Free Probability and Random Matrices

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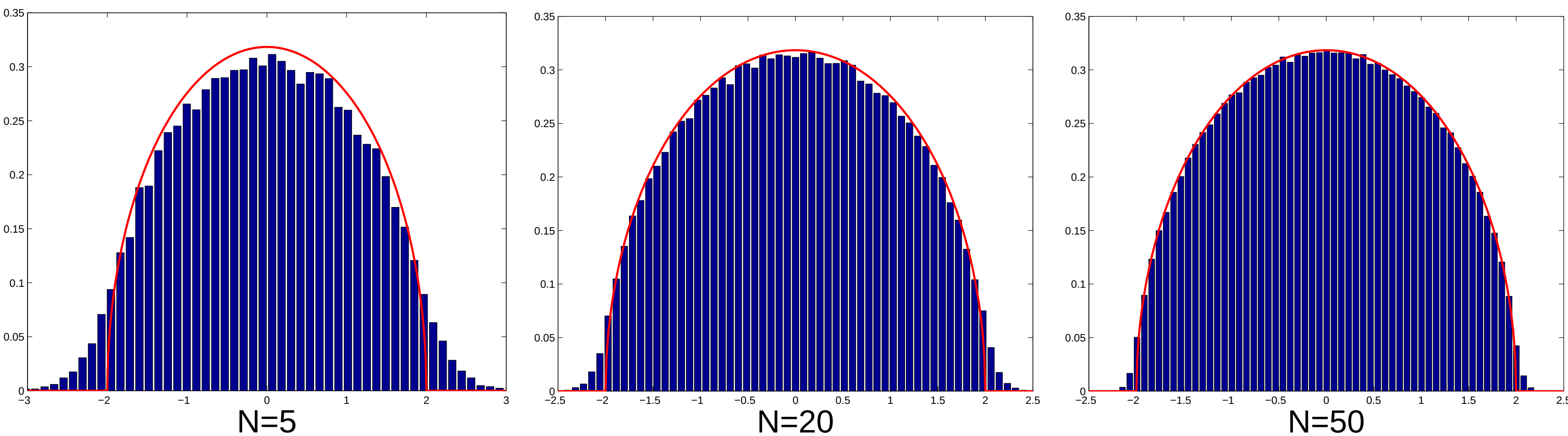
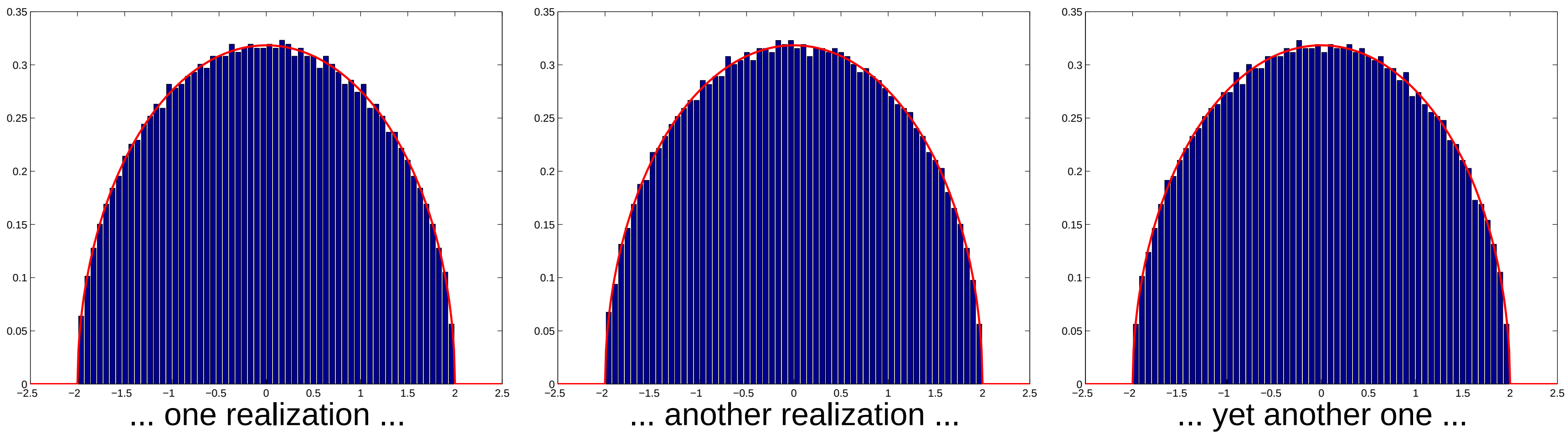
Convergence of averaged eigenvalue distribution of $N \times N$ **Gaussian random matrices** to **Wigner's semicircle**



number of realizations $= 10000$

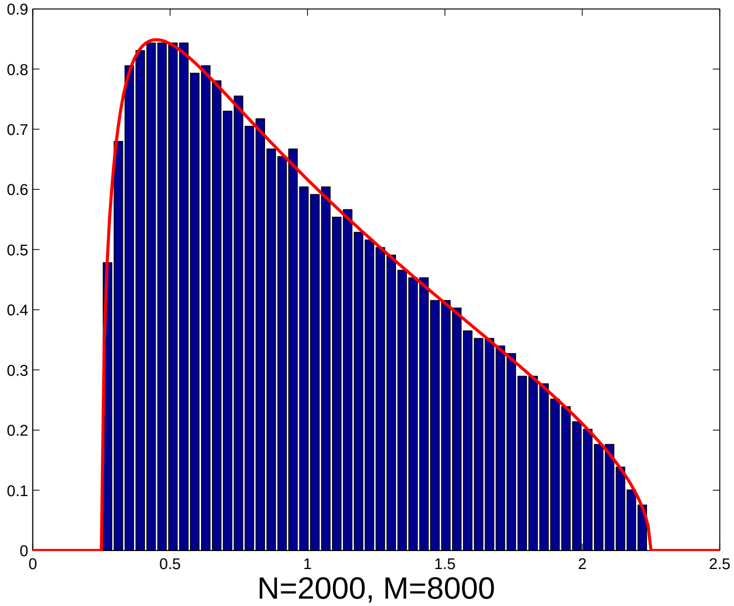
But we also have almost sure convergence of **Gaussian random matrices** to **Wigner's semicircle**.

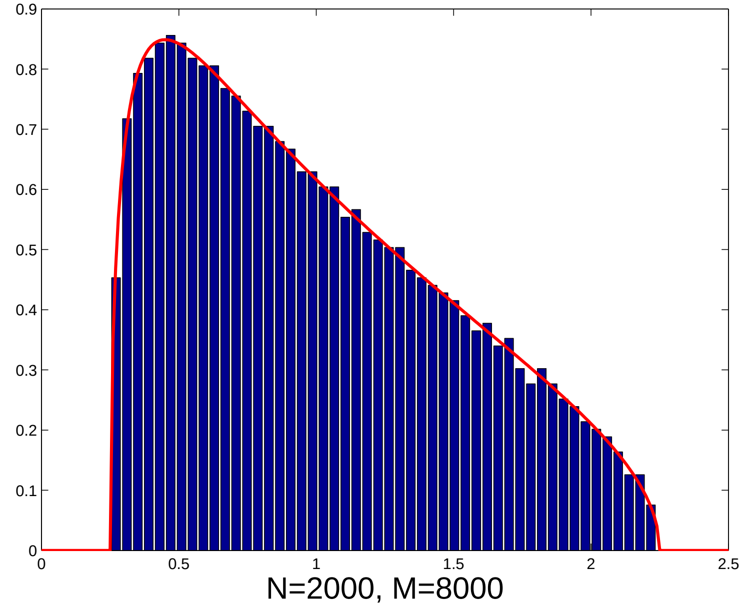
$N = 4000$



Consider **Wishart random matrix** $A = XX^*$, where X is $N \times M$ random matrix with independent Gaussian entries.
Its eigenvalue distribution converges almost surely towards

Marchenko-Pastur distribution.





$$N = 2000, M = 8000$$

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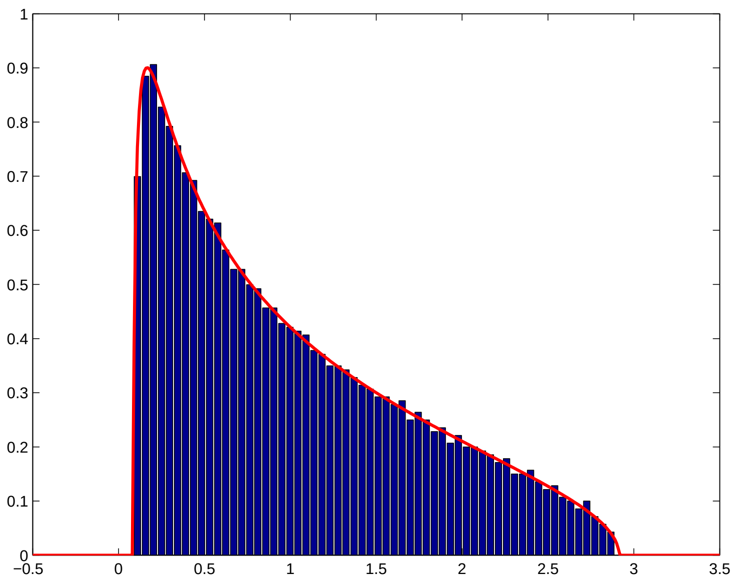
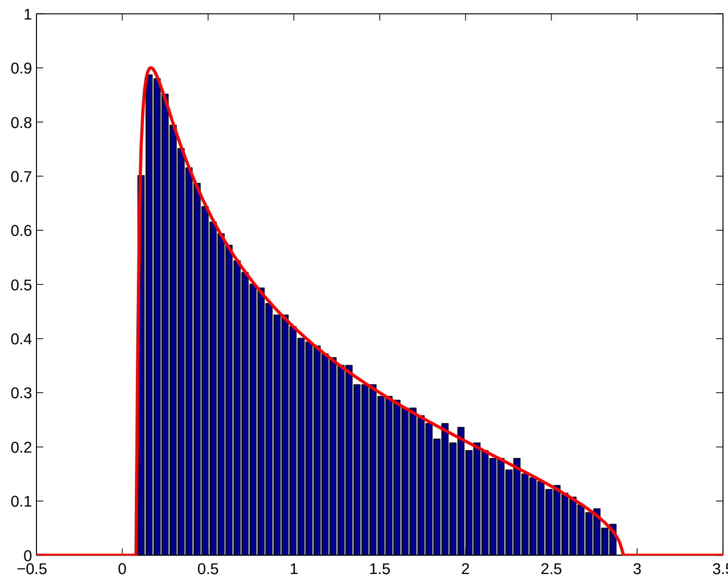


... one realization ...



... another realization ...

$$N = 3000, M = 6000$$

Asymptotic freeness of two Gaussian random matrices A_N, B_N:

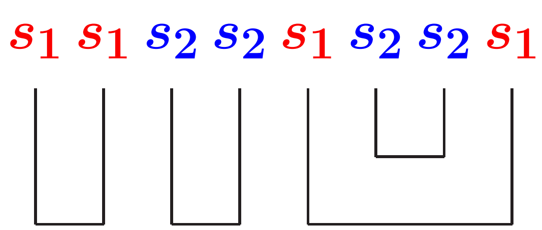
$$A_N, B_N \xrightarrow{\text{distr}} s_1, s_2,$$

where s_1, s_2 are free semicircular elements.

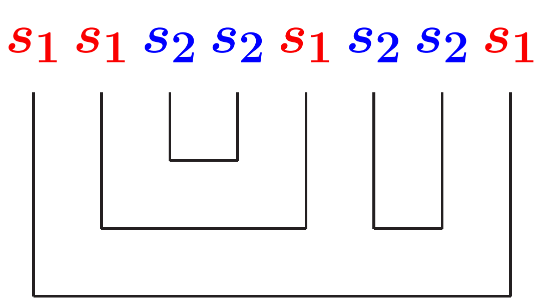
This means, for example, that

$$\lim_{N\to\infty} \mathrm{tr}(A_N A_N \boldsymbol{B}_N \boldsymbol{B}_N A_N \boldsymbol{B}_N \boldsymbol{B}_N A_N) = \varphi(\boldsymbol{s_1 s_1 s_2 s_2 s_1 s_2 s_2 s_1})$$

We have $\varphi(\boldsymbol{s_1 s_1 s_2 s_2 s_1 s_2 s_2 s_1}) = 2$, since there are two non-crossing pairings which respect the color:

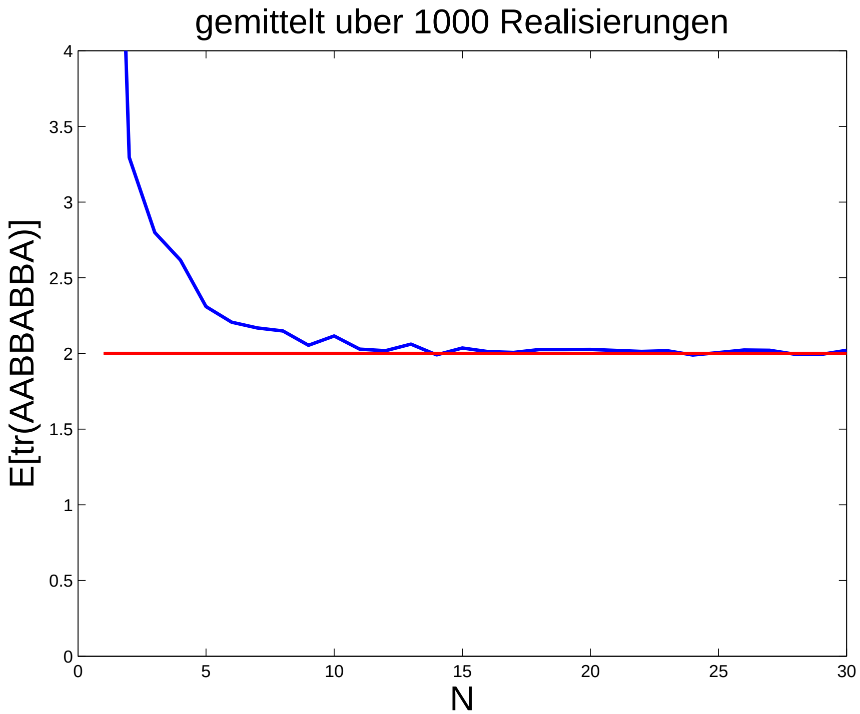


and



$$\mathrm{tr}(A_N A_N B_N B_N A_N B_N B_N A_N)$$

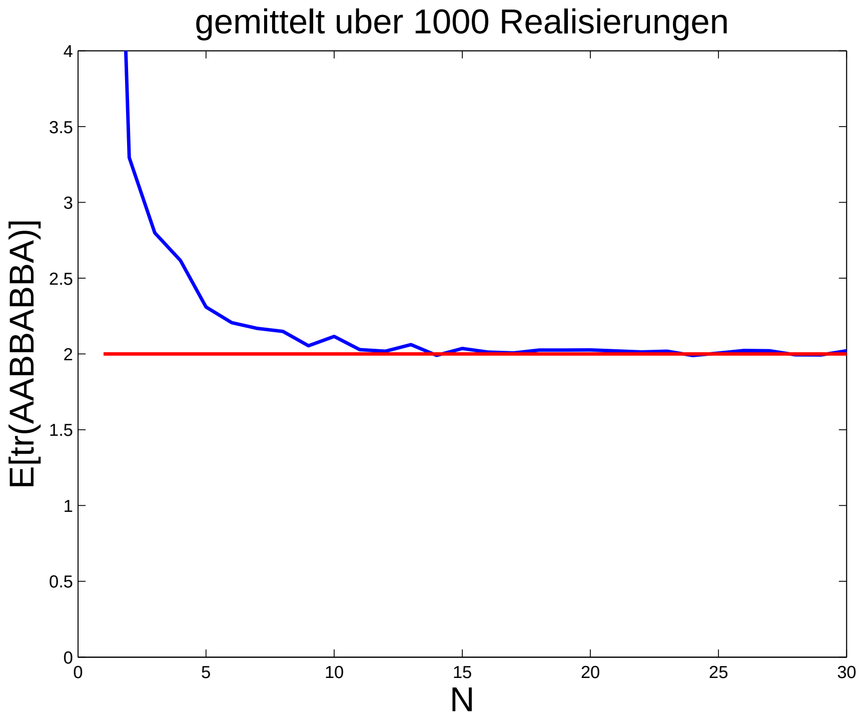
averaged over 1000 realizations

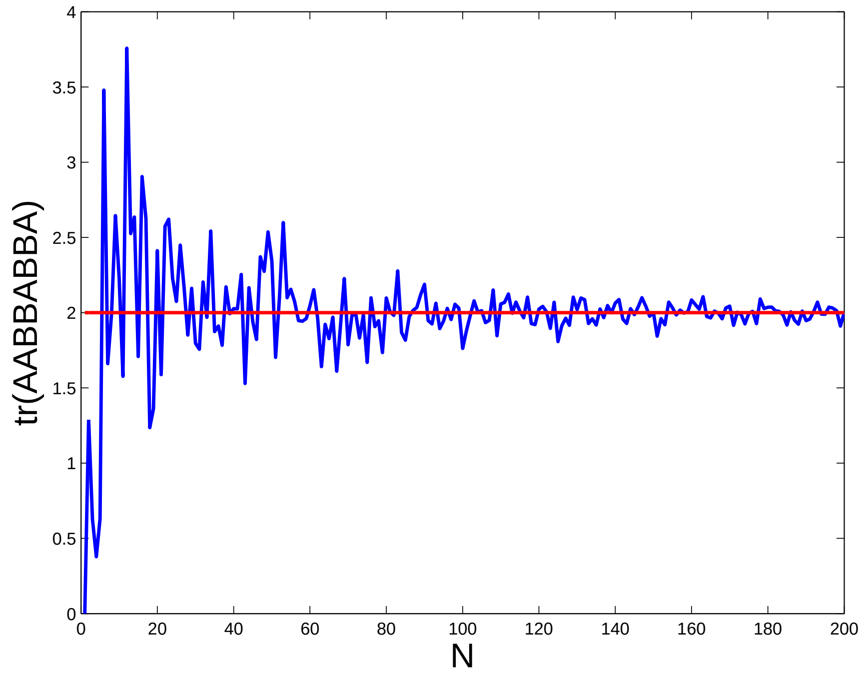


$$\text{tr}(A_N A_N B_N B_N A_N B_N B_N A_N)$$

averaged over 1000 realizations

one realization





The combinatorial relation between moments $(\varphi(A^m))_{m\in\mathbb{N}}$ and free cumulants $(\kappa_m)_{m\in\mathbb{N}}$ can be translated into generating power series.

Put

$$G(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\varphi(A^m)}{z^{m+1}} \qquad \text{Cauchy transform}$$

and

$$\mathcal{R}(z) = \sum_{m=1}^{\infty} \kappa_m z^{m-1} \qquad \textbf{$\mathcal{R}$-transform}$$

Then we have the relation

$$\frac{1}{G(z)} + \mathcal{R}(G(z)) = z.$$

Theorem [Voiculescu 1986, Speicher 1994]:
Let A and B be free. Then one has

$$\mathcal{R}^{A+B}(z) = \mathcal{R}^{A}(z) + \mathcal{R}^{B}(z),$$

or equivalently

$$\kappa_m^{A+B} = \kappa_m^{A} + \kappa_m^{B} \qquad \forall\, m.$$

This, together with the relation between Cauchy transform and $\mathcal{R}$-transform and with the Stieltjes inversion formula, gives an effective algorithm for calculating free convolutions, i.e., the **asymptotic eigenvalue distribution** of sums of random matrices which are **asymptotically free**:

$$\begin{array}{ccccccccc} A & \rightsquigarrow & G^A & \rightsquigarrow & R^A & & & & \\ & & & & \downarrow & & & & \\ & & & & R^A + R^B = R^{A+B} & \rightsquigarrow & G^{A+B} & \rightsquigarrow & A+B \\ & & & & \uparrow & & & & \\ B & \rightsquigarrow & G^B & \rightsquigarrow & R^B & & & & \end{array}$$

Consider independent Gaussian random matrix and Wishart random matrix. They are asypmptotically free.

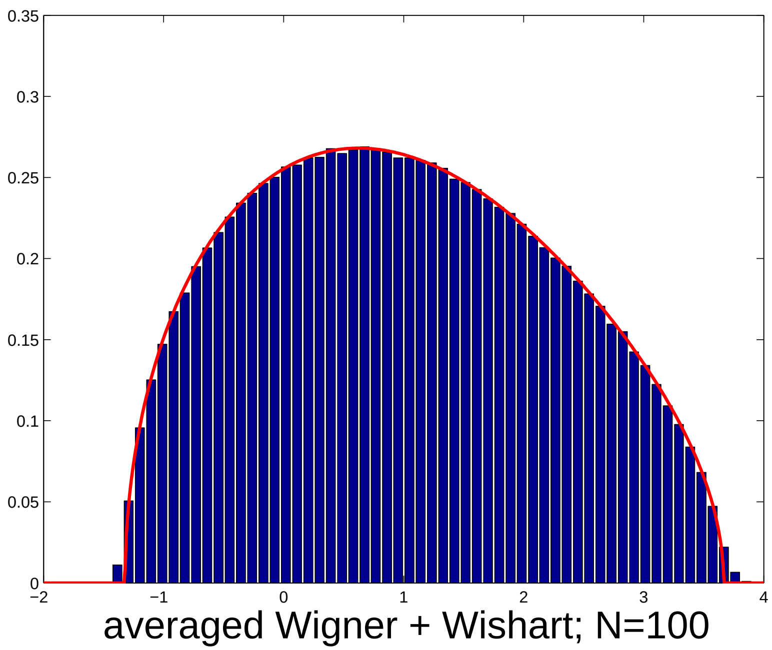
So the asymptotic eigenvalue distribution of

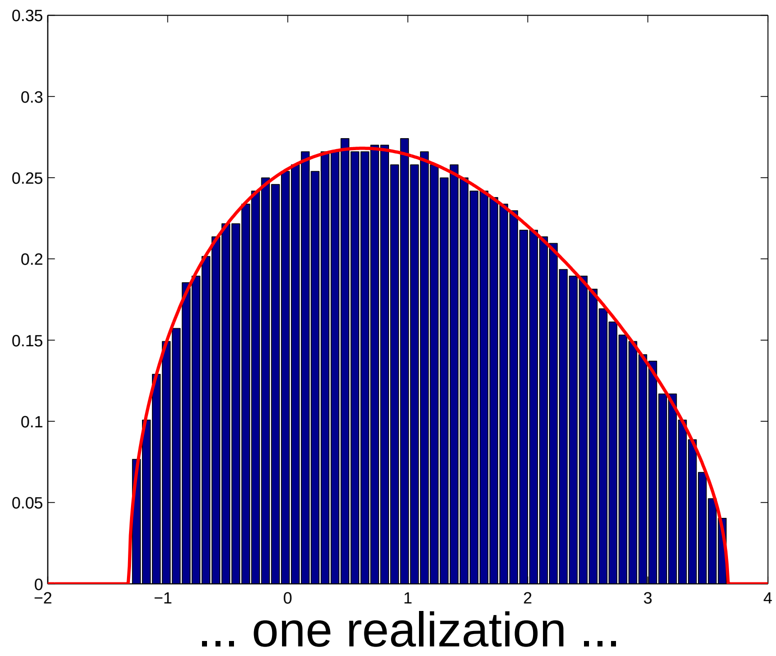
$$\text{Gaussian} + \text{Wishart}$$

should be given by

$$\text{semicircle} \quad \boxplus \quad \text{Marchenko-Pastur}$$

Example: Gaussian + Wishart ($M = 2N$)





trials=4000

N=3000

Consider $A + UBU^*$, where U is Haar random matrix and A and B are diagonal matrices, each with $N/2$ eigenvalues 0 and $N/2$ eigenvalues 1/2.

A and UBU^* are asymptotically free.

Thus the asymptotic eigenvalue distribution of the sum

$$A + UBU^*$$

should be

$$\Big(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{1/2}\Big) \boxplus \Big(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{1/2}\Big) = \text{arcsine}$$

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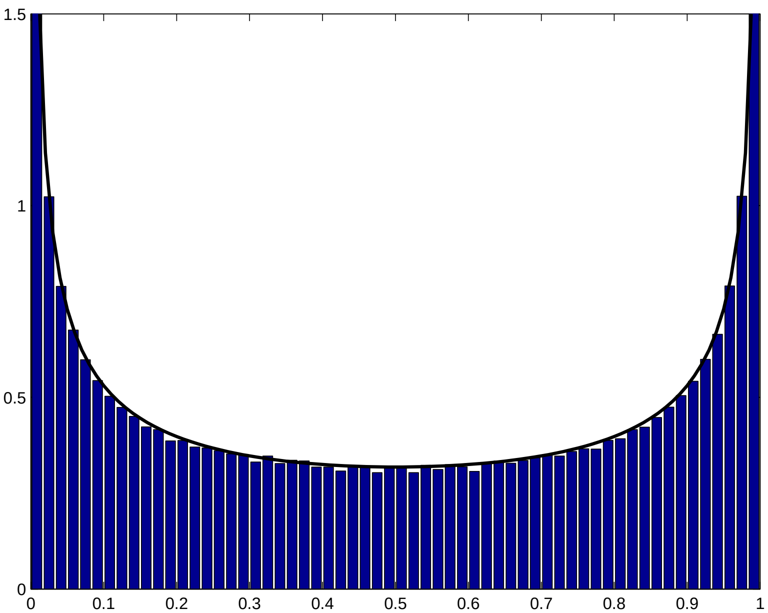
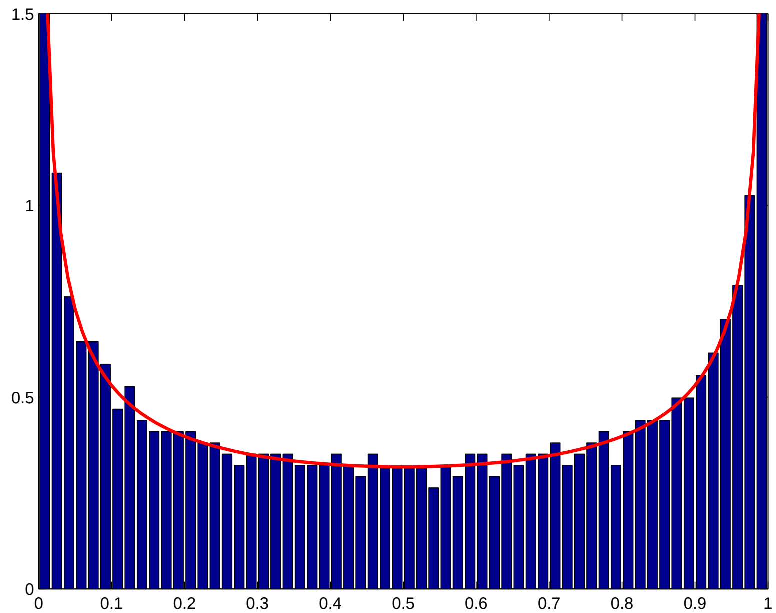


N=64



... one realization ...

trials=5000

N=2048

One has also analytic description for product.

Theorem [Voiculescu 1987, Haagerup 1997, Nica + Speicher 1997]:
Put

$$M_A(z) := \sum_{m=1}^{\infty} \varphi(A^m) z^m$$

and define

$$S_A(z) := \frac{1+z}{z} M_A^{<-1>}(z) \qquad \textbf{\textit{S}-transform of } \boldsymbol{A}$$

Then: If A and B are free, we have

$$S_{AB}(z) = S_A(z) \cdot S_B(z).$$

Consider two independent Wishart random matrices. They are asymptotically free.

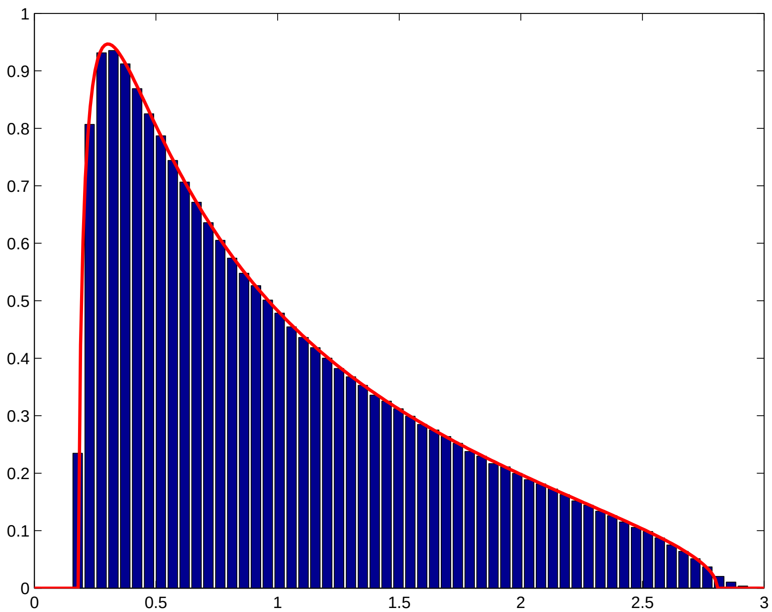
So the asymptotic eigenvalue distribution of

$$\text{Wishart} \times \text{Wishart}$$

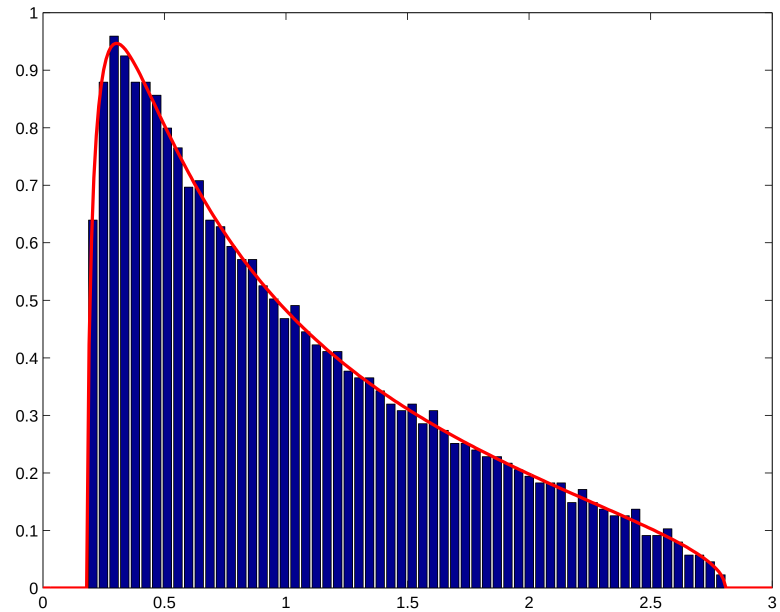
should be given by

$$\text{Marchenko-Pastur} \quad \boxtimes \quad \text{Marchenko-Pastur}$$

Example: Wishart x Wishart ($M = 5N$)



averaged Wishart x Wishart; N=100



... one realization ...

trials=10000

N=2000

THE END