Home-work 13

on lecture dated 19/12/09

1. Justify the following 'change of variable' trick:

   (a) If G is any finite group and if $f : G \to \mathbb{R}$ is any function, show that

   $$\sum_{g\in G} f(g) = \sum_{g\in G} f(g^{-1}), \prod_{g\in G} f(g) = \prod_{g\in G} f(g^{-1})$$

   (b) If $A \in M_n(\mathbb{R})$, verify that

   $$\sum_{\sigma\in S_n} \epsilon(\sigma) \left(\prod_{i=1}^{n} a^i_{\sigma(i)}\right) = \sum_{\pi\in S_n} \epsilon(\pi) \left(\prod_{j=1}^{n} a_j^{\pi(j)}\right)$$

   i.e., that $det(A) = det(A')$.

2. Show that $det(A) \neq 0$ if and only if A is invertible.

3. (a) Show that the matrix

   $$A = \frac{1}{2}\begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

   satisfies $Av_1 = v_1, Av_2 = v_2, Av_3 = v_3, Av_4 = -v_4$, where

   $$v_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, v_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, v_4 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (b) Show that the $\{v_1, v_2, v_3, v_4\}$ of (a) above is a basis for $\mathbb{R}^4$.

   (c) If $S \in L(\mathbb{R}^4)$ is defined by $Se_i = v_i, 1 \leq i \leq 4$, where $\{e_1, e_2, e_3, e_4\}$ denotes the standard basis of $\mathbb{R}^4$, deduce that S is invertible, that

   $$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

   and, in particular, that $det(A) = -1$.