves **positivity**. Its main features are listed in the next two results.

Theorem: (a) The following conditions on an element $x \in A$ are equivalent:

1. $x = x^*$ and $sp(x) \subset [0, \infty)$
2. $\exists y = y^* \in A$ such that $x = y^2$
3. $\exists z \in A$ such that $x = z^* z$

Such x's are said to be 'positive'; the set A_+ of positive elements of A is a 'positive cone' (proved using (1) above).

(b) If $x \in A_+$, then the y of (2) above may be chosen to be positive, and such a 'positive square root of x' is unique, and in fact $y = x^{\frac{1}{2}}$.

Proposition: (a) Let $\phi \in A^*$. T.F.A.E.:

1. $\phi(A_+) \subset \mathbb{R}_+$

2. $\|\phi\| = \phi(1)$

Such ϕ's are said to be positive (linear functionals); the set A^*_+ of of positive elements of A^* is a 'positive cone'.

(b) (*Cauchy-Schwarz inequality*)

$$|\phi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y) \quad \forall \phi \in A^*_+, x, y \in A \ .$$

Gelfand-Naimark-Segal (GNS) construction:

Theorem: T.F.A.E.:

1. $\phi \in A_+^*$

2. there exists a triple $(\mathcal{H}, \pi, \Omega)$ (essentially unique) of a Hilbert space $\mathcal{H}$, a representation π of A on $\mathcal{H}$ (i.e., $\pi : A \to \mathcal{L}(\mathcal{H})$ is a homomorphism of C^*-algebras), and a vector $\Omega \in \mathcal{H}$ such that

   - $\phi(x) = \langle \pi(x)\Omega, \Omega \rangle \ \forall x \in A$

   - Ω is a cyclic vector in the sense that $\mathcal{H} = \{\pi(x)\Omega : x \in A\}^-$

(It is not uncommon to write $\mathcal{H} = L^2(A, \phi)$.)

Sketch of proof: The equation

$$\langle x, y \rangle_\phi = \phi(y^* x)$$

defines a semi-inner product on A (i.e., satisfies all requirements of an inner product except possibly positive - definiteness). Let $\mathcal{N}_\phi = \{x \in A : \|x\|_\phi^2 = \langle x, x \rangle_\phi = 0\}$. The fact that ϕ satisfies Cauchy-Schwarz inequality implies that $\mathcal{N}_\phi$ is a left-ideal in A (i.e., a subspace which is closed under left multiplication by any element of A).

Then $A/\mathcal{N}_\phi$ is a genuine inner product space, whose completion is the desired $\mathcal{H}_\phi$, while the equation

$$\pi_0(x)(y + \mathcal{N}_\phi) = xy + \mathcal{N}_\phi$$

happens to define a bounded operator $\pi_0(x)$ on $A/\mathcal{N}_\phi$; define $\pi_\phi(x)$ to be its unique continuous extension to $\mathcal{H}_\phi$.

Lemma: If $x = x^* \in A$, there exists $\phi \in A^*_+$ such that $|\phi(x)| = \|x\|$

Proof: Let $A_0 = C^*(\{x\})$. Then pick $\phi_0 \in \widehat{A_0} \subset A_0^*$ such that $|\phi_0(x)| = \|x\|$.

Use Hahn-Banach thm. to find $\phi \in A^*$ such that $\phi|_{A_0} = \phi_0$ and $\|\phi\| = \|\phi_0\| (= \phi_0(1) = \phi(1))$. It follows that $\phi \in A^*_+$.

Lemma: Any (unital) homomorphism of C^*-algebras is norm-decreasing.

Proof: If $\pi : A \to B$ is a (unital) homomorphism of C^*-algebras, then clearly $\pi(GL(A) \subset GL(B)$; in particular, if $x = x^* \in A$, we see that $(\pi(x)$ is also self-adjoint, and $sp(\pi(x)) \subset sp(x)$, so)

$$\|\pi(x)\| = r(\pi(x)) \leq r(x) = \|x\| \ ;$$

and for general $z \in A$,

$$\|\pi(z)\| = \|\pi(z)^*\pi(z)\|^{\frac{1}{[2}} \leq \|z^*z\|^{\frac{1}{[2}} \leq \|z\| \ .$$

$\square$

If we write π_x for the above GNS representation of A on, say, $\mathcal{H}_x$, then $\oplus_{\{x=x^*\in A\}}\pi_x$ is easily verified to be an isometric representation of A.

Rings of Operators (a.k.a. von Neumann algebras):

Introduced in - and referred to, by them, as - *Rings of Operators* in 1936 by F.J. Murray and von Neumann, because - in their own words:

the elucidation of this subject is strongly suggested by

- *our attempts to generalise the theory of unitary group-representations, and*

- *various aspects of the quantum mechanical formalism*

Def 1: A vNa is the commutant of a unitary group representation: i.e.,

$$M = \{x \in \mathcal{L}(\mathcal{H}) : x\pi(g) = \pi(g)x \ \forall g \in G\}$$

Note that $\mathcal{L}(\mathcal{H})$ is a Banach *-algebra w.r.t. $\|x\| = \sup\{\|x\xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}$ ('operator norm') and 'Hilbert space adjoint'.

Defs: (a) $S' = \{x' \in \mathcal{L}(\mathcal{H}) : xx' = x'x \ \forall x \in S\}$, for $S \subset \mathcal{L}(\mathcal{H})$

(b) SOT on $\mathcal{L}(\mathcal{H})$: $x_n \to x \Leftrightarrow \|x_n\xi - x\xi\| \to 0 \ \forall \xi$ (i.e., $x_n\xi \to x\xi$ strongly $\forall \xi$)

(c) WOT on $\mathcal{L}(\mathcal{H})$: $x_n \to x \Leftrightarrow \langle x_n\xi - x\xi, \eta\rangle \to 0 \forall \xi, \eta$ (i.e., $x_n\xi \to x\xi$ weakly $\forall \xi$)

(Our Hilbert spaces are always assumed to be **separable**.)

von Neumann's double commutant theorem: Let M be a unital self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$. TFAE:

(i) M is SOT-closed

(ii) M is WOT-closed

(iii) $M = M'' = (M')'$ $\square$

Def 2: A vNa is an M as in DCT above.

The equivalence of definitions 1 and 2 is a consequence of the spectral theorem and the fact that any norm-closed unital *-subalgebra A of $\mathcal{L}(\mathcal{H})$ is linearly spanned by the set $\mathcal{U}(A) = \{u \in A : u^*u = uu^* = 1\}$ of its **unitary** elements.

Some consequences of DCT:

(a) A von Neumann algebra is closed under all 'canonical constructions':

for instance, if $x \to \{1_E(x) : E \in \mathcal{B}_{\mathbb{C}}\}$ is the spectral measure associated with a normal operator x, then $x \in M \Leftrightarrow 1_E(x) \in M \ \forall \ E \in \mathcal{B}_{\mathbb{C}}$.

(*Reason:* $1_E(uxu^*) = u1_E(x)u^*$ for all unitary u; so implication $\Rightarrow$ follows from

$$\begin{aligned} x \in M, u' \in \mathcal{U}(M') &\Rightarrow u'1_E(x)u'^* = 1_E(u'xu'^*) \\ &\Rightarrow 1_E(x) \in \Big(\mathcal{U}(M')\Big)' = M \quad) \end{aligned}$$

(b) For implication $\Leftarrow$, uniform approximability of bounded measurable functions by simple functions implies (by the spectral theorem) that

$$M = [\mathcal{P}(M)] = (span\ \mathcal{P}(M))^- \quad (*),$$

where $\mathcal{P}(M) = \{p \in M : p = p^2 = p^*\}$ is the set of projections in M.

Suppose $M = \pi(G)'$ as before. Then

$$p \leftrightarrow ran\ p$$

establishes a bijection

$$\mathcal{P}(M) \leftrightarrow G\text{-stable subspaces}$$

So, for instance, eqn. (*) shows that

$$(\pi(G))'' = \mathcal{L}(\mathcal{H}) \Leftrightarrow M = \mathbb{C} \ \Leftrightarrow \ \pi \text{ is irreducible}$$

Under this correspondence, between sub-reps of π and $\mathcal{P}(M)$, (unitary) equivalence of sub-reps of π translates to *Murray-von Neumann equivalence* on $\mathcal{P}(M)$:

$$p \sim_M q \Leftrightarrow \exists u \in M \text{ such that } u^*u = p,\ uu^* = q$$

More generally, define

$$p \preceq_M q \Leftrightarrow \exists p_0 \in \mathcal{P}(M) \text{ such that } p \sim_M p_0 \leq q$$

Proposition: TFAE:

1. If $p, q \in \mathcal{P}(M)$, either $p \preceq_M q$ or $q \preceq_M p$.

2. M has trivial center: $Z(M) = M \cap M' = \mathbb{C}$

Such an M is called a **factor**. $\square$

$M = \pi(G)'$, G finite, is a factor iff π is isotypical.

In general, any vNa is a 'direct integral' of factors.

Say a projection $p \in \mathcal{P}(M)$ is **infinite rel M** if $\exists p_0 \in \mathcal{P}(M)$ such that $p \sim_M p_0 \lneq p$; otherwise, call p **finite** (rel M).

Say M is finite if 1 is finite.

Murray von-Neumann classification of factors: A factor M is said to be of type:

1. I if there is a minimal non-zero projection in M.

2. II if it contains non-zero finite projections, but no minimal non-zero projection.

3. III if it contains no non-zero finite projection.

Def. 3: (Abstract Hilbert-space-free def) M is a vNa if

- M is a C^*-algebra (i.e., a Banach *-algebra satisfying $\|x * x\| = \|x\|^2 \ \forall\ x$)

- M is a dual Banach space: i.e., $\exists$ a Banach space M_* such that $M \cong M_*^*$ as a Banach space.

Example: $M = L^\infty(\Omega, \mathcal{B}, \mu)$. Can also view it as acting on $L^2(\Omega, \mathcal{B}, \mu)$ as multiplication operators. (In fact, every commutative vNa is isomorphic to an $L^\infty(\Omega, \mathcal{B}, \mu)$.)

Fact: The predual M_* of M is unique up to isometric isomorphism. (So, (by Alaoglu), $\exists$ a canonical loc. cvx. (weak-*) top. on M w.r.t. which the unit ball of M is compact. This is called the **σ-weak topology** on M.

A linear map between vNa's is called **normal** if it is continuous w.r.t. the σ-weak topologies on domain and range.

The morphisms in the category of vNa's are unital normal $*$-homomorphisms.

The algebra $\mathcal{L}(\mathcal{H})$, for any Hilbert space $\mathcal{H}$, is a vNa - with pre-dual being the space $\mathcal{L}_*(\mathcal{H})$ of trace-class operators.

Any σ-weakly closed $*$-subalgebra of a vNa is a vNa.

Gelfand-Naimark theorem: Any vNa is isomorphic to a vN-subalgebra of some $\mathcal{L}(\mathcal{H})$. (So the abstract and concrete ($=$ tied down to Hilbert space) definitions are equivalent.)

In some sense, the most interesting factors are the so-called *type II_1 factors* (= finite type II factors).

Theorem: Let M be a factor. TFAE:

1. M is finite.

2. $\exists$ a **trace** tr_M on M - i.e., linear functional satisfying:

   - $tr_M(xy) = tr_M(yx)$ $\forall x, y \in M$ (*trace*)

   - $tr_M(x^*x) \geq 0 \forall x \in M$ (*positive*)

   - $tr_M(1) = 1$ (*normaliised*)

Such a trace is automatically unique, and *faithful* - i.e., it satisfies $tr_M(x^*x) = 0 \Leftrightarrow x = 0$

For $p, q \in \mathcal{P}(M)$, M a finite factor, TFAE:

1. $p \sim_M q$

2. $tr_M p = tr_M q$

3. $\exists u \in \mathcal{U}(M)$ such that $upu^* = q$.

If $dim_{\mathbb{C}} M < \infty$, then $M \cong M_n(\mathbb{C}) = \mathcal{L}(\mathbb{C}^n)$ for a unique n.

If $dim_{\mathbb{C}} M = \infty$, then M is a II_1 factor, and in this case, $\{tr_M p : p \in \mathcal{P}(M)\} = [0, 1]$.

So II_1 factors are the arena for continuously varying dimensions; they got von Neumann looking at *continuous geometries*.

Henceforth, M will be a II_1 factor.

Def: An M-module is a separable Hilbert space $\mathcal{H}$, equipped with a vNa morphism $\pi : M \to \mathcal{L}(\mathcal{H})$. Two M-modules are isomorphic if there exists an invertible (equivalently, unitary) M-linear map between them.

Proposition: $\exists$ a complete isomorphism invariant

$$\mathcal{H} \mapsto dim_M \mathcal{H} \in [0, \infty]$$

of M-modules such that:

1. $\mathcal{H} \cong \mathcal{K} \Leftrightarrow dim_M \mathcal{H} = dim_M \mathcal{K}$.

2, $dim_M(\oplus_n \mathcal{H}_n) = \sum_n dim_M \mathcal{H}_n$.

3. For each $d \in [0, \infty]$, $\exists$ an M-module $\mathcal{H}_d$ with $dim_M \mathcal{H}_d = d$.

The equation

$$\langle x, y\rangle = tr_M(y^* x)$$

defines an inner-product on M. Call the completion $L^2(M, tr_M)$. Then $L^2(M, tr_M)$ is an $M - M$ bimodule with left- and right- actions given by multiplication.

$\mathcal{H}_1 = L^2(M, tr_M)$.

If $0 \leq d \leq 1$, then $\mathcal{H}_d = L^2(M, tr_M).p$ where $p \in \mathcal{P}(M)$ satisfies $tr_M p = d$.

$\mathcal{H}_d$ is a finitely generated projective module if $d < \infty$.

In particular $K_0(M) \cong \mathbb{R}$.

The hyperfinite II_1 factor R: Among II_1 factors, pride of place goes to the ubiquitous hyperfinite II_1 factor R. It is characterised as the unique II_1 factor which has any one of several properties, such as injectivity and approximate finite-dimensionality (= hyperfiniteness).

Thus, $\exists$ a unique II_1 factor R which contains an increasing sequence of finite-dimensional $*$-subalgebras

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$$

such that $\cup_n A_n$ is σ-weakly dense in R.

Examples of II_1 factors: Let $\lambda : G \to \mathcal{U}(\mathcal{L}(\ell^2(G)))$ denote the 'left-regular representation' of a countable infinite group G, and let $LG = (\lambda(G))''$. Then LG is a II_1 factor iff every conjugacy class of G other than $\{1\}$ is infinite.

$L\Sigma_\infty \cong R$, while $L\mathbb{F}_2$ is not hyperfinite.

Big open problem: is $L\mathbb{F}_3 \cong L\mathbb{F}_2$?

The study of bimodules over II_1 factors is essentially equivalent to that of 'subfactors'.

$$({}_N\mathcal{H}_M \leftrightarrow \pi_l(N) \subset \pi_r(M)'.)$$

A **subfactor** is a unital inclusion $N \subset M$ of II_1 factors. For a subfactor as above, Jones defined the *index of the subfactor* to be

$$[M : N] = dim_N L^2(M, tr_M)$$

and proved:

$$[M : N] \in [4, \infty] \cup \{4cos^2(\frac{\pi}{n} : n \geq 3\}$$

A subfactor $N \subset M$ satisfies $N' \cap M = \mathbb{C}$ iff $L^2(M, tr_M)$ is irreducible as an $N-M$ bimodule. Such a subfactor is called **irreducible**.

It is known that if a subfactor $N \subset M$ has finite index, then N is hyperfinite if and only if M is. In this case, call the subfactor hyperfinite.

Very little is known about the set $\mathcal{I}_R^0$ of possible index values of irreducible hyperfinite subfactors.

Some known facts:

(a) (Jones) $\mathcal{I}_R = [4, \infty] \cup \{4cos^2(\frac{\pi}{n}) : n \geq 3\}$ and $\mathcal{I}_R^0 \supset \{4cos^2(\frac{\pi}{n} : n \geq 3\}$

(b) $\left(\frac{N+\sqrt{N^2+4}}{2}\right)^2, \left(\frac{N+\sqrt{N^2+8}}{2}\right)^2 \in \mathcal{I}_R^0 \ \forall N \geq 1$

(c) $(N + \frac{1}{N})^2$ is the limit of an increasing sequence in $\mathcal{I}_R^0$.

Open problems:

(a) Is $\mathcal{I}_R^0$ countable?

(b) Does there exist $\epsilon > 0$ such that

$$\mathcal{I}_R^0 \cap (4, 4+\epsilon) = \emptyset?$$

Crossed products and examples of factors:

The **left-regular representation** of a countable group G is the association

$$G \ni t \mapsto \lambda_t \in \mathcal{U}(\mathcal{L}(\ell^2(G)))$$

given by

$$(\lambda_t \xi)(s) = \xi(t^{-1}s) \ .$$

Here, of course, $\ell^2(G)$ denotes the Hilbert space of square-summable functions $\xi : G \to \mathbb{C}$. If we write

$$1_s(t) = \delta_{s,t} \ ,$$

then clearly $\{1_s : s \in G\}$ is an o.n.b. for $\ell^2(G)$.

We shall identify operators on $\ell^2(G)$ with their matrices w.r.t. this o.n.b.; thus

$$\mathcal{L}(\ell^2(G)) \ni x \leftrightarrow ((x(s,t))) \ ,$$

where $x(s,t) = \langle x1_t, 1_s \rangle$; for example,

$$\lambda_u(s,t) = \langle \lambda_u 1_t, 1_s \rangle = \langle 1_{ut}, 1_s \rangle = \delta_{s,ut} \ .$$

Suppose G acts on a von Neumann algebra $M \subset \mathcal{L}(\mathcal{H})$; i.e., assume:
(i) α_t is a *-automorphism of M for each $t \in G$, and
(ii) $G \ni t \mapsto \alpha_t \in Aut(M)$ is a group homomorphism.

Then the **crossed-product construction** is (analogous to the 'semi-direct product construction' for groups and) results in a von Neumann algebra $\tilde{M} \subset \mathcal{L}(\tilde{\mathcal{H}})$, a normal representation $\pi : M \to \mathcal{L}(\tilde{\mathcal{H}})$, and a unitary group representation $\lambda : G \to \mathcal{U}(\mathcal{L}(\tilde{\mathcal{H}}))$, such that:

(a) $\tilde{M} = (\pi(M) \cup \lambda(G))''$; and

(n) $\lambda(u)\pi(x) = \pi(\alpha_u(x))\lambda(u) \ \forall \ u \in G, x \in M.$

It turns out that the isomorphism type of $\tilde{M}$ is independent of the choices of $\tilde{\mathcal{H}}, \pi$ and λ.

The model of the crossed-product we shall use is as follows:

$$\tilde{\mathcal{H}} = \ell^2(G;\mathcal{H}) \cong \ell^2(G) \otimes \mathcal{H} \cong \oplus_{t\in G}\mathcal{H} \ ,$$

where

$$\tilde{\xi} \leftrightarrow \sum_t \tilde{1}_t \otimes \tilde{\xi}(t) \leftrightarrow ((\tilde{\xi}(t)))$$

As before, we identify operators $\tilde{x} \in \mathcal{L}(\tilde{\mathcal{H}})$ with their 'operator-matices $((\tilde{x}(s,t)))$ - defined by the requirement that

$$\langle \tilde{x}(s,t)\xi, \eta\rangle = \langle \tilde{x}(1_t \otimes \xi), (1_s \otimes \eta)\rangle \ ,$$

or equivalently,

$$(\tilde{x}\tilde{\xi})(s) = \sum_t \tilde{x}(s,t)\tilde{\xi}(t) \ .$$

Define

$$\begin{aligned} (\pi(x)\tilde{\xi})(s) &= (\alpha_{s^{-1}}(x)\tilde{\xi}(s)) \\ (\lambda(u)\tilde{\xi})(s) &= \tilde{\xi}(u^{-1}s) \end{aligned}$$

In matricial terms, we see that

$$\begin{aligned}(\pi(x))(s,t) &= \delta_{s,t}\alpha_{t^{-1}}(x)\\ (\lambda(u))(s,t) &= \delta_{s,ut} id_{\mathcal{H}}\end{aligned}$$

In fact, we find that

$$\tilde{M} = \{\mathcal{L}(\tilde{\mathcal{H}}) \ni \tilde{x} : \exists x(s) \in M, s \in G \text{ such that } \tilde{x}(s,t) = \alpha_{t^{-1}}(x(st^{-1})) \ \forall s,t \in G\} \ .$$

Remark: It is customary to denote the crossed product by $M \rtimes_\alpha G$. (If α is the trivial action on $\mathbb{C}$ - with $\alpha_t = id_{\mathbb{C}}$ $\forall t$ - then

$$\mathbb{C} \rtimes_\alpha G \cong L(G) = (\lambda(G))''.)$$

Aside on abelian vNa's: Any abelian vNa is isomorphic to some $A = L^\infty(\Omega, \mathcal{B}, \mu)$, with μ a probability measure.

Any automorphism $\theta \in Aut(A)$ is of the form $\theta(f) = f \circ T$ for some non-singular automorphism of $(\Omega, \mathcal{B}, \mu)$ (meaning a bi-measurable bijection $T : \Omega \to \Omega$ such that $\mu \circ T^{-1}$ and μ have the same null sets). Further, TFAE:

1. $\mu(\{\omega \in \Omega : T\omega = \omega\}) = 0$

2. $\theta = \theta_T$ is 'free' in the sense of the next definition.

Def.: (a) An automorphism $\theta \in Aut(M)$ is said to be **free** if, for $x \in M$,

$$xy = \theta(y)x \ \forall y \in M \Leftrightarrow x = 0 \ .$$

(b) An action $\alpha : G \to M$ is said to be **free** if α_t is free for all $t \neq 1$.

(c) An action $\alpha : G \to M$ is said to be **ergodic** if

$$M^\alpha := \{x \in M : \alpha_t(x) = x \ \forall t \in G\} == \mathbb{C}.$$

Note: If $M = A$ is abelian as before, the action is given by

$$\alpha_t(f) = f \circ T_t^{-1}$$

for an action $t \to T_t$ of G as non-singular automorphisms of $(\Omega, \mathcal{B}, \mu)$; and the action α is ergodic in the above sense iff the action $t \mapsto T_t$ is ergodic in the classical sense, meaning

$$E \in \mathcal{B}, \ \mu(E \Delta T_t E) = 0 \ \forall t \in G \Rightarrow \mu(E) \cdot \mu(\Omega \setminus E) = 0.$$

Proposition: TFAE:

(i) $\pi(M)' \cap \tilde{M} \subset \pi(Z(M))$

(ii) The action α is free. $\square$

Proposition: Suppose the action is free. Then, TFAE:

(i) $\tilde{M}$ is a factor.

(ii) The restricted action $\alpha_Z : G \to Aut(Z(M)$ is ergodic.

Corollary: If $G \ni t \mapsto T_t$ is a free ergodic action of G as non-singular automorphisms of $(\Omega, \mathcal{B}, \mu)$, then $M = L^\infty(\Omega, \mathcal{B}, \mu) \rtimes_\alpha G$ is a factor - where, of course, $\alpha_t(f) = f \circ T_t^{-1}$.

The type of this factor is described below.

Theorem:(MvN)

Let A, G, T_t, M be as in the previous corollary. Then:

(a) M is of type III iff there does not exist a σ-finite measure ν which has the same null sets as μ and is left invariant by each T_t.

(b) Suppose M is not type III, and that ν is a G-invarian measure which is mutually absolutely continuous with μ. Then,

(i) M is of type I iff ν has atoms (or, is equivalently, purely atomic).

(ii) M is of type II iff ν is non-atomic.

(iii) M is a finite factor iff ν is a finite measure.

□

(I_n) If $G = \mathbb{Z}_n$ acts transitively on $\Omega = \{1, 2, \cdots, n\}$ and if μ denotes counting measure on Ω, then $M \cong M_n(\mathbb{C})$.

(I_∞) If $G = \mathbb{Z}$ acts transitively on $\Omega = \{1, 2, \cdots, n\}$ and if μ denotes counting measure on Ω, then $M \cong \mathcal{L}(\ell^2(\mathbb{Z})))$.

(II) If a countable dense subgroup G of a locally compact group Ω acts by translation on Ω, and if μ denotes Haar measure on Ω, then M is a type II factor, which is of type II_1 iff Ω is compact.

(III) The $ax + b$ group G acts naturally on $\mathbb{R}$ in an ergodic and free manner. Further $G_0 = \{g \in G : g$ preserves Lebesgue measure$\}$ is a proper subgroup (corresponding to $a = 1$) which also acts ergodically on $\mathbb{R}$. It follows that no measure mutually absolutely continous with Lebesgue measure is left invariant by all of G. So M is of type III in this case.

Fact: If M is a factor, and $\theta \in Aut(M)$, TFAE:

(i) θ is free.

(ii) θ is 'outer': i.e., there does not exist $u \in \mathcal{U}(M)$ such that $\theta = Ad_u$ - i.e., $\theta(x) = uxu^*$ for all $x \in M$

Corollary: If $\alpha : G \to Aut(M)$ is an **outer action** - i.e., if α_t is outer for each $t \neq 1$ - then $M \rtimes_\alpha G$ is a factor. If M is a II_1 factor and G is a finite group, then $M \rtimes_\alpha G$ is also a II_1 factor.

Facts: (i) If $G = U_n(\mathbb{C})$, then G admits an outer action on the hyperfinite II_1 factor R;

(ii) In particular, any finite group admits an outer action on R.

Theorem:

Let G, H be finite groups. Then TFAE:

(i) There is an 'isomorphism of hyperfinite subfactors'

$$(R \subset R \rtimes_\alpha G) \cong (R \subset R \rtimes_\beta H)$$

(ii) There is an isomorphism of groups

$$G \cong H.$$

Subfactors:

The standard module: Assume M is a 'finite' vNa, with faithful trace tr_M; then $L^2(M, tr_M)$ has a distinguished dense subspace $\mathcal{D} = \{\hat{x} : x \in M\}$ such that

$$\langle \hat{x}, \hat{y} \rangle = tr_M(y^* x) \ \forall x, y \in M$$

and that $L^2(M, tr_M)$ is an $M - M$ bimodule, with

$$a \cdot \hat{x} \cdot b = \widehat{axb} \ .$$

(The reason for the hats is that we shall identify an $a \in M$ with the unique operator $a \in \mathcal{L}(L^2(M, tr_M))$ such that

$$a\hat{x} = \widehat{ax} \ \forall \hat{x} \in \mathcal{D}$$

and we will need to distinguish between the operator a and the vector $\hat{a}$. Thus we view M as contained in $\mathcal{L}(L^2(M, tr_M)$.)

Since tr_M is a trace, it follows that the mapping $\widehat{x} \mapsto \widehat{x^*}$ defines a conjugate-linear norm preserving self-map of $\mathcal{D}$ which is its own inverse, and consequently extends uniquely to an anti-unitaty involution of $L^2(M, tr_M)$), which is usually denoted by J (or J_M, if it is necessary to draw attention to the dependence on M) and called **the modular conjugation of M**.

The definitions imply that $Ja*J\widehat{x} = \widehat{xa}$, so that

$$a \cdot \widehat{x} \cdot b = aJb^*J\widehat{x} \ ,$$

thus establishing the easy half of part (a) of:

Proposition: (baby version of the celebrated **Tomita-Takesaki theorem**)

(a) $JMJ(= \{JaJ : a \in M\}) = M'$; and

(b) $\mathcal{D}$ is precisely the collection of **bounded vectors**, meaning that a $\xi \in^2 (M, tr_M)$ belongs to $\mathcal{D}$ iff $\exists K > 0$ such that $\|a\xi\| \leq K\|\widehat{a}\| \ \forall \ a \in M$.

Suppose now that $N \subset M$ is a vN subalgebra. Notice then that there is an (isometric) identification of $L^2(N, tr_N)$ as a subspace $L^2(M, tr_M)$ (where we write $tr_M|_N = tr_N$). Let e_N denote the orthogonal projection of $L^2(M, tr_M)$ onto $L^2(N, tr_N)$.

If we express operators on $L^2(M, tr_M)$ as 2×2 operator-matrices w.r.t. the decomposition $L^2(M, tr_M) = L^2(N, tr_N) \oplus ker e_N$, we see that

$$e_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, J_M = \begin{bmatrix} J_N & 0 \\ 0 & J_1 \end{bmatrix}, M \ni a = ((a_{ij})) ,$$

where J_1 is some antiunitary operator on $ker\ e_N$ and $a_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$, with $\mathcal{H}_1 = ran\ e_N$ and $\mathcal{H}_2 = ker\ e_N$.

To keep track of various identifications, it will help if we write $\pi_l^M(a)$ to denote the operator of left multiplication by a on $L^2(M, tr_M)$. Then if $a \in N$, note that $\pi_l^M(a)(\mathcal{H}_1) \subset \mathcal{H}_1$, and we find that in this case, we must have $a_{12} = a_{21} = 0$ and $a_{11} = \pi_l^N(a)$.

More generally, for any $a \in M$, the fact that $J_M \pi_l^M(M) J_M \in \pi_l^M(M)'$ implies that

$$\begin{bmatrix} J_N a_{11} J_N & J_N a_{12} J_1 \\ J_1 a_{21} J_N & J_1 a_{22} J_1 \end{bmatrix} \in \pi_l^M(M)' \subset \pi_l^M(N)' ,$$

and in particular, $J_N a_{11} J_N \in \pi_l^N(N)' = J_N \pi_l^N(N) J_N$: i.e., $a_{11} \in \pi_l^N(N)$.

So we have a well-defined map $E_N : M \to N$ such that

$$a \in M \Rightarrow a_{11} = \pi_l^N(E_N(a)) .$$

Proposition: The (linear) maps E_N (resp., e_N) satisfy, for arbitrary $x \in M, a, b \in N$:

(i) $\widehat{E_N x} = e_N \widehat{x}$;

(ii) $E_N x = x \Rightarrow x \in N \Rightarrow x e_N = e_N x$ (projection);

(iii) $E_N(axb) = a(E_N x)b$ (N-bilinear)

(iv) $E_N(x^*x) > 0 \Rightarrow x \neq 0$ (faithful & positive)

(v) $tr_N \circ E_N = tr_M$ (trace-preserving) $\square$

The map E_N is called the trace-preserving **conditional expectation** of M onto N - because, if $M = L^\infty(\Omega, \mathcal{B}, \mu)$, any vN subalgebra is of the form $N = L^\infty(\Omega, \mathcal{B}_0, \mu)$ for some sub-σ-algebra $\mathcal{B}_0$, and E_M agrees with the conditional expectation familiar from classical probability theory.

The first step in the analysis of subfactors is the so-called **basic construction** due to Jones. (This notion makes sense in greater generality than the case we state, but we shall only need this.)

Proposition: Suppose $N \subset M$ is a subfactor. Define $M_1 = J_M N' J_M$. Then

(a) M_1 is also a factor and $M \subset M_1$.

(b) M_1 is a II_1 factor iff $[M : N] < \infty$; in this case, $[M_1 : M] = [M : N]$ and $dim_{\mathbb{C}}(N' \cap M) < \infty$; and hence

$$[M : N] \leq 4 \Rightarrow \; N' \cap M = \mathbb{C} \; .$$

(c) $M_1 = (M \cup \{e_N\})''$. $\square$

We abbreviate the content of (c) above and say that

$$N \subset M \subset^{e_N} M_1$$

is the basic construction. Thus, applied to a subfactor $(N =)M_{-1} \subset (M =)M_0$ of finite index, say d, the basic construction yields another subfactor $M_0 \subset M_1 = \langle M, e_1 (= e_N)\rangle$ also of index d.

The tower of the basic construction:

And, as Jones says, we should 'push a good thing along', and inductively construct :

$N = M_{-1} \subset M = M_0 \subset^{e_1} M_1 \subset^{e_2} M_2 \cdots \subset^{e_n} M_n \cdots$

We then find that:

(a) Each M_n is a II_1 factor.

(b) e_n implements the CE of M_{n-1} onto M_{n-2}, meaning

$$e_n x_{n-1} e_n = E_{M_{n-2}}(x_{n-1}) e_n$$

(c) $[M_{k+l} : M_l] = \lambda^k$, and $M_l \subset M_{k+l} \subset M_{k+2l}$ is an instance of the basic construction.

So, to the subfactor $N \subset M$ is canonically associated the grid $((A_{ij} = M_i' \cap M_j))$ of finite-dimensional C^*-algebras - with $A_{ij} \subset A_{kl}$ whenever $-1 \leq k \leq i \leq j \leq l$ - which comes equipped with a consistent 'trace tr' - which agrees on A_{ij} with tr_{M_j}.

Owing to a certain periodicity of order 2 - $A_{ij} \cong A_{i+2,j+2}$ - it turns out that the entire data of this grid is already contained in the first two rows (for $i = 0, 1$, namely the grid:

$$(\dagger) \quad \begin{array}{ccccccc} N' \cap N & \subset & N' \cap M & \cdots & N' \cap M_N & \cdots \\ & & \cup & & \cup & \cdots \\ & & M' \cap M & \cdots & M' \cap M_N & \cdots \end{array}$$

This grid, equipped with the trace tr (cf. the first para above), is called the **standard invariant** of the subfactor $N \subset M$.

Write $\pi(A)$ for the set of minimal (non-zero) projections in the centre $Z(A)$ of a finite dimensional C^*-algebra A. The Wedderburn-Artin theorem then guarantees the existence of a function $d : \pi(A) \to \mathbb{N}$ such that

$$A \cong \oplus_{p \in \pi(A)} M_{d(p)}(\mathbb{C})$$

Further, $\pi(A)$ parametrises the set of inequivalent irreducible representations of A thus : if $a \leftrightarrow \oplus_p a_p$ under the above isomorphism, then $\pi_p(a) = a_p$. If $\phi : A \to B$ is a (unital) inclusion of finite-dimensional C^*-algebras, define the associated non-negative integer-valued 'inclusion matrix' Λ with rows and columns indexed by $\pi(A)$ and $\pi(B)$ respectively thus: if p_0 is a minimal projection of A such that $p_0 \leq p$, then $\Lambda(p,q) = tr_{M_{d(q)}(\mathbb{C})} \pi_q(\phi(p_0))$ (= 'the no. of times that $\pi_q|_A$ contains π_p').

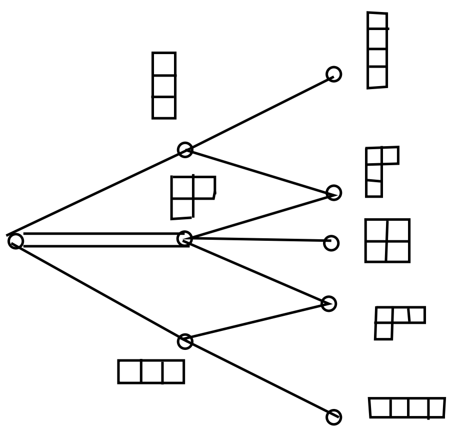
It is a fact that two inclusions $A_i \subset B_i, i = 1,2$ are isomorphic iff they have the same inclusion matrix.

Non-negative integer matrices may be viewed as adjacency matrices of bipartite graphs; so inclusions may be described by bipartite graphs.

Successive inclusion 'graphs' can be glued together into the **Bratteli diagram** of a tower. Thus the representation theory of the Σ_n's shows that the Bratteli diagram for

$$\mathbb{C} \subset \mathbb{C}\Sigma_3 \subset \mathbb{C}\Sigma_4$$

is:



The standard invariant may be viewed as having three ingredients:

(i) the tower $\{N' \cap M_n : n \geq -1\}$;

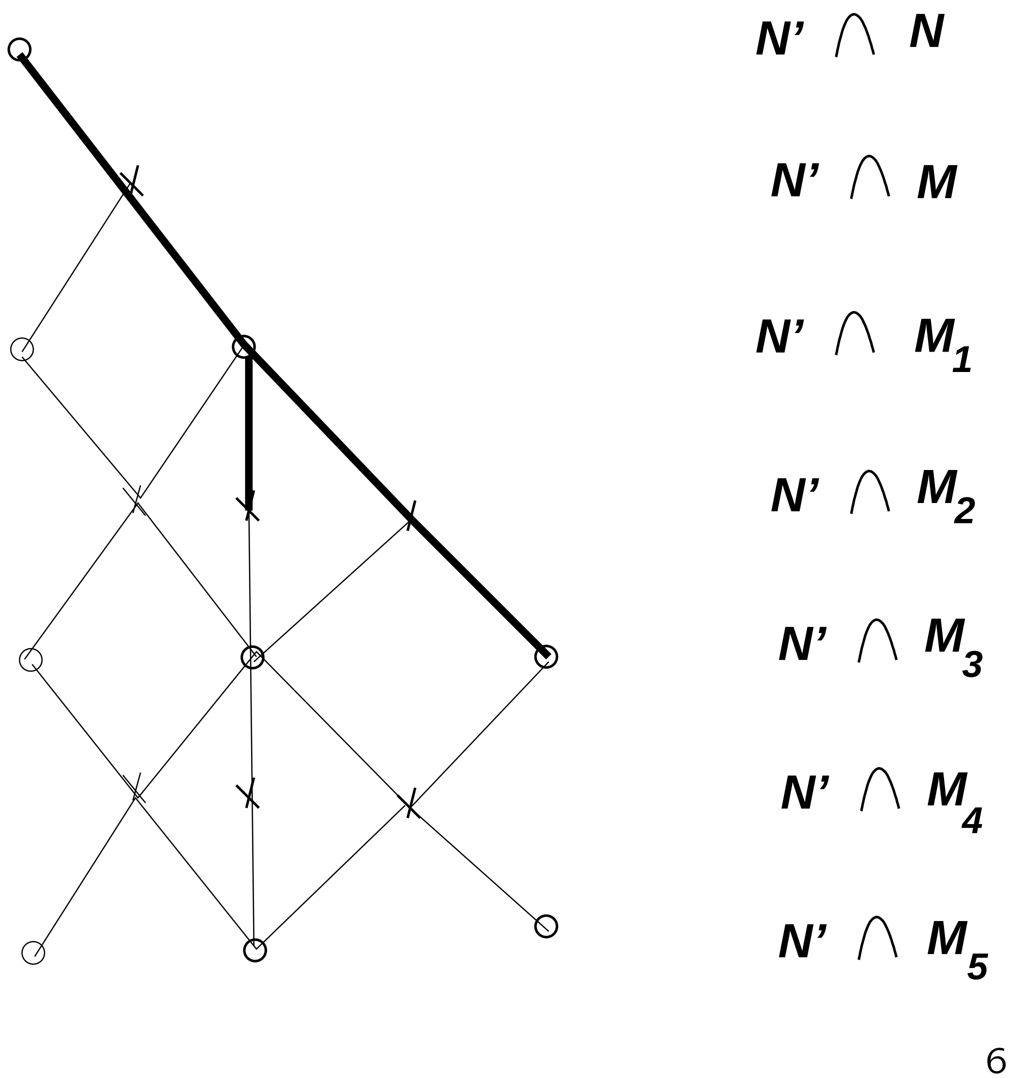
(ii) the tower $\{M' \cap M_n : n \geq 0\}$; and

(iii) the data of how the former tower is included in the latter.

These three ingredients are described by the so-called **principal graph, dual principal graph** and the **flat connection** associated to the subfactor, respectively.

The principal graph invariant:

The presence of the Jones projections $\{e_n : n \geq 1\}$ in the tower of the basic construction causes the presence of a certain *reflection symmetry* in the Bratteli diagram of the tower $\{N' \cap M_n : n \geq -1\}$. We illustrate this with an example.



Thus, if we write A_n for $N' \cap M_n$, then at each stage, the 'inclusion graph' for $A_n \subset A_{n+1}$ contains a reflection of that of $A_{n-1} \subset A_n$ and a possibly 'new part'. The graph obtained by retaining only the 'new parts' is the *principal graph.* (In the above example, the preincipal graph is the Coxeter graph E_6.) It should be clear that the Bratteli diagram for the entire tower $\{A_n\}$ can be recaptured from the principal graph.

Since the 'dual principal graph' associated to the subfactor $N \subset M$ is just the principal graph associated to $M \subset M_1$, we see that the dual principal graph also exhibits the same *reflection symmetry* as the principal graph.

A discussion of flat connections is beyond the scope of these lectures; but we will say that it imposes constraints on when a pair of graphs can arise as the principal and the dual principal graphs of a subfactor. We will state a few sample results though.

Suppose Γ and Γ' denote the principal and dual graphs associated with a subfactor. Then:

(a) Γ is finite iff Γ' is finite; and in this case,

$$\|A(\Gamma)\| = \|A(\Gamma')\| = [M : N]^{\frac{1}{2}} ,$$

where $A(\Gamma)$ denotes the 'adjacency matrix' of Γ.

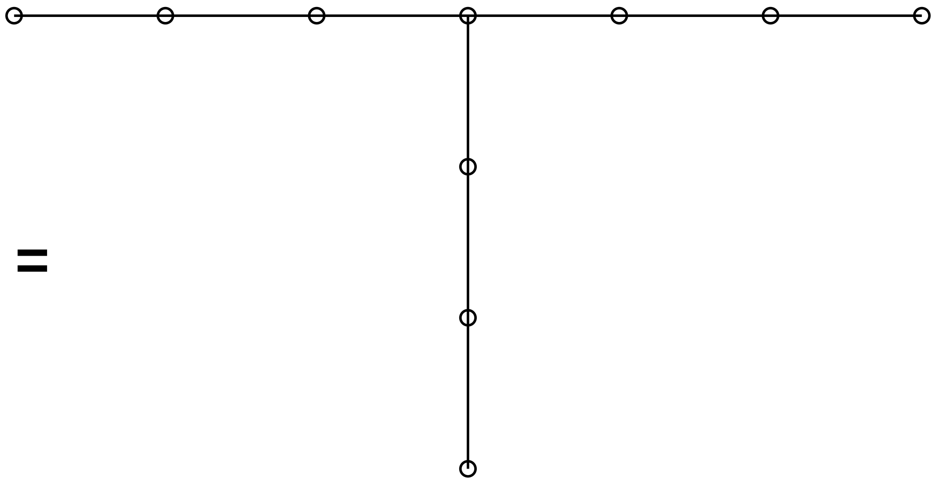
Thus if Γ denotes the E_6 graph (with 6 vertices), then

$$A(\Gamma) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

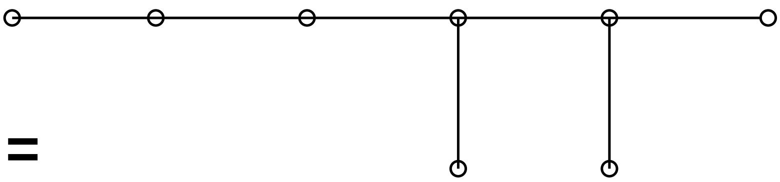
(b) If $[M : N] < 4$, then Γ is isomorphic to Γ' and to one of the Coxeter graphs A_n, D_{2n}, E_6 or E_8.

(c) There exists a subfactor, of index $\frac{5+\sqrt{13}}{2}$, with

Γ =



Γ' =

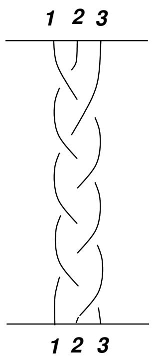


and if $4 < [M : N] < \frac{5+\sqrt{13}}{2}$, then the principal graph is infinite, and in fact, isomorphic to A_∞.

From subfactors to knot invariants:

Jones' polynomial invariant of knots is a consequence of the connection between subfactors and braid groups. So we shall digress with a sortie into braid groups and knots.

To an Indian, the term 'braid' can be motivated by the following '3-strand braid':

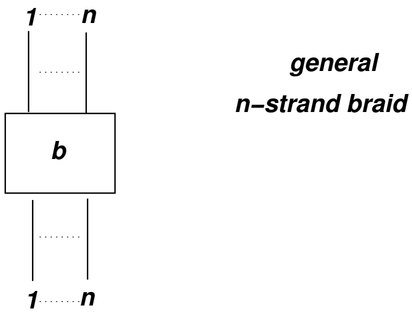


3–strand braid

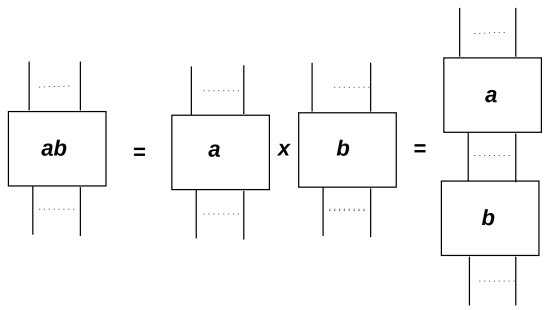
$(= (b_1^{-1} b_2)^k)$

Informal definition of an *n-strand braid*: two parallel rods with n hooks each, and n strands with one end of each strand tied to a hook on one of the rods - with two braids being identified if they can be (homotopically) deformed into one another.

We shall think of an n-strand braid b as follows - with ‘all the action taking place’ within the box:

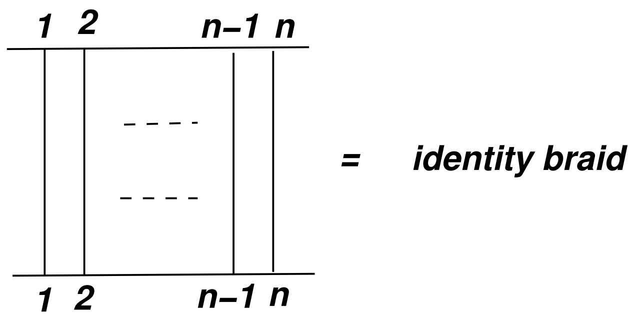


The collection B_n of all n-strand braids is equipped with a product thus:

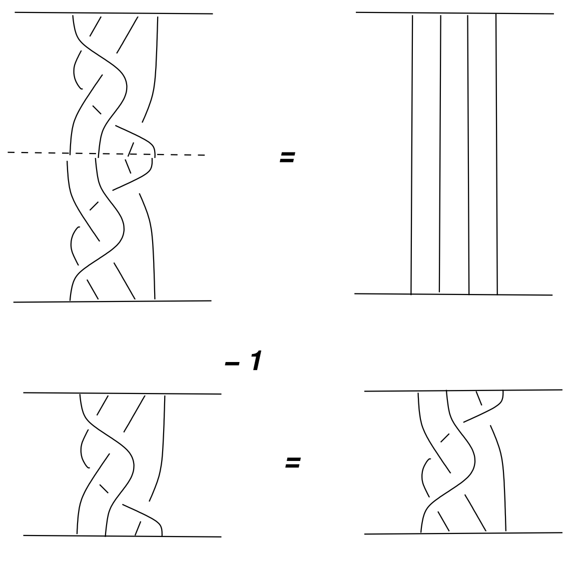


(To verify that this multiplication is associative, we need the assumption that homotopic braids are the same.)

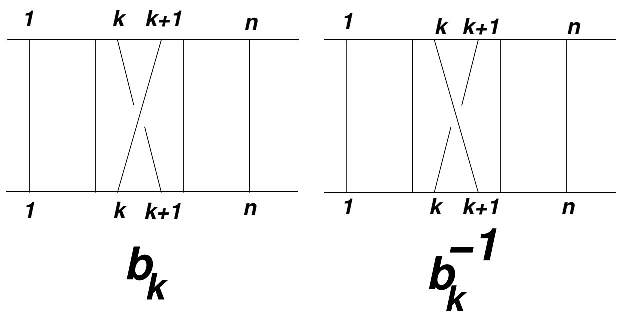
B_n turns out to be a group; the identity element $1_n \in B_n$ is given by



while the inverse of a braid is obtained by reflecting in a horizontal mirror placed at the level of the lower rod of the braid, thus:



Since braids can be built up ‘one crossing at a time’ it is clear that B_n is generated , as a group, by the braids $b_1, b_2, \cdots, b_{n-1}$ shown below - together with their inverses:

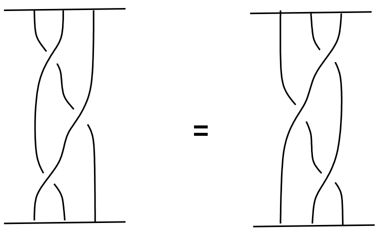


The b_j's satisfy the following relations:

- $b_i b_j = b_j b_i$ if $|i - j| \geq 2$

$b_1 \; b_3$ $\qquad$ $b_3 \; b_1$

- $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for all $i < n - 1$



$b_1 b_2 b_1$ $\qquad$ $b_2 b_1 b_2$

In order to describe a celebrated theorm by Artin on the braid group, we briefly digress into *presentations of (finitely generated) groups*.

Recall that **the free group with generators** $\{g_1, \cdots, g_n\}$ if for any set $\{h_1, \cdots, h_n\}$ of elements in any group H, there exists a *unique* homomorphism $\phi : G \to H$ with the property that $\phi(g_k) = h_k$ for each $k = 1, \cdots, n$. Such a group exists, is unique up to isomorphism, and is denoted by the symbol

$$G = \langle g_1, \cdots, g_n \rangle \ .$$

For example, $\mathbb{Z} = \langle 1 \rangle$ is **the** free group on one generator.

A group G is said to have **presentation**

$$G = \langle g_1, \cdots, g_n | r_1, \cdots, r_m \rangle$$

if:

(i) it is generated by the set $\{g_1, \cdots, g_n\}$

(ii) the g_i's satisfy each **relation** r_j for $j = 1, \cdots, m$; and

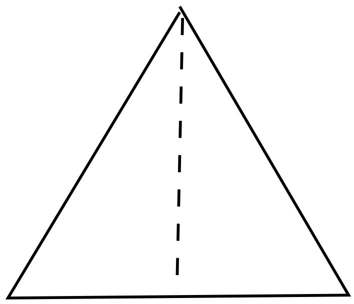
(iii) for any set $\{h_1, \cdots, h_n\}$ of elements in any group H, which 'satisfy each of the relations $r_1, \cdots, r_m$', there exists a *unique* homomorphism $\phi : G \to H$ with the property that $\phi(g_k) = h_k$ for each $k = 1, \cdots, n$.

Such a group exists, and is unique up to isomorphism.

Examples of presentations

(i) $C_n = \langle g | g^n = 1 \rangle$ is **the** cyclic group of order n.

(ii) $D_n = \langle g, t | g^n = 1, tgt^{-1} = t^{-1} \rangle$ is the **dihedral group** of symmetries of an n-gon.
(D_n has $2n$ elements.)



g = rotation by 120°

t = reflection about an altitude

The Braid group is often referred to as *Artin's Braid Group*, partly because of the following theorem he proved:

Theorem: (Artin) B_n has the presentation

$$B_n = \langle b_1, \cdots, b_{n-1} | r_1, r_2 \rangle \ ,$$

where

- (r_1) $b_i b_j = b_j b_i$ if $|i - j| \geq 2$
- (r_2) $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for all $i < n - 1$

It is a fact that the permutation group Σ_n has the presentation

$$\Sigma_n = \langle t_1, \cdots, t_{n-1} | r_1, r_2, r_3 \rangle \ ,$$

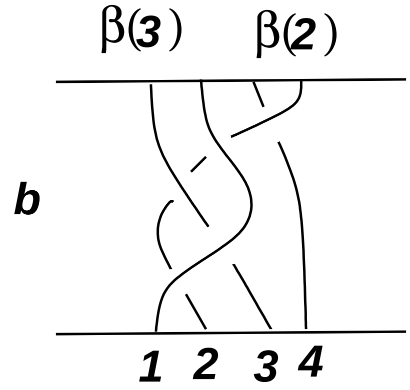
where r_1, r_2 are the braid relations above, and

$$(r_3) \text{ is } t_i^2 = 1 \text{ for all } i < n \ .$$

(We may choose t_i to be $(i\ i+1)$.)

Remarks: (a) There exists a unique homomorphism $\phi : B_n \to \Sigma_n$ such that $\phi(b_i) = t_i$ for each i. (ϕ is onto, and hence Σ_n is a quotient of B_n.)

In fact, $\phi(b) = \beta$, where

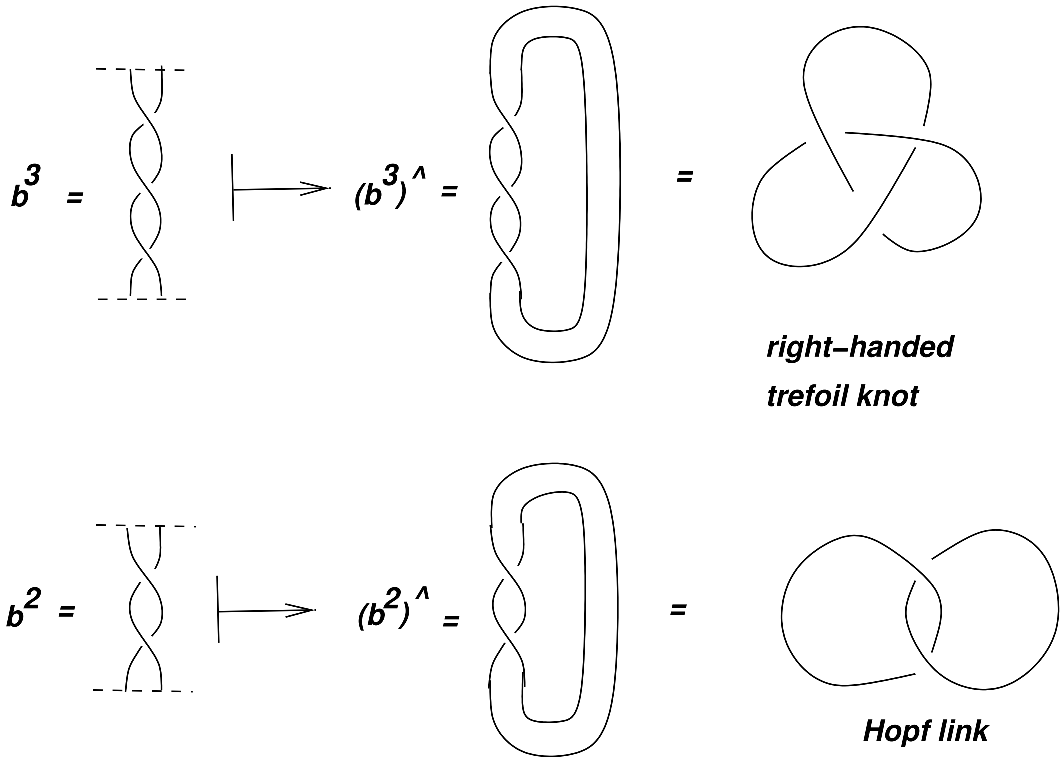


(b) There exist 1-1 homomorphisms $B_n \hookrightarrow B_{n+1}$ given by $b_k^{(n)} \mapsto b_k^{(n+1)}$ for each $k < n$.

(c) The generators b_i are all pairwise conjugate in B_n; in fact, if $b = b_1 b_2 \cdots b_n$, then $b b_i b^{-1} = b_{i+1}$ $\forall i < n-1$. (For example:

$$b_1 b_2 b_3 \cdot b_1 = b_1 b_2 b_1 b_3 = b_2 \cdot b_1 b_2 b_3)$$

The **closure** of a braid $b \in B_n$ is obtained by sticking together the strings connected to the j-th pegs at the top and bottom. The result is a many component **knot** (also called a **link**) $\hat{b}$.



Two theorems make this ‘closure operation’ useful:

Theorem (**Alexander**):

Every *tame* link is the closure of some braid (on some number of strands).

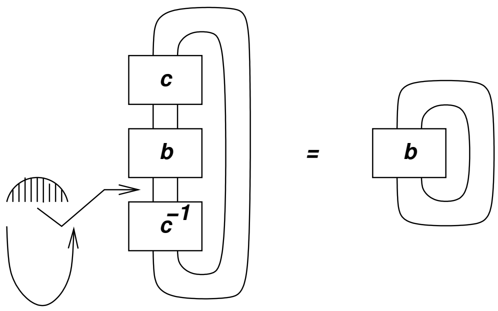
Theorem(**Markov)**:

Two braids have equivalent closures iff you can pass from one to the other by a finite sequence of moves of one of two types (the so-called ‘Markov moves’).

(Two links are ‘equivalent’ if each may be continuously deformed into the other: the two should be *ambient isotopic* in $\mathbb{R}^3$.)

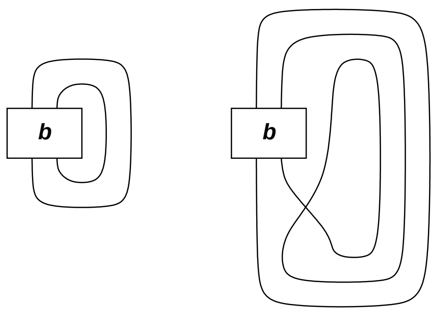
Type I Markov move:

$$c^{(n)}b^{(n)}(c^{(n)})^{-1} \sim b^{(n)}$$



Type II Markov move:

$$b^{(n)} \sim b^{(n+1)}(b_n^{(n+1)})^{-1}$$



Def: A *link invariant* (taking values in some set S) is a function

$$\mathcal{L} \ni L \mapsto \phi_L \in S$$

such that $\phi_{L_1} = \phi_{L_2}$ whenever $L_1 \sim L_2$. $\square$

The theorems of Alexander and Markov give us a strategy for constructing link invariants: simply define $\phi_L = \phi_n(a)$ if $L = \hat{a}$ for some $a \in B_n$, where $\{\phi_N : B_n \to S : n \geq 1\}$ is any family of functions which satisfy, for all n:

1. $$\phi_N(cbc^{-1}) = \phi_n(b) \ \forall \ b, c \in B_n.$$

2. $$\phi_n(a^{(n)}) = \phi_{n+1}(a^{(n+1)}(b_n^{(n+1)})^{\pm 1})$$

   for all $a^{(n)} \in B_n$.

And the Jones projections from subfactor theory permit us to put this atrategy into practice. Recall that a subfactor $N \subset M$ with $[M : N] = d < \infty$ gives rise to a sequence $\{e_n : n \geq 1\}$ of projections which have the following properties:

(a) $e_n e_m = e_m e_n$ if $|m - n| > 1$;

(b) $e_n e_{n\pm 1} e_n = d^{-1} e_n$;

(c) there is a faithful positive trace tr defined on the unital *-subalgebra A_∞ generated by $\{e_n : n \geq 1\}$, such that

$$tr(x e_{n+1}) = d^{-1} tr(x)$$

whenever x is in the unital algebra A_n generated by $\{e_1, \cdots, e_n\}$.

Comparing the braid relations and the relations (a),(b) satisfied by the Jones projections, we see - after a little algebra - that if we define

$$g_i = C[(q+1)e_i - 1] \ ,$$

- for any $C \neq 0$, and $i \geq 1$ - then the g_i's satisfy the braid relations, povided $q \in \mathbb{C}$ satisfies

$$q + q^{-1} + 2 = d \ ;$$

and so we have a homomorphism π_n from B_n into the group of invertible elements of A_n such that

$$\pi_n(b_i) = g_i \ \forall 1 \leq i \leq n \ .$$

Motivated by condition 2. of the last page, we choose the constant C such that $tr \ g_{n+1} = tr \ g_{n+1}^{-1}$; this forces $C = q^{\frac{1}{2}}$ and

$$\begin{aligned} & tr \ \pi_{n+1}(a^{(n+1)}(b_N^{(n+1)})^{\pm 1}) \\ & \quad = \ [-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{-1}] tr \ \pi_n(a^{(n)}) \ \forall a^{(n)} \in B_n \end{aligned}$$

Note that

$$\tau = 4cosh^2 z \Leftrightarrow q = e^{\pm 2z} ,$$

and in particular τ^{-1} is a possible finite index value iff

$$q \in Q = \{e^{\pm \frac{2\pi i}{n}} : n \geq 3\} \cup [1, \infty)$$

In conclusion, we find that for each $q \in Q$, there exists a link invariant (in fact an invariant of *oriented links*)

$$\mathcal{L} \ni L \mapsto \phi^q_L \in \mathbb{C}$$

such that if π_n, C etc. are associated to $\tau = (q + q^{-1} + 2)^{-1}$ as above, then

$$\phi^q_{\widehat{a^{(n)}}} = [-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})]^{n-1} tr\ \pi_n(a^{(n)})$$

It is customary to write $V_l(q)$ for what we denoted above by ϕ^q_L, so as to draw attention to the function $q \mapsto V_L(q)$.

We list some remarkable properties of this function - commonly referred to as the *one-variable Jones polynomial* - below.

Proposition: Let L be any oriented link.

(a) If L has an odd number of components, then $V_L(q)$ is a Laurent polynomial in q; and if L has an even number of components, then $V_L(q)$ is $q^{\frac{1}{2}}\times$ a Laurent polynomial in q.

(b) If $\tilde{L}$ denotes the 'mirror-reflection' of L, then

$$V_{\tilde{L}}(q) = V_L(q^{-1}).$$

(c) $V_{U_n}(q) = [-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})]^{n-1}$, where U_n denotes the *unlink* on n components.

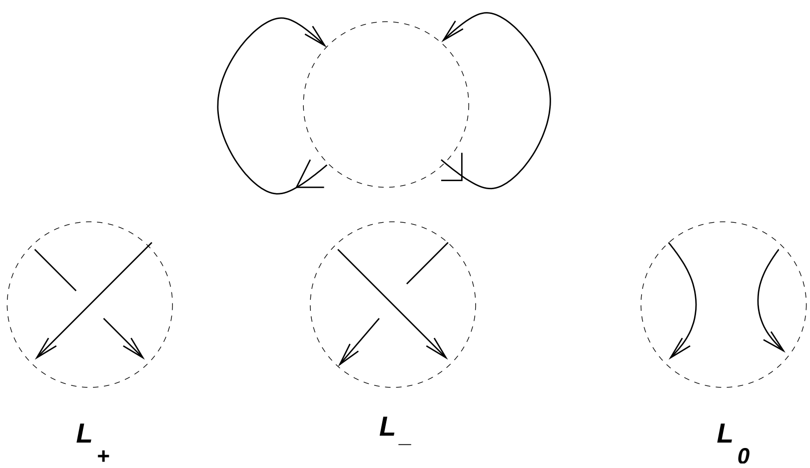
(d)

$$q^{-1}V_{L_+}(q) - qV_{L_-}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_{L_0}(q)$$

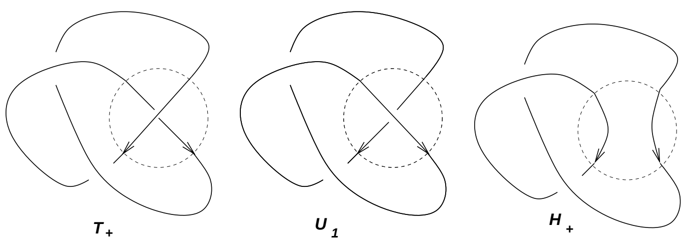
for any *skein-related triple* L_+, L_-, L_0.

(e) the invariant V is uniquely determined by properties (c) and (d) above.

Three links L_+, L_-, L_0 are said to be **skein-related** if they may be represented by link-diagrams which are identical except at one crossing, where they look like:



An instance of such a triple is given by:



where

$$L_+ = T_+, L_- = U_1,\ \ L_0 = H_+$$