Approximation properties and absence of Cartan subalgebra for free Araki-Woods factors

Cyril Houdayer
(Joint work with Éric Ricard)

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Organization of the talk

1. Definition of the free Araki-Woods factors $\Gamma(H_{\mathbf{R}}, U_t)''$.
   Construction, basic examples and earlier classification results.
2. Approximation properties and structural results for $\Gamma(H_{\mathbf{R}}, U_t)''$.
3. Applications to the classification problem of type III_1 factors.

Free Gaussian process

- **Real** Hilbert space, **orthogonal** representation of $\mathbf{R}$

$$(H_{\mathbf{R}}, U_t)$$

$$H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \text{ and } U_t = A^{it}$$

- **Isometric** embedding $H_{\mathbf{R}} \hookrightarrow H$, $K_{\mathbf{R}} = j(H_{\mathbf{R}})$.

$$j(\zeta) = \left(\frac{2}{1 + A^{-1}}\right)^{1/2} \zeta$$

$$\overline{K_{\mathbf{R}} + iK_{\mathbf{R}}} = H \text{ and } K_{\mathbf{R}} \cap iK_{\mathbf{R}} = \{0\}$$

- Closed densily defined conjugate-linear **involution** on H.

$$I(e + if) = e - if, \forall e, f \in K_{\mathbf{R}}$$

(Observe that $I^* I = A^{-1}$).

Free Gaussian process

- **Full Fock space**

$$\mathcal{F}(H) = \mathbf{C}\Omega \oplus \bigoplus_{n \geq 1} H^{\otimes n}$$

- **Left creation operators**. For $\xi \in H$, $\ell(\xi) : \mathcal{F}(H) \to \mathcal{F}(H)$

$$\begin{cases} \ell(\xi)\Omega = \xi \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \end{cases}$$

- **Semicircular element** (whose spectrum is $[-2\|\xi\|, 2\|\xi\|]$)

$$W(\xi) = \ell(\xi) + \ell(\xi)^*$$

- **Free quasi-free state**

$$\chi = \langle \cdot \Omega, \Omega \rangle$$

Free Araki-Woods factors

Definition (Shlyakhtenko 1997)

Free Araki-Woods von Neumann algebra associated with $(H_{\mathbf{R}}, U_t)$

$$\Gamma(H_{\mathbf{R}}, U_t)'' := \{W(e) : e \in K_{\mathbf{R}}\}'' \subset \mathbf{B}(\mathcal{F}(H))$$

Note that $\Gamma(\mathbf{R}, 1)'' \simeq L(\mathbf{Z})$.

Theorem (Functorial property)

$$\Gamma(\bigoplus_{j \in J} H_{\mathbf{R}}^j, \bigoplus_{j \in J} U_t^j)'' = *_{j \in J} \Gamma(H_{\mathbf{R}}^j, U_t^j)''$$

where the latter free product is taken along free quasi-free states.

It follows immediately that if (U_t) is trivial then

$$\Gamma(H_{\mathbf{R}}, 1)'' \simeq \underbrace{L(\mathbf{Z}) * \cdots * L(\mathbf{Z})}_{\dim(H_{\mathbf{R}}) \text{ times}} = L(\mathbf{F}_{\dim(H_{\mathbf{R}})})$$

Free Araki-Woods factors

Example

If $H_{\mathbf{R}} = \mathbf{R}^2$ and $U_t = \begin{pmatrix} \cos(2\pi \log(\lambda)t) & -\sin(2\pi \log(\lambda)t) \\ \sin(2\pi \log(\lambda)t) & \cos(2\pi \log(\lambda)t) \end{pmatrix}$, then $\Gamma(H_{\mathbf{R}}, U_t)''$ is a type III_λ factor.

More generally

Theorem (Shlyakhtenko 1997)

If $\dim(H_{\mathbf{R}}) \geq 2$, *then* $M = \Gamma(H_{\mathbf{R}}, U_t)''$ *is a full factor and*

1. M *is of type* II_1 *iff* (U_t) *is trivial:* $M = L(\mathbf{F}_{\dim(H_{\mathbf{R}})})$
2. M *is of type* III_λ, $0 < \lambda < 1$, *iff* (U_t) *is* $\frac{2\pi}{|\log(\lambda)|}$*-periodic*
3. M *is of type* III_1 *otherwise*

Classification when (U_t) is almost periodic

Theorem (Shlyakhtenko 1997)

Assume that (U_t) is not trivial and **almost periodic**

$$U_t = \mathbf{R}^k \oplus \bigoplus_n \begin{pmatrix} \cos(2\pi \log(\lambda_n) t) & -\sin(2\pi \log(\lambda_n) t) \\ \sin(2\pi \log(\lambda_n) t) & \cos(2\pi \log(\lambda_n) t) \end{pmatrix}$$

Then $\Gamma(H_{\mathbf{R}}, U_t)''$ is completely classified up to $*$-isomorphism and only depends on the countable subgroup of $\mathbf{R}_+$ generated by (λ_n).

Classification of $\Gamma(H_{\mathbf{R}}, U_t)''$ in the non-almost periodic case is an extremely hard problem!

Complete metric approximation property

Definition (Haagerup 1979)

A von Neumann algebra M is said to have the **complete metric approximation property** (CMAP) if $\exists$ a net $\Phi_n : M \to M$ of normal finite rank completely bounded maps such that

- $\Phi_n \to \mathrm{Id}$ pointwise $*$-strongly
- $\limsup_n \|\Phi_n\|_{\mathrm{cb}} = 1$

Example

- M amenable (hyperfinite)
- $M = L(G)$, where G is a free group and more generally a lattice in $\mathrm{SL}_2(\mathbf{R})$, $\mathrm{SL}_2(\mathbf{C})$, $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$.

CMAP for free Araki-Woods factors

Theorem A (H-Ricard 2010)

All the free Araki-Woods factors $\Gamma(H_{\mathbf{R}}, U_t)''$ have the CMAP.

Idea of proof. Recall that the GNS-space of $(\Gamma(H_{\mathbf{R}}, U_t)'', \chi)$ is

$$\mathbf{C}\Omega \oplus \bigoplus_{n \geq 1} H^{\otimes n}$$

1. Using **radial multipliers**, we can project onto words $W(\xi_{\underline{i}})$ of bounded length with a good control of the $\| \cdot \|_{\mathrm{cb}}$-norm.
2. The **second quantization** allows us to project (with cp maps) onto words with letters ξ in finite dimensional subspaces.
3. By composing, we get normal finite rank completely bounded maps (Φ_n) that do the job.

Cartan subalgebra

Definition

A subalgebra $A \subset M$ is said to be a **Cartan subalgebra** if

1. A is maximal abelian, i.e. $A' \cap M = A$
2. There exists a faithful normal conditional expectation $E : M \to A$
3. The normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates the von Neumann algebra M

By Feldman-Moore, any such Cartan subalgebra $A \subset M$ arises as

$$L^\infty(X, \mu) = A \subset M = L(\mathcal{R}, \omega)$$

where $\mathcal{R}$ is a **nonsingular** equivalence relation on (X, μ) and ω is a scalar 2-cocycle for $\mathcal{R}$.

Group actions $\Gamma \curvearrowright (X, \mu)$

- G infinite countable discrete group
- (X, μ) nonatomic standard measure space
- $G \curvearrowright (X, \mu)$ **nonsingular** action:

$$\mu(\mathcal{V}) = 0 \iff \mu(g \cdot \mathcal{V}) = 0, \forall g \in G, \forall \mathcal{V} \subset X$$

Definition

- $G \curvearrowright (X, \mu)$ is **free** if $\forall g \neq e$,

$$\mu(\{x \in X : g \cdot x = x\}) = 0$$

- $G \curvearrowright (X, \mu)$ is **ergodic** if $\forall \mathcal{V} \subset X$,

$$G \cdot \mathcal{V} = \mathcal{V} \Longrightarrow \mu(\mathcal{V}) = 0, 1$$

Nonsingular **orbit** equivalence relation $\mathcal{R}(G \curvearrowright X)$

Group actions $\Gamma \curvearrowright (X, \mu)$

Example

1. **Compact** action. Let K be a compact group with $G < K$ a countable dense subgroup. Let $G \curvearrowright (K, \mathrm{Haar})$ by left multiplication.
2. **Bernoulli** shift. Let G be an infinite group and $(X, \mu) = ([0,1]^G, \mathrm{Leb}^G)$. Then $G \curvearrowright [0,1]^G$ is defined by
$$s \cdot (x_h)_{h \in G} = (x_{g^{-1}h})_{h \in G}$$
3. The **linear** actions $\mathrm{SL}_n(\mathbf{Z}) \curvearrowright (\mathbf{T}^n, \lambda^n)$ (of type II_1), and $\mathrm{SL}_n(\mathbf{Z}) \curvearrowright (\mathbf{R}^n, \lambda^n)$ (of type II_∞), for $n \geq 2$.
4. The **projective** action $\mathrm{SL}_n(\mathbf{Z}) \curvearrowright \mathbf{P}^{n-1}(\mathbf{R})$ (of type III_1), for $n \geq 2$.

Global dichotomy result

Theorem B (H-Ricard 2010)

Let $M = \Gamma(H_{\mathbf{R}}, U_t)''$ and $N \subset M$ be a diffuse subalgebra (for which $\exists$ a faithful normal conditional expectation $E : M \to N$).
Then either N is hyperfinite or N has no Cartan subalgebra.

Global dichotomy result

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Ozawa-Popa (2007) showed that $L(\mathbf{F}_n)$ is **strongly solid**, i.e. $\forall P \subset L(\mathbf{F}_n)$ diffuse amenable, the normalizer $\mathcal{N}_{L(\mathbf{F}_n)}(P)$ generates an amenable von Neumann algebra. This strengthened two well-known indecomposability results for free group factors:

- Voiculescu (1994): $L(\mathbf{F}_n)$ has no Cartan decomposition
- Ozawa (2003): $L(\mathbf{F}_n)$ is **solid** ($\forall A \subset L(\mathbf{F}_n)$ diffuse, $A' \cap L(\mathbf{F}_n)$ is amenable)

Our proof is a generalization of theirs and relies on Theorem A.

Close Encounters of the Third Kind

Given a type III von Neumann algebra M, there is a canonical construction of the **noncommutative flow of weights**

$$(M \subset M \rtimes_\sigma \mathbf{R}, \theta, \mathrm{Tr})$$

where

1. σ is the **modular** group and θ the **dual** action
2. The **core** $M \rtimes_\sigma \mathbf{R}$ is of type II_∞ and Tr is a semifinite trace
3. $\mathrm{Tr} \circ \theta_s = e^{-s} \mathrm{Tr}, \forall s \in \mathbf{R}$
4. $(M \rtimes_\sigma \mathbf{R}) \rtimes_{(\theta_s)} \mathbf{R} \simeq M \overline{\otimes} \mathbf{B}(L^2(\mathbf{R}))$

This construction does **not** depend on the choice of a state ψ.

$$M \text{ is a type } \mathrm{III}_1 \text{ factor} \iff M \rtimes_\sigma \mathbf{R} \text{ is a type } \mathrm{II}_\infty \text{ factor}$$

Malleable deformation on $\Gamma(H_{\mathbf{R}}, U_t)''$

On $H_{\mathbf{R}} \oplus H_{\mathbf{R}}$, the rotations

$$V_s = \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix}$$

commute with $U_t \oplus U_t$. The second quantization $\alpha_s = \Gamma(V_s)$ on

$$\Gamma(H_{\mathbf{R}} \oplus H_{\mathbf{R}}, U_t \oplus U_t)'' = \Gamma(H_{\mathbf{R}}, U_t)'' * \Gamma(H_{\mathbf{R}}, U_t)''$$

satisfies

$$\alpha_1(x * 1) = 1 * x, \forall x \in \Gamma(H_{\mathbf{R}}, U_t)''$$

(α_s) is a **malleable deformation** in the sense of Popa. Moreover (α_s) can be extended to $M \rtimes_{\sigma^\chi} \mathbf{R}$ by letting $\alpha_{s|L(\mathbf{R})} = \mathrm{Id}_{L(\mathbf{R})}$.

Weak compactness

Definition (Ozawa-Popa 2007)

Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ). We say that $\mathcal{R}$ is **weakly compact** if $\exists$ a sequence (ν_n) of Borel probability measures on $X \times X$ such that $\nu_n \sim \mu \times \mu$, $\forall n \in \mathbf{N}$ and

1. ${\pi^i}_* \nu_n = \mu$, $\forall i = 1, 2$ (where $\pi^i : (x_1, x_2) \mapsto x_i$)
2. $\lim_n \int_{X \times X} (f_1 \otimes f_2) \, \mathrm{d}\nu_n = \int_X f_1 f_2 \, \mathrm{d}\mu$, $\forall f_1, f_2 \in L^\infty(X)$
3. $\lim_n \|\nu_n - (\theta \times \theta)_* \nu_n\| = 0$, $\forall \theta \in [\mathcal{R}]$

Example

If $G \curvearrowright X$ is compact, then $\mathcal{R}(G \curvearrowright X)$ is weakly compact.

Theorem (Ozawa-Popa 2007)

Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ). If $L(\mathcal{R})$ has the CMAP, then $\mathcal{R}$ is weakly compact.

Idea of proof of Theorem B

- Let $M = \Gamma(H_{\mathbf{R}}, U_t)''$. By contradiction, let $N \subset M$ be a diffuse nonamenable subalgebra which has a Cartan subalgebra $A \subset N$.
- Choose a state ψ on M such that $A \subset N^\psi$. It follows that

$$A \overline{\otimes} \lambda^\psi(\mathbf{R})'' \subset N \rtimes_{\sigma^\psi} \mathbf{R}$$

  is a Cartan subalgebra.
- Using Ozawa-Popa's techniques ((α_s) + weak compactness + semifinite intertwining techniques), one shows that

$$\text{a corner of } A \overline{\otimes} \lambda^\psi(\mathbf{R})'' \text{ embeds into a corner of } \lambda^\chi(\mathbf{R})''$$

- One finally shows that this contradicts the fact that A is diffuse.

On the classification of type III_1 factors

Definition

A (separable) factor M is said to be **full** if $\mathrm{Inn}(M)$ is a closed subgroup of $\mathrm{Aut}(M)$. Then $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ is a Polish group.

By Connes' classical results, define the modular map

$$\delta : \mathbf{R} \ni t \mapsto \pi(\sigma_t^{\varphi}) \in \mathrm{Out}(M)$$

Definition (Connes 1974)

Let M be a type III_1 factor. The invariant $\tau(M)$ is defined as the weakest topology on $\mathbf{R}$ that makes the map δ continuous.

If $\tau(M)$ is the usual topology on $\mathbf{R}$, then M fails to have a **discrete decomposition**, i.e. of the form $M = \mathrm{II}_\infty \rtimes G$

In the '70s, Connes constructed type III_1 factors that fail to have such a discrete decomposition using his τ invariant.

Connes' construction

Let μ be a finite Borel measure on $\mathbf{R}_+$ such that $\int x \, \mathrm{d}\mu(x) < \infty$. Normalize μ so that $\int (1 + x) \, \mathrm{d}\mu(x) = 1$. Define the unitary representation (U_t) of $\mathbf{R}$ on $L^2(\mathbf{R}_+, \mu)$ by

$$(U_t \xi)(x) = x^{it} \xi(x)$$

Define on $P = \mathbf{M}_2(L^\infty(\mathbf{R}_+, \mu))$ the state φ by

$$\varphi \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \int f_{11}(x) \, \mathrm{d}\mu(x) + \int x f_{22}(x) \, \mathrm{d}\mu(x)$$

On the classification of type III_1 factors

Let $\mathbf{F}_n$ be acting by Bernoulli shift on

$$\mathcal{P}_\infty = \overline{\bigotimes_{g \in \mathbf{F}_n}} (P, \varphi).$$

Theorem (Connes 1974)

Assume (U_t) is not periodic. Then $\mathcal{N} = \mathcal{P}_\infty \rtimes \mathbf{F}_n$ is a full factor of type III_1 and $\tau(\mathcal{N})$ is the weakest topology that makes the map $t \mapsto U_t$ $*$-strongly continuous. In particular, if (U_t) is the left regular representation, then $\mathcal{N}$ has no discrete decomposition.

The following answers a question of Shlyakhtenko and Vaes.

Corollary C

$\mathcal{N}$ is not $*$-isomorphic to any free Araki-Woods factor.

Indecomposability results for $M \rtimes_\sigma \mathbf{R}$

Theorem D (H-Ricard 2010)

Let $M = \Gamma(H_{\mathbf{R}}, U_t)''$ where (U_t) is neither trivial nor periodic. Let $N = p(M \rtimes_\sigma \mathbf{R})p$, where p is a finite projection.

- *For any $A \subset N$ maximal abelian, $\mathcal{N}_N(A)''$ is amenable.*
- *If (U_t) is mixing or $U_t = \mathbf{R} \oplus V_t$, with (V_t) mixing, then N is strongly solid.*

We obtain new examples of strongly solid II_1 factors N (with CMAP and Haagerup property), such that $N \neq L(\mathbf{F}_t)$ and $\mathcal{F}(N) = \mathbf{R}_+$.