Home-work 11

on lecture dated 28/11/09

1. (Assume the fact that the determinant of an $n \times n$ matrix, for arbitrary n, may be obtained by 'expanding along any row"; thus, assume the formula

$$\begin{aligned} det(A) &= \sum_{i,j=1}^{n} (-1)^{i+j} a^i_j det(A^i_j) \\ &= \sum_{j=1}^{n} a^i_j c^j_i \ , \qquad (0.1) \end{aligned}$$

where c^i_j denotes the so-called (i,j)**-th cofactor** of A, defined by

$$c^i_j = (-1)^{i+j} det(A^j_i) \ ,$$

with A^i_j denoting the submatrix of A obtained by deleting the i-th row and j-th column.)

   (a) Show that $\sum_{j=1}^{n} a^k_j c^j_i = 0$ for $1 \leq i \neq k \leq n$.

   (b) Deduce that if $C(A) = ((c^i_j))$ denotes the matrix of cofactors, then

$$AC(A) = det(A) I_n$$

   where, of course, I_n denotes the identity matrix.

   (c) Deduce that if $det(A) \neq 0$, then A is invertible and

$$A^{-1} = (det(A))^{-1} C(A).$$

2. Illustrate by example that if A is any 3×3 matrix, and if E is any kind of elementary 3×3 matrix, then

$$det(EA) = det(E) det(A).$$

3. Use equation (0.1) to show that the expansion of the determinant of a 3×3, resp., a 4×4, whose entries are distinct symbols, contains exactly 6, resp., 24, terms. What about 5×5?