# *Cellularity and the Jones basic construction*

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# *Introduction*

The goal of this work is to study certain finite dimensional algebras that arise in invariant theory, knot theory, subfactors, QFT, and statistical mechanics.

The algebras in question have parameters; for generic values of the parameters, they are semisimple, but it is also interesting to study non–semisimple specializations. It turns out that operator algebra ideas — specifically, **the Jones basic construction** — are **still useful in the non–semisimple case.**

## *Introduction – cont.*

More explicitly, we develop a framework for studying several important examples of **pairs of towers of algebras**,

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots$$

such as

- $A_n =$ Brauer algebra, $Q_n =$ symmetric group algebra.
- $A_n =$ BMW algebra, $Q_n =$ Hecke algebra.
- $A_n =$ cyclotomic BMW algebra, $Q_n =$ cyclotomic Hecke algebra.
- $A_n =$ partition algebra, $Q_n =$ “stuttering" sequence of symmetric group algebras.
- etc.

# Introduction, cont. 2

Some properties of these examples:

1. The algebras are **generically semisimple**, and in the generic semisimple case, the tower $(A_n)_{n\geq 0}$ is obtained from the tower $(Q_n)_{n\geq 0}$ "by repeated Jones basic constructions."
2. More explicitly, this means that there is a generic (integral) ground ring $R$ for $A_n$ (independent of $n$). Every instance of $A_n$ over a ground ring $S$ is a specialization: $A_n^S = A_n^R \otimes_R S$. With $F$ the field of fractions of $R$, the algebra $A_n^F$ is semisimple.
3. $A_n$ has an ideal $J_n$ such that $Q_n \cong A_n/J_n$, and $J_{n+1}^F$ is isomorphic to a Jones basic construction for the pair $A_{n-1}^F \subset A_n^F$.

## *Introduction, cont. 3*

4. All this results from a method due to **Wenzl** in the 80's for showing the generic semisimplicity and determining the generic branching diagram of the tower $(A_n^F)_{n\geq 0}$. (Wenzl applied this to Brauer and BMW algebras.)
5. One is also interested in non–semisimple specializations (e.g. symmetric group in characteristic $p$, Hecke algebras at roots of unity), and here the **framework of *cellularity*, due to Graham and Lehrer**, is helpful.
6. For studying cellularity of the $A_n$, it suffices to work over the generic ground ring $R$, since cellularity is preserved under specialization.
7. In the examples, it is known that the cellular structures of the $Q_n$'s are *coherent*, which means well behaved with respect to induction and restriction.

# *Introduction, cont. 4*

8. A reflection of this is that one has cellular bases indexed by paths on the generic branching diagram of $(Q_n)_{n\geq 0}$ that are well behaved with respect to restriction (**path bases**).
9. In our examples, cellularity of the algebras $A_n$ was already known, but previous proofs generally did not give coherence of the cellular structures, nor path bases, or only gave such results by methods special to particular examples.

# *Introduction, cont. 5*

In this work we have found a cellular analogue of Wenzl's construction which applies uniformly to all the examples, and produces coherent cellular structures and (cellular) path bases for the $A_n$'s.

Advantages:

- Easy to apply in examples, gives efficient proof of cellularity.
- Considerable simplification of previous work in the case of cyclotomic BMW algebras.
- Relates cellular structure to Jones basic construction: Cell modules of $A_n$ either "come from" $Q_n$ or from $A_{n-2}$.
- Produces path bases of $A_n$
- One can also give a criterion for lifting Jucys–Murphy elements from $Q_n$ to $A_n$.

## *Cellularity, a tool for the non–semisimple theory*

Next, we want to introduce a tool for studying the non–semisimple case, namely the theory of *cellularity*, due to Graham and Lehrer.

## *What is cellularity?*

Let $A$ be an algebra with involution $*$ over an integral domain $S$. $A$ is said to be cellular if there exists a finite partially ordered set $(\Lambda, \geq)$ and for each $\lambda \in \Lambda$, a finite index set $\mathcal{T}(\lambda)$, such that

- $A$ has an $S$–basis $\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Lambda; \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$.
- For every order ideal $\Gamma$ of $\Lambda$,

  $$A(\Gamma) := \operatorname{span}\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Gamma, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$$

  is a $*$–ideal of $A$. In particular, write $A^{\lambda}$ for $A(\{\mu : \mu \geq \lambda\})$ and $\breve{A}^{\lambda}$ for $A(\{\mu : \mu > \lambda\})$.
- $(c^{\lambda}_{\mathfrak{s},\mathfrak{t}})^* \equiv c^{\lambda}_{\mathfrak{t},\mathfrak{s}}$ modulo $\breve{A}^{\lambda}$.
- For each $\lambda \in \Lambda$, there is an $A$–module $\Delta^{\lambda}$, free as $S$–module, with basis $\{c^{\lambda}_{\mathfrak{t}} : \mathfrak{t} \in \mathcal{T}(\lambda)\}$, such that the map $\alpha^{\lambda} : A^{\lambda}/\breve{A}^{\lambda} \to \Delta^{\lambda} \otimes_R (\Delta^{\lambda})^*$ defined by $\alpha^{\lambda} : c^{\lambda}_{\mathfrak{s},\mathfrak{t}} + \breve{A}^{\lambda} \mapsto c_{\mathfrak{s}} \otimes (c^{\lambda}_{\mathfrak{t}})^*$ is an $A$–$A$ bimodule isomorphism.

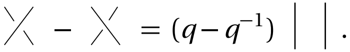
## *What is cellularity?, cont.*

- This whole apparatus is called a cell datum. The $R$–basis is called a cellular basis.
- The modules $\Delta^\lambda$ are called *cell modules*. When the ground ring is a field, and the algebra $A$ is semisimple, these are exactly the simple modules.
- In general, general, $\Delta^\lambda$ has a canonical bilinear form. With $\mathrm{rad}(\lambda)$ the radical of this form, and with the ground ring a field, $\Delta^\lambda/\mathrm{rad}(\lambda)$ is either zero or simple, and all simples are of this form.

# *Cellularity – Example, the Hecke algebras*

*Definition 1*

*The Hecke algebra $H_n^S(q^2)$ over $S$ is the quotient of the braid group algebra over $S$ by the Hecke skein relation:*




*Fact:*

The Hecke algebras $H_n^S(q^2)$ are cellular, with $\Lambda = Y_n$, the set of Young diagrams with $n$ boxes, ordered by *dominance*, and $\mathcal{T}(\lambda)$ the set of standard tableaux of shape $\lambda$. The cell modules are known as *Specht modules*. The cellular structure is due to Murphy. See, for example, A. Mathas, *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*, AMS University Lecture Series.

# *Cellularity – Basis free formulation*

*Definition 2*

*A* $\Lambda$–cell net *is an (order preserving) map from the set of order ideals of* $\Lambda$ *to the set of* $*$*–ideals of* $A$, $\Gamma \mapsto A_\Gamma$, *with several natural properties, the most important being:*

*For each* $\lambda \in \Lambda$, *there is an* $A$*–module* $M^\lambda$, *finitely generated and free as an* $S$*–module, such that whenever* $\Gamma \subseteq \Gamma'$ *are order ideals of* $\Lambda$, *with* $\Gamma' \setminus \Gamma = \{\lambda\}$, *then there exists an isomorphism of* $A$*–*$A$*–bimodules*

$$\alpha : A_{\Gamma'}/A_\Gamma \longrightarrow M^\lambda \otimes_S i(M^\lambda),$$

*satisfying* $i \circ \alpha = \alpha \circ i$, *where* $i$ *is the map, induced from* $*$, *which interchanges left and right* $A$*–modules.*

# *Cellularity – basis free formulation – continued*

*Proposition 3*

*Let $A$ be an $R$–algebra with involution, and let $(\Lambda, \geq)$ be a finite partially ordered set. Then $A$ has a cell datum with partially ordered set $\Lambda$ if, and only if, $A$ has a $\Lambda$–cell net.*

Of course, then $M^\lambda$ will turn out to be the cell module corresponding to $\lambda$.

# *Coherence – Introduction*

It is a general principle that representation theories of the Hecke algebras $H_n(q)$ or of the symmetric group algebras $KS_n$ should be considered all together, that induction/restriction between $H_n$ and $H_{n-1}$ plays a role in building up the representation theory.

*Coherence* of cellular structures is the cellular version of this principle.

# Coherence

*Definition 4*

*A sequence* $(A_n)_{n\geq 0}$ *of cellular algebras, with cell data* $(\Lambda_n, \ldots)$ *is* coherent *if for each* $\mu \in \Lambda_n$*, the restriction of* $\Delta^\mu$ *to* $A_{n-1}$ *has a filtration*

$$\mathrm{Res}(\Delta^\lambda) = F_t \supseteq F_{t-1} \supseteq \cdots \supseteq F_0 = (0),$$

*with* $F_j/F_{j-1} \cong \Delta^{\lambda_j}$ *for some* $\lambda_j \in \Lambda_{n-1}$*, and similarly for induced modules.*

*The sequence* $(A_n)_{n\geq 0}$ *is* strongly coherent *if, in addition,*

$$\lambda_t < \lambda_{t-1} < \cdots < \lambda_1$$

*in* $\Lambda_{n-1}$*, and similarly for induced modules.*

# Strong coherence

Given a strongly coherent sequence $(A_n)_{n\geq 0}$ of cellular algebras, with mild additional assumptions, satisfied in our examples, one always has *path bases* of cell modules and of the algebras themselves.

*Example of strong coherence:* The sequence of Hecke algebras $H_n(q)$ is a strongly coherent sequence of cellular algebras. This results from combining theorems of Murphy, Dipper-James, and Jost from the 80's

# Main theorem

*Theorem 5*

*Suppose* $(A_n)_{n\geq 0}$ *and* $(Q_n)_{n\geq 0}$ *are two sequence of* $*$ *- algebras over* $R$*. Let* $F$ *be the field of fractions of* $R$*. Assume:*

1. $(Q_n)_{n\geq 0}$ *is a (strongly) coherent tower of cellular algebras.*
2. $A_0 = Q_0 = R$, $A_1 \cong Q_1$.
3. *For each* $n \geq 2$, $A_n$ *has an essential idempotent* $e_{n-1} = e^*_{n-1}$ *and*

$$0 \longrightarrow A_n e_{n-1} A_n \longrightarrow A_n \longrightarrow Q_n \longrightarrow 0,$$

   *is a short exact sequence of* $*$ *- algebras.*

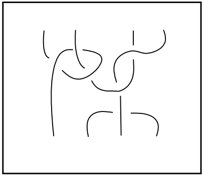
4. $A_n^F = A_n \otimes_R F$ *is split semisimple.*
5. (a) $e_{n-1}$ commutes with $A_{n-1}$, and $e_{n-1}A_{n-1}e_{n-1} \subseteq A_{n-2}e_{n-1}$, (b) $A_n e_{n-1} = A_{n-1}e_{n-1}$ and $x \mapsto xe_{n-1}$ is injective from $A_{n-1}$ to $A_{n-1}e_{n_1}$, and (c) $e_{n-1} = e_{n-1}e_n e_{n-1}$.

*Then* $(A_n)_{n\geq 0}$ *is a (strongly) coherent tower of cellular algebras.*

## *The BMW algebras*

A chief example is $A_n =$ BMW (Birman-Murakami-Wenzl) algebra on $n$ strands, and $Q_n =$ Hecke algebra on $n$ strands.

The BMW algebra is an algebra of braid like objects, namely *framed*

$(n, n)$*–tangles*:  .

Tangles can be represented by quasi-planar diagrams as shown here. Tangles are multiplied by stacking (like braids).

# Definition of BMW algebras

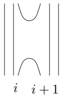
**Definition 6**

*Let $S$ be a commutative unital ring with invertible elements $\rho, q, \delta$ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The BMW algebra $W_n^S$ is the $S$–algebra of framed $(n, n)$–tangles, modulo the Kauffman skein relations:*

1. *(Crossing relation)* $\quad - \quad = (q^{-1} - q)\Big( \quad - \quad \Big).$
2. *(Untwisting relation)* $= \rho$ and $= \rho^{-1}$ .
3. *(Free loop relation)* $T \cup \bigcirc = \delta\, T,$ *where $T \cup \bigcirc$ is the union of a tangle $T$ and an additional closed loop with zero framing.*

# *The BMW algebras, cont.*

- $W_n^S$ imbeds in $W_{n+1}^S$. On the level of tangle diagrams, the embedding is by adding one strand on the right.
- The BMW algebras have an $S$–linear algebra involution, acting by turning tangle diagrams upside down.
- The following tangles generate the BMW algebra

$e_i =$ 

and $g_i =$ 

.

The element $e_i$ is an essential idempotent with $e_i^2 = \delta e_i$.
One has $e_i e_{i\pm1} e_i = e_i$.

## *The BMW algebras, cont. 2*

The ideal $J_n$ generated by one or all $e_i$'s in $W_n^S$ satisfies $W_n^S/J_n \cong H_n^S(q^2)$ where $H_n^S(q^2)$ is the Hecke algebra.

There is a generic ground ring for the BMW algebras, namely

$$R = \mathbb{Z}[\boldsymbol{q}^{\pm 1}, \boldsymbol{\rho}^{\pm 1}, \boldsymbol{\delta}^{\pm 1}]/J,$$

where $J$ is the ideal generated by

$$\boxed{\boldsymbol{\rho}^{-1} - \boldsymbol{\rho} - (\boldsymbol{q}^{-1} - \boldsymbol{q}^{-1})(\boldsymbol{\delta} - 1)}$$

and where the bold symbols denote indeterminants.

# *BMW – Generic ground ring, cont. 3*

The generic ground ring $R$ is an integral domain, with field of fractions $F = \mathbb{Q}(\boldsymbol{q}, \boldsymbol{\rho})$, and $\boldsymbol{\delta} = (\boldsymbol{\rho}^{-1} - \boldsymbol{\rho})/(\boldsymbol{q}^{-1} - \boldsymbol{q}) + 1$ in $F$.

For every instance of the BMW over a ring $S$ with parameters $\rho, q, \delta$, one has $W_n^S \cong W_n^R \otimes_R S$.

The BMW algebras over $F$ are semisimple (theorem of Wenzl). Outline of proof: Assume $W_k^F$ is s.s. for $k \leq n$. The ideal $J_{n+1} \subseteq W_{n+1}^F$ is the Jones basic construction for $W_{n-1}^F \subseteq W_n^F$, so is also s.s. The quotient $W_{n+1}^F / J_{n+1}$ is the Hecke algebra $H_n^F(\boldsymbol{q}^2)$, which is s.s. since $\boldsymbol{q}$ is not a root of 1. Hence $W_{n+1}^F$ is s.s.

## *The BMW algebras, application of our theorem*

Now let's see what's involved in applying the theorem to the BMW algebras (with $A_n = W_n^R$, and $Q_n = H_n^R(q^2)$). Hypothesis (1) is the (strong) coherence of the sequence of Hecke algebras, which is a significant theorem about Hecke algebras. Hypothesis (4) on the semisimplicity of $W_n^F$ is Wenzl's theorem. Everything else is elementary, and already contained in Birman-Wenzl.

All the other examples work pretty much the same way.

# *Some idea of the proof*

The proof is inductive and is a cellular version of Wenzl's semisimplicity proof. Suppose we know that $A_k$ is cellular (and satisfies all the conclusions of the theorem) for $k \leq n$.

Then we want to show the same for $A_{n+1}$. The main point is to show that $J_{n+1} = A_{n+1} e_n A_{n+1} = A_n e_n A_n$ is a "cellular ideal" in $A_{n+1}$. (This suffices to show cellularity, because we also have that $A_{n+1}/J_{n+1} \cong Q_{n+1}$ is cellular by hypothesis, and extensions of cellular algebras by cellular ideals are also cellular.)

# *Some idea of the proof–cont.*

We have a $\Lambda_{n-1}$–cell net $\Gamma \mapsto J(\Gamma) := \mathrm{span}\{c^{\lambda}_{\mathfrak{s},\mathfrak{t}} : \lambda \in \Gamma, \mathfrak{s}, \mathfrak{t} \in \mathscr{T}(\lambda)\}$. Now we want to show that

$$\Gamma \mapsto \hat{J}(\Gamma) := A_n e_n J(\Gamma) A_n$$

is a $\Lambda_{n-1}$–cell net in $A_n e_n A_n$. Along the way to doing this we also have to show that

$$J'(\Gamma) := A_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} A_n \cong A_n e_n J(\Gamma) A_n$$

and in particular $A_n \otimes_{A_{n-1}} A_n \cong A_n e_n A_n$, and if $\Gamma_1 \subseteq \Gamma_2$, then also $J'(\Gamma_1)$ imbeds in $J'(\Gamma_2)$. This is a bit tricky because $A_n e_n A_n$ is not a unital algebra and $A_n$ is not projective as $A_{n-1}$–module.

# *Last slide!*

Now if $\lambda \in \Lambda_{n-1}$ and $\Gamma_1 \subseteq \Gamma_2$, with $\Gamma_2 \setminus \Gamma_1 = \lambda$, then

$$\begin{aligned} \hat{J}(\Gamma_2)/\hat{J}(\Gamma_1) &\cong J'(\Gamma_2)/J'(\Gamma_1) \\ &\cong A_n \otimes_{A_{n-1}} J(\Gamma_1)/J(\Gamma_2) \otimes_{A_{n-1}} A_n \\ &\cong A_n \otimes_{A_{n-1}} (\Delta^\lambda \otimes_R (\Delta^\lambda)^*) \otimes_{A_{n-1}} A_n \\ &\cong (A_n \otimes_{A_{n-1}} \Delta^\lambda) \otimes_R (\Delta^\lambda)^* \otimes_{A_{n-1}} A_n) \end{aligned}$$

Now we need that $M^\lambda = A_n \otimes_{A_{n-1}} \Delta^\lambda$ is free as $R$–module, to verify the crucial property in the definition of a cell net. But as $A_n$–module, $M^\lambda$ is $\mathrm{Ind}(\Delta^\lambda)$, and by an induction assumption on coherence of the cellular structures on $(A_k)_{k \leq n}$, this has a filtration by cell modules for $A_n$, so is free as an $R$–module.