Home-work 2

on lecture dated 12/10/09

Recall the following definition:

DEFINITION 0.1. *A set V is said to be a* **(real) vector space** *if it admits two operations,* vector addition *and* scalar operation *satisfying the following conditions:*

- **Vector addition:** *To every pair u, v of elements of V, - called vectors - there is associated a unique vector, called their sum and denoted $u + v$, such that the following conditions hold, for all $u, v, w \in V$:*
  1. *(*commutativity of addition*)* $u + v = v + u$
  2. *(*associativity of addition*)* $(u + v) + w = u + (v + w)$
  3. *(*existence of zero*) there exists a vector in V, called zero, and denoted by 0, such that $u + 0 = u$ for all $u \in V$*
  4. *(*existence of negatives*) for each vector $v \in V$, there exists a vector in V, denote by $-v$, with the property that $v + (-v) = 0$*
- **Scalar multiplication :** *To each real number α (called a scalar) and vector v, there is associated a unique vector, denoted by αv, such that the following conditions hold, for all $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$:*
  1. *(*associativity of scalar multiplication*)* $\alpha(\beta v) = (\alpha\beta)v$
  2. *(*identity axiom*)* $1v = v$
  3. distributivity of scalar multiplication over vector addition $\alpha(u + v) = \alpha u + \alpha v$ *and* $(\alpha + \beta)v = \alpha v + \beta v$

1. Show, using only the axioms above that $(u + v) + (x + y) = (u + (y + (v + x)))$ for all $x, y, u, v \in V$
2. Show, using only the axioms above that

   $$u + v = v + w \Rightarrow u = w$$

   In particular, deduce that for any vector v, the negative $-v$ is uniquely determined by v, and that $-v = (-1)v$.

3. If we write $u - v$ for $u + (-v)$, verify that

$$u - (v - w) = u - v + w$$

4. On $\mathbb{R}^2$, define two operations

$$\begin{aligned} u \oplus v &= u - v \\ \alpha * v &= -\alpha v \end{aligned}$$

   where the symbols on the right side have their usual meanings. Which axioms of a vector space are satisfied by $(\mathbb{R}^2, \oplus, *)$.

5. Verify that the following prescriptions satisfy all the above axioms and are hence examples of vector spaces:

   (a) $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R} \forall i\}$, with addition and scalar multiplication defined by

$$\begin{aligned} (x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\ \alpha(x_1, x_2, \cdots, x_n) &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n) \end{aligned}$$

   (b) The set $\mathbb{R}[t] = \{a_0 + a_1 t + \cdots a_n t^n : n \geq 0, a_i \in \mathbb{R}\}$ of polynomials with real coefficients, with the natural addition and scalar multiplication. Thus, for example,

$$(2t + \frac{22}{7}t^4) + 3(1 + t + t^2) = 3 + 5t + 3t^2 + \frac{22}{7}t^4.$$

   (c) The set of $m \times n$ real matrices defined by

$$M_{m\times n}(\mathbb{R}) = \{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R} \text{ for } 1 \leq i \leq m, 1 \leq j \leq n\}$$

   with vector operations defined in the natural (entrywise) manner.