# Complete invariants for complex semisimple Hopf algebras

Sumanth Datt[1], Vijay Kodiyalam[2] and V.S. Sunder[3]
University of Hyderabad[1] and Institute of Mathematical Sciences[2,3]
INDIA

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**Abstract**

We obtain a complete (and finite) list of isomorphism invariants of complex semisimple Hopf algebras of a fixed dimension. We do this by proving a generalisation of a theorem due to Procesi and Razmyslov (which, in turn, was used to prove Artin's conjecture).

## 1 Introduction

In this paper, we consider the problem of distinguishing two complex semisimple Hopf algebras of dimension $n$ specified in terms of their structure constants with respect to some bases. As a solution to this problem, we give a finite list of polynomials in the structure constants that are isomorphism invariant and that distinguish the isomorphism classes.

The methods are those of classical invariant theory [Wyl] supplemented by the diagrammatic formalism of Hopf algebras due to Kuperberg [Kpr] as expounded by Kauffman and Radford [KffRdf]. We also rely on the theorem of Stefan [Stf] that there are only finitely many complex semisimple Hopf algebras of any fixed dimension.

In §2 we show that complex semisimple Hopf algebras (of dimension $n$) form a nonsingular subvariety of the variety of complex bialgebras (of dimension $n$) and that polynomial invariants separate their isomorphism classes. We devote §3 - which is self contained - to a proof of a result in invariant theory that generalises the Procesi-Razmyslov theorem proving Artin's conjecture. §4 describes the finite list of polynomial invariants which distinguish Hopf algebras. A final §5 contains some remarks, examples and questions.

## 2 The variety of semisimple Hopf algebras

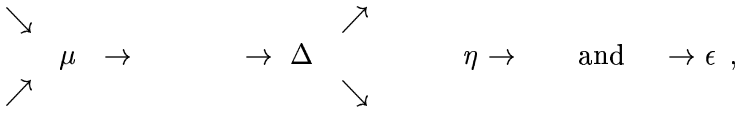
For the rest of this paper, we fix a positive integer $n$ which will be the dimension of the bialgebras and Hopf algebras that we consider. Let $V$ be a complex vector space of dimension $n$ and $v_1, v_2, \cdots v_n$ be a fixed basis of $V$.

A bialgebra structure on $V$ is specified by giving its structure constants with respect to this basis. With the usual notations $\mu, \Delta, \eta$ and $\epsilon$ for the multiplication, comultiplication, unit and counit maps respectively, a bialgebra structure on $V$ is specified by giving complex numbers $\mu^i_{jk}$, $\Delta^{jk}_i$, $\eta^i$ and $\epsilon_i$ - here and in the sequel, all indices range from 1 to $n$ and we will use the Einstein summation convention where each index that occurs as an 'upper' index and a 'lower' index in a product is summed over its range - that satisfy the following equations:

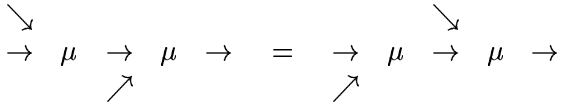
$$\begin{aligned}
\mu^t_{jk}\mu^i_{tl} &= \mu^t_{kl}\mu^i_{jt} \\
\Delta^{jt}_i\Delta^{kl}_t &= \Delta^{tl}_i\Delta^{jk}_t \\
\eta^t\mu^j_{it} = \ \delta^j_i \ &= \eta^t\mu^j_{ti} \\
\Delta^{tj}_i\epsilon_t = \ \delta^j_i \ &= \Delta^{jt}_i\epsilon_t \\
\mu^t_{ij}\Delta^{kl}_t &= \Delta^{pq}_i\Delta^{rs}_j\mu^k_{pr}\mu^l_{qs} \\
\eta^t\Delta^{ij}_t &= \eta^i\eta^j \\
\mu^t_{ij}\epsilon_t &= \epsilon_i\epsilon_j \\
\eta^t\epsilon_t &= 1
\end{aligned}$$

Thus the bialgebra structures on $V$ form an affine variety $B \subseteq \mathbb{A}^d_{\mathbb{C}}$ where $d = 2n^3 + 2n$.

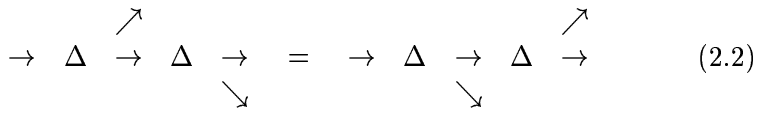
These equations are easier to appreciate in the symbolic notation due to Kuperberg [Kpr] as explained in Kauffman and Radford [KffRdf]. We will give a very brief summary of this. The bialgebra structure maps are represented as:




while the equations defining the variety $B$ are symbolically written:




(2.1)



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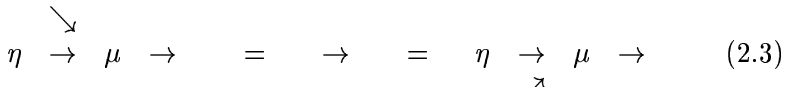

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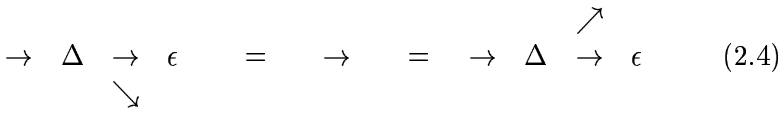

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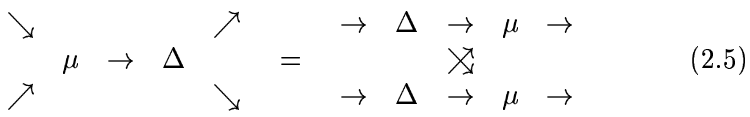

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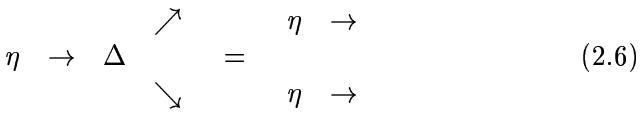

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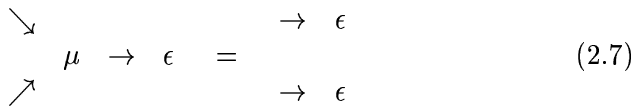


(2.7)

(2.8)

Equations (2.1) - (2.8) are to be interpreted thus. Each is an equality of one or more 'pictures'. A picture with $k$ inputs and $l$ outputs represents a map from $V^{\otimes k}$ to $V^{\otimes l}$. By convention, the inputs for each picture are read anticlockwise and the outputs clockwise. A general endomorphism $\rho$ of $V$ is represented by $\rightarrow \rho \rightarrow$ while the identity endomorphism of $V$ is represented by $\rightarrow$.

There is also a structure constant interpretation for such a picture as a tensor with $l$ upper and $k$ lower indices. A picture that has a 'bound' arrow - one that is neither an input nor an output - involves a contraction of a tensor. In a structure constant interpretation of a pictorial equation, each arrow is decorated with an index, bound arrows correspond to summing over the corresponding index, and the

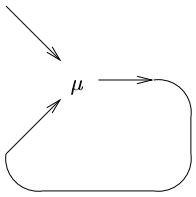



Figure 1: An element of $V^*$

equation is deemed to hold for all values of the indices of the 'free' arrows.

We will illustrate these interpretations for a picture that will play a particularly important role in the sequel. Consider the picture in Figure 1 which has one input and no outputs and so represents an element of $V^*$. In terms of structure constants, this corrresponds to the picture of Figure 2 which is read as $\mu^t_{it}$. The free arrow here is

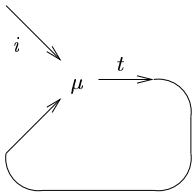



Figure 2: Structure constant interpretation

labelled by $i$ and the bound arrow by $t$.

Fix a point $(\mu, \Delta, \eta, \epsilon)$ on $B$. This gives a bialgebra structure on $V$ for which, for instance, $v_i v_j = \mu^t_{ij} v_t$. The trace of $v_i$ in the left regular representation is therefore $\mu^t_{it}$. Hence the picture in Figure 1 represents the trace on $V$ in its left regular representation.

Similarly, it may be verified that the picture in Figure 3 represents

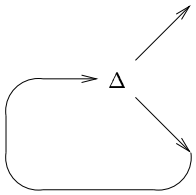



Figure 3: An element of $V$

the trace on $V^*$ in its left regular representation. In order to simplify drawing various pictures that we will need, we will henceforth use $\rightarrow \phi$ for the picture in Figure 1 and $h \rightarrow$ for the picture in Figure 3.

There is a natural action of the group $G = GL_n(\mathbb{C})$ on $\mathbb{A}^d_{\mathbb{C}}$ defined as follows: For a point $(\mu, \Delta, \eta, \epsilon) = (\mu^i_{jk},\ \Delta^{jk}_i,\ \eta^i, \epsilon_i) \in \mathbb{A}^d_{\mathbb{C}}$ and $g \in G$, define $g \cdot (\mu, \Delta, \eta, \epsilon) = (\tilde{\mu}, \tilde{\Delta}, \tilde{\eta}, \tilde{\epsilon})$ where

$$\begin{aligned}
\tilde{\mu}^i_{jk} &= g^i_p (g^{-1})^q_j (g^{-1})^r_k \mu^p_{qr}, \\
\tilde{\Delta}^{jk}_i &= (g^{-1})^p_i g^j_q g^k_r \Delta^{qr}_p, \\
\tilde{\eta}^i &= g^i_p \eta^p, \quad \text{and} \\
\tilde{\epsilon}_i &= (g^{-1})^p_i \epsilon_p.
\end{aligned}$$

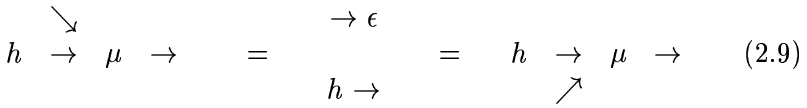
It is easy to see that this action carries $B$ onto itself and that points of $B$ lie in the same $G$-orbit precisely when they correspond to isomorphic bialgebra structures on $V$.

We will be interested in the points on $B$ that correspond to semisimple Hopf algebra structures. We summarise some well known facts about such Hopf algebras in the following proposition. See [LrsRdf] and [LrsRdf2] for proofs. Recall that a two-sided integral in a Hopf algebra is an element $h$ such that $hx = \epsilon(x)h = xh$ for each $x$ in the algebra.

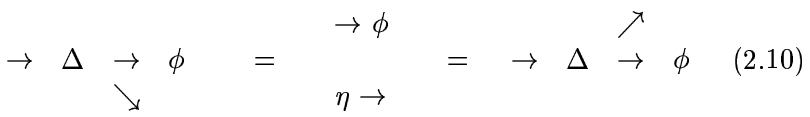
PROPOSITION 1. *Let $H$ be a complex semisimple Hopf algebra of dimension $n$ with antipode $S$ and let $H^*$ be the dual Hopf algebra. Let $\phi \in H^*$ (resp. $h \in H$) be the trace on $H$ (resp. $H^*$) in its left regular representation. Then,*
*(a) $H^*$ is also semisimple,*
*(b) $\phi$ (resp. $h$) is a two-sided integral for $H^*$ (resp. $H$),*
*(c) $\phi(h) = n$, and*
*(d) $S$ is involutive, i.e., $S^2 = id_H$.* □

Let $SCH$ be the subset of $B$ of all points that give semisimple Hopf algebra structures on $V$.

LEMMA 2. *For a point $(\mu, \Delta, \eta, \epsilon) \in B$, the following two conditions are equivalent :*
*(i) $(\mu, \Delta, \eta, \epsilon) \in SCH$.*
*(ii) The following pictorial equations hold:*

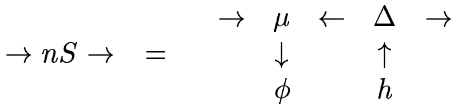



(2.9)



$$h \;\rightarrow\; \phi \;=\; n. \tag{2.11}$$

*Proof.* Since the pictorial equations are equivalent to the requirements that $\phi$ and $h$ be two-sided integrals for $H$ and $H^*$ respectively with $\phi(h) = n$, Proposition 1(b,c) show that $(i) \Rightarrow (ii)$. To see the reverse implication, consider a point $(\mu, \Delta, \eta, \epsilon) \in B$ satisfying the equations (2.9) - (2.11) and define an endomorphism $S$ of $V$ by the equation:



The calculation in Figure 4 below shows that $\sum_{(x)} S(x_{(1)})x_{(2)} = \epsilon(x)\ 1$ (in 'Sweedler's notation'),

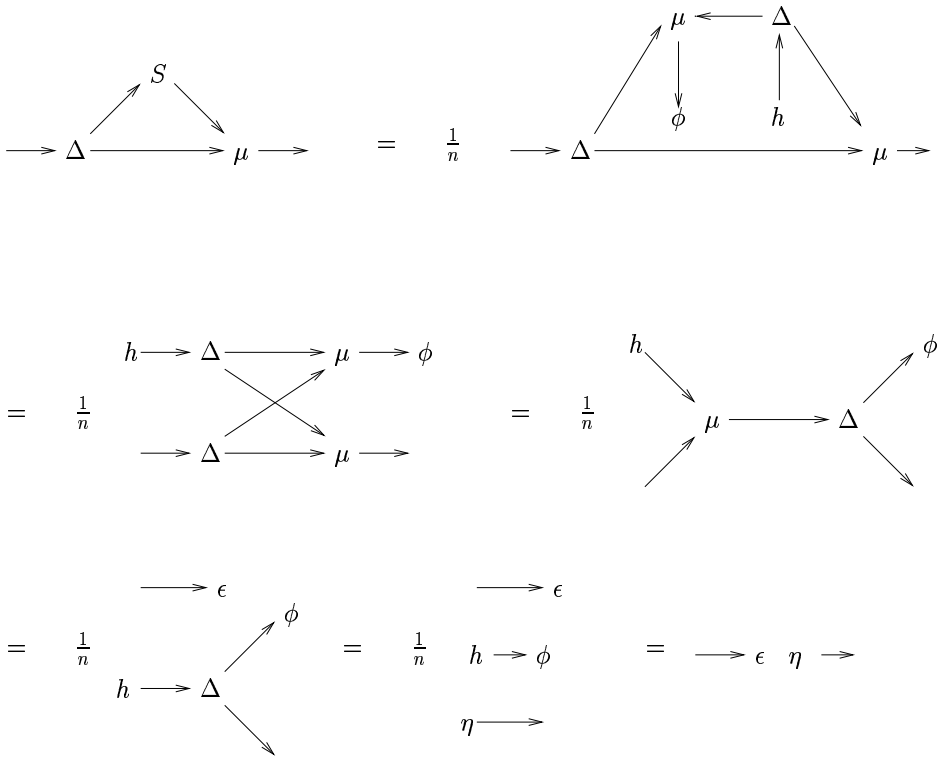


Figure 4: Antipode verification

while another such similar calculation shows that $\sum_{(x)} x_{(1)}S(x_{(2)}) = \epsilon(x).1$. Hence this bialgebra structure on $V$ admits an antipode and is therefore a Hopf algebra structure on $V$. Consider now the calculation in Figure 5 where the first equality follows from equation (2.4) and the second since the trace of the identity endomorphism of $V$ is $n$. This shows that $\epsilon(h) = n \neq 0$ and therefore by [LrsSwd] the Hopf

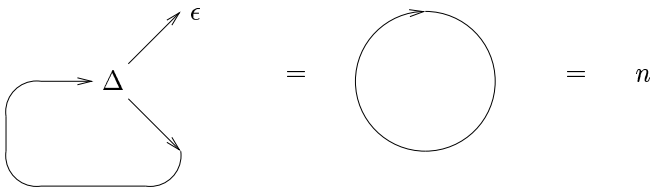



Figure 5: Semisimplicity verification

algebra structure on $V$ is semisimple. Thus $(ii) \Rightarrow (i)$. □

COROLLARY 3. *Every point of SCH satisfies the following pictorial equation:*

$$\begin{array}{ccccccccccc} \to & \mu & \leftarrow & \Delta & \to & \mu & \leftarrow & \Delta & \to & & \\ & \downarrow & & \uparrow & & \downarrow & & \uparrow & & = & n^2\ (\ \to\ ) \\ & \phi & & h & & \phi & & h & & & \end{array} \qquad (2.12)$$

*Proof.* The lemma states an equality of two endomorphisms of $V$ - the right side being $n^2$ times the identity endomorphism , and the left side being the square of the endomorphism

$$\begin{array}{ccccccc} \to & \mu & \leftarrow & \Delta & \to & & \\ & \downarrow & & \uparrow & & ; & \\ & \phi & & h & & & \end{array}$$

this latter endomorphism is, however, seen to be nothing but $nS$ (see the displayed picture defining $nS$ in the proof of Lemma 2). An appeal to Proposition 1(d) completes the proof. □

PROPOSITION 4. *The subset SCH is a nonsingular G-stable (closed) subvariety of B that is a union of finitely many closed G-orbits.*

*Proof.* Since isomorphic bialgebra structures on $V$ lie in the same $G$-orbit, $SCH$ is a union of $G$-orbits and hence $G$-stable, while Lemma 2 shows that $SCH$ is a closed subvariety of $B$. By the results of Stefan - see Corollary 1.5, Corollary 1.6, Theorem 2.1 of [Stf] - and

Proposition 1(a), there are only finitely many orbits of semisimple Hopf algebras in $B$ each of which is open in $B$ and therefore also in $SCH$. Thus each such orbit is also closed in $SCH$ and being the finite disconnected union of closed nonsingular orbits, $SCH$ is itself nonsingular. □

Consider the dual action of the group $G$ on the coordinate ring $R = \mathbb{C}[\mu^i_{jk}, \Delta^{jk}_i, \eta^i, \epsilon_i]$ of $\mathbb{A}^d_{\mathbb{C}}$, which is a polynomial ring in $d = 2n^3 + 2n$ variables. Let $R^G$ denote the ring of invariants. Given a bialgebra $A$ of dimension $n$ and an element $f \in R^G$ one may 'evaluate $f$ on $A$' by taking the structure constants of $A$ with respect to an arbitrary basis as the coordinates of a point on $B$ and evaluating $f$ at that point. The result, which we will denote $f(A)$, is independent of the chosen basis since a change of basis corresponds to moving in a $G$-orbit on $B$ and $f \in R^G$.

COROLLARY 5. *Two complex semisimple Hopf algebras $H_1$ and $H_2$ of dimension $n$ are isomorphic if and only if for each $f \in R^G$, we have $f(H_1) = f(H_2)$.*

*Proof.* If $H_1$ and $H_2$ are isomorphic, then they have the same structure constants with respect to appropriately chosen bases and therefore for each $f \in R^G$, $f(H_1) = f(H_2)$. Conversely, if $H_1$ and $H_2$ are not isomorphic, then their structure constants with respect to any choice of bases belong to different $G$-orbits in $SCH$. Since the $G$-orbits in $SCH$ are closed by Proposition 4, it follows from what [MmfFgrKrw] refers to as the 'only really important geometric property implied by the reductivity of $G$' - see Corollary 1.2 of Chapter 1, §2 - that there is an $f \in R^G$ that is 1 on $H_1$ and 0 on $H_2$. □

REMARK 6. *The results of this section hold,* mutatis mutandis, *when $SCH$ is the subset of semisimple and cosemisimple Hopf algebra structures - and therefore the choice of notation $SCH$ - of the variety $B$ of bialgebra structures on an $n$-dimensional vector space over an algebraically closed field of arbitrary characteristic.*

## 3 Invariants of tensors

Let $V$ be a complex vector space of dimension $n$, and let $G = GL(V)$. For non-negative integers $t$ and $b$, let $V^t_b$ be the $G$-module $V^{\otimes\, t} \otimes (V^*)^{\otimes\, b}$.

Given tuples $(t_i, b_i)$ of non-negative integers, for $i = 1, 2, \cdots, k$, consider the $G$-module defined by $W$ $(= W(\{(t_i, b_i) : i = 1, 2, \cdots, k\}))$

$= \oplus_{i=1}^{k} V_{b_i}^{t_i}$. We wish to describe, in this section, the polynomial invariants of the $G$-module $W$ - by which is meant the following: regard $W$ as an affine variety with a $G$ action and consider the dual action on the coordinate ring $\mathbb{C}[W^*]$; a polynomial invariant of $W$ is just a $G$-invariant element of $\mathbb{C}[W^*]$.

Explicitly, choose a basis $v_1, \cdots, v_n$ of $V$ and let $v^1, \cdots v^n$ be the dual basis of $V^*$. Then, a basis of $V_b^t$ is given by all $v_{u_1} \otimes v_{u_2} \otimes \cdots \otimes v_{u_t} \otimes v^{l_1} \otimes \cdots \otimes v^{l_b}$ where the indices $u_1, \cdots u_t, l_1, \cdots, l_b$ all range from 1 to $n$. Let $T_{l_1 \cdots l_b}^{u_1 \cdots u_t}$ be the coordinate function on $V_b^t$ that gives the coefficient of $v_{u_1} \otimes v_{u_2} \otimes \cdots \otimes v_{u_t} \otimes v^{l_1} \otimes \cdots \otimes v^{l_b}$. Thus $\mathbb{C}[(V_b^t)^*]$ is identified with the polynomial ring $\mathbb{C}[T_{l_1 \cdots l_b}^{u_1 \cdots u_t}]$ in $n^{b+t}$ variables. The group $G$ is identified with $GL_n(\mathbb{C})$ using the basis of $V$. The dual action of $G$ on this polynomial ring is then given by:

$$g(T_{l_1 \cdots l_b}^{u_1 \cdots u_t}) = \prod_{i=1}^{t} (g^{-1})_{\tilde{u}_i}^{u_i} \prod_{j=1}^{b} (g)_{l_j}^{\tilde{l}_j} \; T_{\tilde{l}_1 \cdots \tilde{l}_b}^{\tilde{u}_1 \cdots \tilde{u}_t},$$

where, of course, the summation convention is used.

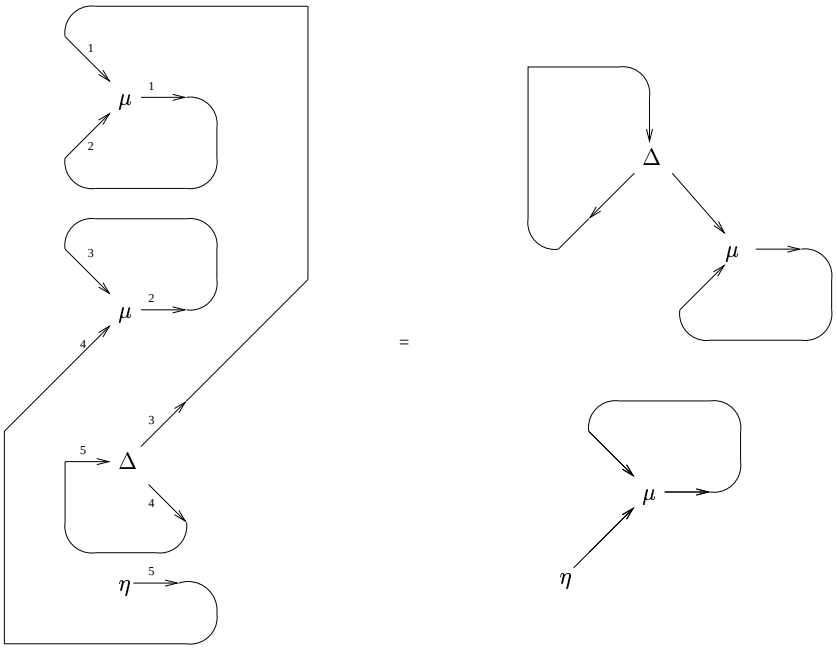
More generally, for $W = W(\{(t_i, b_i) : i = 1, 2, \cdots, k\})$, we identify the coordinate ring $\mathbb{C}[W^*]$ with the polynomial ring $\mathbb{C}[T(i)_{l_1 \cdots l_{b_i}}^{u_1 \cdots u_{t_i}}]$ in $\sum_{i=1}^{k} n^{b_i + t_i}$ variables. Here, $i$ ranges from 1 to $k$ and all the indices of the $T(i)$ from 1 to $n$. This polynomial ring has a $\mathbb{N}^k$-grading where $\deg(T(i)_{l_1 \cdots l_{b_i}}^{u_1 \cdots u_{t_i}}) = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ - where the 1 is in the $i^{\text{th}}$ place - independent of the sub- and super-scripts. The $G$-action preserves this grading and so the ring of invariants is a graded subring of $\mathbb{C}[W^*]$.

Fix $W = W(\{(t_i, b_i) : i = 1, 2, \cdots, k\})$. By a **picture invariant on** $W$ we shall mean the following: it is determined by the data of (a) a $k$-tuple of non-negative integers $\underline{m} = (m_1, \cdots, m_k)$ such that $\sum_{i=1}^{k} m_i t_i = \sum_{i=1}^{k} m_i b_i = N$ for some $N \in \mathbb{N}$, and (b) a permutation $\sigma \in \Sigma_N$ - the symmetric group on $N$ letters. The associated picture invariant is the following element of $\mathbb{C}[W^*]$:

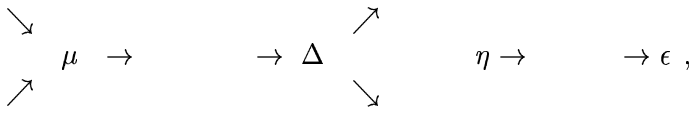
$$\prod_{i=1}^{k} \left( \prod_{j=1}^{m_i} T(i)_{r_{\left(\sum_{p<i} m_p b_p + (j-1)b_i + 1\right)}, \cdots, r_{\left(\sum_{p<i} m_p b_p + j b_i\right)}}^{r_{\sigma\left(\sum_{p<i} m_p t_p + (j-1)t_i + 1\right)}, \cdots, r_{\sigma\left(\sum_{p<i} m_p t_p + j t_i\right)}} \right)$$

i.e., we take $N$ dummy indices $r_1, \cdots, r_N$, take a product of $m_1$ $T(1)$'s, $m_2$ $T(2)$'s, $\cdots$, $m_k$ $T(k)$'s and write the lower indices in order and the upper indices in the permuted order given by $\sigma$. It should be clear that this 'picture invariant' is homogeneous of degree $(m_1, \cdots, m_k)$ in the $\mathbb{N}^k$-grading. (We will soon show - see Proposition 7 - that picture invariants are indeed invariant.)

By means of one example - which is the main case of interest for our purposes - we will explain how picture invariants are represented by pictures. Suppose that $W = V_2^1 \oplus V_1^2 \oplus V_0^1 \oplus V_1^0$. We will use $\mu, \Delta, \eta$ and $\epsilon$ instead of $T(1), T(2), T(3)$ and $T(4)$. Consider, for instance, the 4-tuple $(2,1,1,0)$ for which $2(1,2) + 1(2,1) + 1(1,0) + 0(0,1) = (5,5)$ and the permutation $(123)(45) \in \Sigma_5$. The picture invariant associated to this data is equal to $\mu^{r_2}_{r_1 r_2} \mu^{r_3}_{r_3 r_4} \Delta^{r_1 r_5}_{r_5} \eta^{r_4}$. To this 'picture invariant', we shall associate the following picture:




In the picture on the left, we have numbered the input and output arrows so as to make clear the role of the permutation in drawing the invariant. So briefly, a picture invariant - in this case, i.e., when $k = 4$ and $W$ is specified by the tuple $\{(1,2),(2,1),(1,0),(0,1)\}$ as above - is constructed by taking, in order, a collection of basic pictures of the types




the numbers of each of which are specified by the 4-tuple, and then joining the $i^{th}$ output arrow to the $\sigma(i)^{th}$ input arrow for each $i$ to get a 'closed picture', i.e., one with no free arrows. We will not dis-

tinguish between a picture invariant and any picture that represents it.

Note that if a picture invariant is disconnected - as in the example considered - then, its components also define picture invariants, the product of all of which gives the full picture invariant.

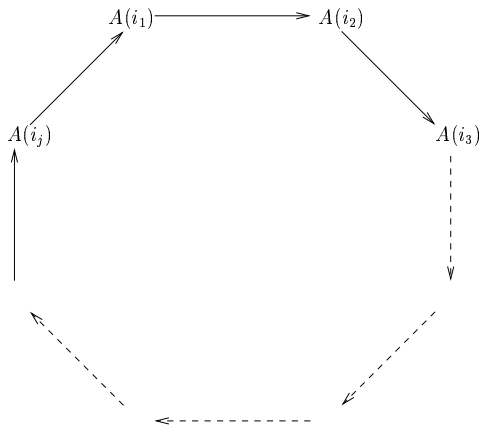
We may now state our main observation about the picture invariants.

PROPOSITION 7. *Let $V$ be a finite dimensional complex vector space and $(t_i, b_i)$ for $i = 1, 2, \cdots, k$ be tuples of non-negative integers. Let $W = \oplus_{i=1}^{k} V_{b_i}^{t_i}$ and set $R = \mathbb{C}[W^*]$. Then, $R^G$, for the $G = GL(V)$ action, is linearly spanned by the picture invariants on $W$.*

Before proving this, we pause to point out a corollary - see Theorem 1.3 of [Prc] and [Rzm].

COROLLARY 8. *The ring of invariants of the group $GL_n(\mathbb{C})$ acting by simultaneous conjugation on $k$ square matrices $A(1), \cdots, A(k)$ is linearly spanned by monomials in the $tr(A(i_1)A(i_2)\cdots A(i_j))$ where $A(i_1)A(i_2)\cdots A(i_j)$ is any possible (non-commutative) monomial.*

*Proof.* This corresponds to choosing all the $k$ tuples to be equal to $(1, 1)$. The basic pictures in this case are $\rightarrow A(i) \rightarrow$ and so any connected picture invariant must be as illustrated. In terms of




the entries of the matrices $A(i)$, this picture invariant evaluates to $tr(A(i_1)A(i_2)\cdots A(i_j))$. A general picture invariant that is possibly disconnected is therefore a monomial in the $tr(A(i_1)A(i_2)\cdots A(i_j))$ and now an appeal to Proposition 7 completes the proof. □

*Proof of Proposition 7.* We need to see that the picture invariants span the invariant ring $\mathbb{C}[W^*]^G$. Note that $C[W^*] = Sym_{\mathbb{C}}(W^*) = \oplus_{d\geq 0} Sym^d_{\mathbb{C}}(W^*) = \oplus_{d\geq 0} \oplus_{\{(m_1,\cdots,m_k):\sum_i m_i = d\}} \otimes_{i=1}^k Sym^{m_i}_{\mathbb{C}}((V^{t_i}_{b_i})^*)$ - as $G$-modules. Hence it suffices to see that picture invariants span each $\left(\otimes_{i=1}^k Sym^{m_i}_{\mathbb{C}}((V^{t_i}_{b_i})^*)\right)^G$.

As the natural map of $\otimes_{i=1}^k ((V^{t_i}_{b_i})^*)^{\otimes m_i}$ onto $\otimes_{i=1}^k Sym^{m_i}_{\mathbb{C}}((V^{t_i}_{b_i})^*)$ is a $G$-map, the reductivity of $G$ implies that $\left(\otimes_{i=1}^k ((V^{t_i}_{b_i})^*)^{\otimes m_i}\right)^G$ maps onto $\left(\otimes_{i=1}^k Sym^{m_i}_{\mathbb{C}}((V^{t_i}_{b_i})^*)\right)^G$. Clearly, $\otimes_{i=1}^k ((V^{t_i}_{b_i})^*)^{\otimes m_i}$ is isomorphic as a $G$-module to $V^N_M$ where $N = \sum_i m_i b_i$ and $M = \sum_i m_i t_i$.

We now appeal to the fact from classical invariant theory - see Theorem 4.3.1 in [GdmWll] - that non-zero $GL(V)$ invariants exist in $V^N_M$ only if $N = M$ and in that case the space of invariants is spanned by all $v_{r_1} \otimes v_{r_2} \otimes \cdots v_{r_N} \otimes v^{r_{\sigma(1)}} \otimes v^{r_{\sigma(2)}} \otimes \cdots v^{r_{\sigma(N)}}$ as $\sigma$ ranges over $\Sigma_N$.

Chasing through the isomorphisms, the images of the $G$-invariants in $V^N_N$ are seen to be precisely the picture invariants, thereby completing the proof. □

# 4 Invariants of semisimple Hopf algebras

Let $V$ be a finite dimensional complex vector space, $G = GL(V)$ and $W = V^1_2 \oplus V^2_1 \oplus V^1_0 \oplus V^0_1$ for which we label the coordinate tensors $\mu, \Delta, \eta$ and $\epsilon$.

Corollary 5 may be restated to say that isomorphism classes of complex semisimple Hopf algebras are separated by the polynomial invariants of $W$. By Proposition 7, these are exactly the picture invariants built out of $\mu, \Delta, \eta$ and $\epsilon$. Our goal is to identify a suitable 'small' subset of the picture invariants which accomplishes the same task. In this section, we shall be slightly sloppy and also refer to a scalar multiple of a picture invariant as a picture invariant.

DEFINITION 9. *Two picture invariants on $W$ are said to be equivalent modulo $SCH$ if they agree on $SCH$.*

Pictorially, if a picture invariant can be transformed into another by 'moves' that locally replace a subpicture appearing on one side of an equality in equations (2.1)-(2.12) by one appearing on the other, then, the two picture invariants are equivalent modulo $SCH$. Thus, for instance, Corollary 3 shows that any arrow in a picture may be replaced by a more complicated sub-picture which contains no

directed path from the beginning to the end, so that the resulting picture is equivalent modulo $SCH$ to the initial one.

In order to state our next proposition we will find it convenient to introduce some notation for iterated products and coproducts - see p.108 of [Kpr] - as well as for certain picture invariants. First, let

$$\rightarrow \Delta_1 \rightarrow \;=\; \rightarrow \;=\; \rightarrow \mu_1 \rightarrow$$

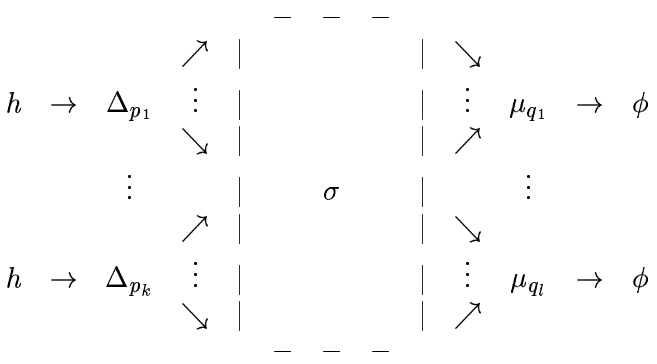
and for $p, q > 1$, inductively define

$$\rightarrow \Delta_p \begin{array}{c}\nearrow\\ \vdots \\ \searrow\end{array} = \rightarrow \Delta \rightarrow \Delta_{p-1} \begin{array}{c}\nearrow\\ \vdots \\ \searrow\end{array}$$

and

$$\begin{array}{c}\searrow\\ \vdots \\ \nearrow\end{array} \mu_q \rightarrow = \begin{array}{c}\searrow\\ \vdots \\ \nearrow\end{array} \mu_{q-1} \rightarrow \mu \rightarrow$$

Also, if $\mathbf{p} = (p_1, \cdots, p_k)$, $\mathbf{q} = (q_1, \cdots, q_l)$ are tableaux of equal size $N$ (say) - i.e., $p_1 \geq \cdots \geq p_k > 0$, $q_1 \geq \cdots \geq q_l > 0$, and $\sum_{i=1}^{k} p_i = \sum_{j=1}^{l} q_j = N$ - and if $\sigma \in \Sigma_N$, we shall define the picture invariant $\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)$ to be the following picture:




where the central 'box' labelled $\sigma$ is meant to indicate that the $i$-th output of the picture to the left of the box is to be joined to the $\sigma(i)$-th input of the picture to the right of the box, and we have used the symbols $h$ and $\phi$ for the pictures associated with them in Figures 3 and 1 respectively.

PROPOSITION 10. *Any picture invariant on $W$ is equivalent modulo $SCH$ to an $\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)$. If the total number of $\mu$'s and $\Delta$'s in a picture invariant is $k$, then we may choose $\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)$ so that $\sum p_i = \sum q_j \leq 13k/2$.*

*Proof.* Begin with a picture invariant, say $P$, on $W$. From its 'equivalence class modulo $SCH$', pick a picture invariant, say $P_1$, for which the total number $k(P_1)$ of $\mu$'s and $\Delta$'s is minimal. Next, pick a picture invariant, say $P_2$, in the 'equivalence class modulo $SCH$' of $P$ such that the total number of $\eta$'s and $\epsilon$'s in $P_2$ is minimal among all picture invariants $Q$ in the 'equivalence class modulo $SCH$' of $P$ for which $k(Q) = k(P_1)$.

We assert that $P_2$ has no $\eta$'s or $\epsilon$'s. For suppose that there is an $\eta$. Its output must go into either a $\mu$ or a $\Delta$ or a $\epsilon$. In these cases, it follows from equations (2.3), (2.6) and (2.8) that the minimality requirements defining $P_2$ (on $\mu$'s and $\Delta$'s in the first two cases, and on $\eta$'s and $\epsilon$'s in the last case) are violated. A similar argument shows that $P_2$ cannot have any $\epsilon$'s either.

Since $P_2$ is a closed picture - i.e., has no free arrows - it is easy to see that the number of $\mu$'s = number of $\Delta$'s $= l$, where $2l = k(P_2) \leq k(P) = k$; from which it follows that the total number of arrows in $P_2$ is $3l$. Now use Corollary 3 to replace each arrow of $P_2$ to get an equivalent picture modulo $SCH$, say $P_3$, with $13l$ each of the $\mu$'s and $\Delta$'s - the original $l$ together with the 4 new ones introduced for each of the $3l$ arrows replaced. (Recall that each $h$ (resp., $\phi$) is a picture with a self-loop containing one $\Delta$ (resp., $\mu$).)

We claim that $P_3$ has no directed loops - except possibly for self loops on the $\mu$'s and $\Delta$'s that are inherent in $h$ and $\phi$. Note first that the only arrows of $P_3$ are the newly introduced ones; and the newly introduced substitutes for the edges of $P_2$ are seen to not contain any edges that can be part of a non-trivial loop. This establishes the claim about 'no loops in $P_3$'. Further, an inspection of the newly introduced substitutes for the edges of $P_2$ also reveals that $P_3$ contains no directed edge from a $\mu$ to a $\Delta$.

To finish the proof it suffices to see that if a picture invariant on $W$ (a) involves only $\mu$'s and $\Delta$'s, (b) has no directed loops except for self loops, and (c) has no directed edge from a $\mu$ to a $\Delta$, then such a picture invariant is necessarily equivalent modulo $SCH$ to a $\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)$.

We prove this as follows. Begin with such a picture invariant and delete all arrows that go from a $\Delta$ to a $\mu$. Consider a connected component of the picture that remains. Each such component contains either only $\mu$'s or only $\Delta$'s. Fix a component, say $C$, containing only $\Delta$'s, say $p$ of them. Each edge of $C$ feeds into a different $\Delta$ (since the 'in-degree' of $\Delta$ is one) so there are exactly $p$ edges.

Let $C_1$ denote the graph obtained by removing self-loops from $C$ and regarding the remaining graph as an undirected graph. We

assert that $C_1$ is a tree. To see this, since it is clearly connected, it is enough to verify that $C_1$ contains no loops. Suppose $L$ were such a loop. Let us associate the ordered pair $(d_{in}, d_{out})$ of 'in-' and 'out'-degrees to every vertex of $L$ when regarded as a vertex of the directed subgraph of $C$ corresponding to $L$. Each such ordered pair must *á priori* be $(1,1), (2,0)$ or $(0,2)$; but our observation about 'no directed loops in $C$' means that not all pairs can be $(1,1)$. So at least one vertex must correspond to $(2,0)$ or $(0,2)$. Since the sum of the in-degrees (as also the out-degrees) of all the vertices of $L$ must be equal to the number of edges of $L$, we may conclude that at least one vertex of $L$ must have in-degree 2; but our graph $C$ contains only $\Delta$'s which have in-degree 1.

Since a tree with $p$ vertices has exactly $(p-1)$ edges, we deduce that $C$ contains exactly one self loop.

Let $C_2$ be the picture obtained from $C$ as a result of adding all those arrows of $P_3$ which emanated from a $\Delta$ of $C$ and terminated in a $\mu$. It is a consequence of co-associativity in Hopf algebras, that $C$ is 'equivalent modulo $SCH$' to the standard picture (independent of the structure of the tree $C_1$):

$$h \quad \rightarrow \quad \Delta_p \quad \begin{array}{c} \nearrow \\ \vdots \\ \searrow \end{array} \cdot$$

(We have been slightly glib in using the expression 'equivalent modulo $SCH$' for general pictures which are not picture invariants (but more general tensors); we trust the meaning should be clear.)

A dual verification shows that a component containing only $\mu$'s - say $q$ of them - is equivalent modulo $SCH$ to the picture

$$\begin{array}{c} \searrow \\ \vdots \\ \nearrow \end{array} \quad \mu_q \quad \rightarrow \quad \phi \cdot$$

Let $k$ denote the number of components (such as $C$ above) which contain only $\Delta$'s , and suppose $p_1 \geq \cdots \geq p_k$ is the non-increasing of the numbers of vertices in these components. Let $l$ and $q_1 \geq \cdots \geq q_l$ denote the corresponding numbers for the 'only $\mu$ components'. It should then be clear that our picture invariant $P_3$ (and hence also $P$) is equivalent modulo $SCH$ to $\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)$ for an appropriately chosen permutation $\sigma$.

Finally, $\Sigma p_i = \Sigma q_j = 13l \leq 13k/2$. $\square$

THEOREM 11. *Two complex semisimple Hopf algebras $H_1$ and $H_2$ of dimension $n$ are isomorphic if and only if for every positive integer $N$, tableaux $\mathbf{p}, \mathbf{q}$ of size $N$ and permutation $\sigma \in \Sigma_N$, we have $\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)(H_1) = \mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)(H_2)$. It suffices to verify this only for $N \leq (2n+1)^{(2n^2+5)}$.*

*Proof.* By Corollary 5, $H_1$ and $H_2$ are isomorphic if and only if $f(H_1) = f(H_2)$ for all $f \in R^G$. Now Proposition 7 and Proposition 10 immediately imply the first assertion of the theorem. The bound on $N$ follows from computational invariant theory - see §4.7 of [DrkKmp]. By Proposition 4.7.16 and Theorem 4.7.4 of [DrkKmp] $R^G$ is generated as an algebra by its elements of degree at most $k = \frac{3}{8}(2n^3+2n)(n+1)^2(2n+1)^{2n^2}$ - the numbers $2n^3+2n$, $n+1$, $2n+1$ and $n^2$ being upper bounds for what they call $r, C, A$ and $m$ respectively. Therefore picture invariants involving at most $k$ $\mu$'s and $\Delta$'s separate isomorphism classes of semisimple Hopf algebras. Now the second assertion of Proposition 10 finishes the proof. □

# 5 Remarks and questions

This section is a collection of a simple example, some possibly naive questions and a possibly rash conjecture.

EXAMPLE 12 (GROUP ALGEBRAS). *Evaluated on a semisimple Hopf algebra $H$, we may write*

$$\mathcal{I}(\mathbf{p}, \mathbf{q}, \sigma)(H) \;=\; \langle \Delta_{p_1}(h) \otimes \cdots \otimes \Delta_{p_k}(h) \mid \sigma \mid \Delta_{q_1}(\phi) \otimes \cdots \otimes \Delta_{q_l}(\phi) \rangle$$

*where this means: compute the elements of $H^{\otimes N}$ and $(H^*)^{\otimes N}$ given by the left and the right sides of the above expression and pair them off by pairing the $i$-th tensor factor on the left with the $\sigma(i)$-th tensor factor on the right. In the case when $H$ is the complex group algebra of a finite group $G$, it is not hard to see that these picture invariants give essentially the data of the number of solutions in $G$ of all systems of equations of the form $m_1 = m_2 = \cdots m_l = 1$ where $1$ is the identity element of $G$ and $m_1, \cdots, m_l$ are monomials in the (non-commuting) variables $X_1, \cdots, X_k$. Theorem 11 then implies that - as can also be seen by a pleasant application of the inclusion-exclusion principle - these numbers determine the group $G$.*

QUESTION 13 (RELATIONS BETWEEN INVARIANTS AND RECONSTRUCTION). *A natural question that arises is what the "second fundamental theorem" for these invariants is. Explicitly, consider a polynomial ring in the infinitely many variables $X_{(\mathbf{p},\mathbf{q},\sigma)}$ and determine*

*the ideal $I_n$ of all polynomials that vanish when evaluated on any n-dimensional semisimple Hopf algebra. A related problem is to reconstruct the Hopf algebra from the invariants.*

CONJECTURE 14 (THE CHARACTERISTIC $p$ CASE). *We conjecture that the picture invariants separate isomorphism classes of semisimple and cosemisimple Hopf algebras over an algebraically closed field of arbitrary characteristic. Note that the analogue to the Procesi-Razmyslov theorem has been proved by Donkin in [Dnk] and it is not clear how to interpret this pictorially. Our "justifications" for making this conjecture are the results of Etingof and Gelaki - see [TngGlk] - on lifting theorems from characteristic p to characteristic* 0.

QUESTION 15 (SUBFACTORS). *The original motivation for considering this problem comes from subfactor theory where the problem we wish to solve is: decide whether or not two finite-depth hyperfinite subfactors are isomorphic. Considering the gauge group action on the space of flat connections on the graph invariants, we expect that a similar invariant theoretic answer must exist. The difference now will be that the groups involved are real Lie groups acting on smooth manifolds. This will be the subject of a future paper.*

QUESTION 16 (THE GENERAL ISOMORPHISM PROBLEM). *Is there an explicit decision procedure for the isomorphism problem for general (not neccessarily semisimple) finite - dimensional complex Hopf algebras ?*

QUESTION 17 (EFFICIENT COMPUTABILITY). *This relates to finding better bounds on the number and size of invariants needed to distinguish semisimple Hopf algebras. In particular, can this be 'done in polynomial time'?*

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## References

[Dnk] S. Donkin, *Invariants of several matrices*, Inventiones Math., 110, (1992) 389 - 401.

[DrkKmp] H. Derksen and G. Kemper, *Computational invariant theory*, Encyc. of Math. Sc. 130, Springer-Verlag, 2002.

[TngGlk] P. Etingof and S. Gelaki, *On the exponent of finite-dimensional Hopf algebras*, Math. Res. Lett., 6, (1999), 131-140.

[GdmWll] R. Goodman and N.R. Wallach, *Representations and invariants of the classical groups*, Encyc. of Math. and Its Appl. 68, Cambridge Univ. Press, 1998.

[KffRdf] L. H. Kauffman and D. E. Radford, *On two proofs of the existence and uniqueness of integrals for finite-dimensional Hopf algebras*, In New trends in Hopf algebra theory, Contemp. Math., 267, (2000) 177-194.

[Kpr] G. Kuperberg, *Non-involutory Hopf algebras and 3-manifold invariants*, Duke. J. Math., 84, (1996) 83-129.

[LrsRdf] G. Larson and D. E. Radford, *Finite dimensional cosemisimple Hopf algebras in characteristic* 0 *are semisimple*, J. Algebra, 117, (1988), 267-289.

[LrsRdf2] G. Larson and D. E. Radford, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math, 110, (1998), 187-195. 117, (1988), 267-289.

[LrsSwd] G. Larson and M. Sweedler, *An asociative orthogonal bilinear form for Hopf alebras*, American Journal of Mathematics, 91, (1969), 71-94.

[MmfFgrKrw] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant theory, Third Edition*, Ergebnisse der Mathematik und ihre Grenzgebiete 34, Springer Verlag, Berlin (1994).

[Prc] C. Procesi, *The invariant theory of* $n \times n$ *matrices*, Adv. in Math., 19, (1976), 306-381.

[Rzm] J. Razmyslov, *Trace identities of matrix algebras via a field of chracteristic zero*, Math. USSR Izvestia (translation), 8, (1974), 727 - 760.

[Stf] D. Stefan, *The set of types of* $n$*-dimensional semisimple and cosemisimple Hopf algebras is finite*, J. Alg., 193, (1997), 571-580.

[Wyl] H. Weyl, *The classical groups: Their invariants and representations*, Princeton University Press, 1997.

e-mail: msdsm@uohyd.ernet.in, vijay@imsc.res.in, sunder@imsc.res.in