A characterization of freeness by invariance under quantum spreading

Stephen Curran

UCLA

Satellite Conference on Operator Algebras
IMSc
August 9, 2010

Distributional symmetries

Definition

A sequence $(X_1, \ldots, X_n)$ of random variables is called

- *Exchangeable* if their joint distribution is invariant under permutations:

$$(X_1, \ldots, X_n) \sim_d (X_{\omega(1)}, \ldots, X_{\omega(n)}), \qquad (\omega \in S_n).$$

- *Spreadable* if their joint distribution is invariant under taking subsequences:

$$(X_1, \ldots, X_k) \sim_d (X_{l_1}, \ldots, X_{l_k}), \qquad (1 \leq l_1 < \cdots < l_k \leq n).$$

- Note that independent and identically distributed (i.i.d) $\Rightarrow$ exchangeable $\Rightarrow$ spreadable.
- Reverse implications fail for finite sequences.

De Finetti's theorem

Theorem (de Finetti '30s)

Any infinite exchangeable sequence of random variables is conditionally i.i.d.

Theorem (Ryll-Nardzewski '57)

Any infinite spreadable sequence of random variables is conditionally i.i.d.

- So for infinite sequences, the conditions of being exchangeable, spreadable and conditionally i.i.d. are all equivalent.
- For finite exchangeable sequences, Diaconis and Freedman have obtained approximate de Finetti type results.

Noncommutative context

- We will consider sequences $(x_1, \ldots, x_n)$ of (self-adjoint) elements of a von Neumann algebra M with a fixed tracial state τ.
- The *joint distribution* of such a sequence is the collection of *noncommutative joint moments*

$$\tau(x_{i_1} \cdots x_{i_k}), \qquad (k \in \mathbb{N}, 1 \leq i_1, \ldots, i_k \leq n)$$

- Alternative notions of "independence" in this context, most notably Voiculescu's *free independence*.
- Exchangeability is no longer strong enough to single out one type of independence. In particular freely independent and identically distributed (f.i.d.) sequences are exchangeable.
- Spreadability also no longer implies exchangeability in this context (Köstler '10).
- More on spreadability and exchangeability in the noncommutative context in the talks of Köstler and Gohm tomorrow.

Quantum exchangeability and the free de Finetti theorem

- S_n is characterized as the automorphism group of $\{1, \ldots, n\}$.
- Wang ('98) showed that $\{1, \ldots, n\}$ also has a "quantum" automorphism group, denoted $A_s(n)$.

Definition (Köstler-Speicher '08)

A sequence $(x_i)_{i \in \mathbb{N}}$ is *quantum exchangeable* if for each $n \in \mathbb{N}$ the joint distribution of $(x_1, \ldots, x_n)$ is invariant under "quantum permutations" coming from $A_s(n)$.

Theorem (Köstler-Speicher '08)

An infinite sequence $(x_i)_{i \in \mathbb{N}}$ in (M, τ) is quantum exchangeable if and only if it is free and identically distributed with respect to a conditional expectation.

- Fails for finite quantum exchangeable sequences (Köstler-Speicher), but still holds in an approximate sense (C. '08).

A free analogue of Ryll-Nardzewski's theorem

The goals in this talk:

- Introduce "quantum increasing sequence spaces" $A_i(k, n)$.
- Develop an appropriate notion of "quantum spreadability".
- Sketch the following free Ryll-Nardzewski theorem:

Theorem (Köstler-Speicher '08, C. '10)

For an infinite sequence $(x_i)_{i\in\mathbb{N}}$ in (M, τ), the following are equivalent:

1. $(x_i)_{i\in\mathbb{N}}$ *is quantum exchangeable.*
2. $(x_i)_{i\in\mathbb{N}}$ *is quantum spreadable.*
3. $(x_i)_{i\in\mathbb{N}}$ *is free and identically distributed with respect to a conditional expectation.*

Compact quantum groups (of Kac type)

Let G be a compact group.

- $C(G)$ captures the topological structure of G (Gelfand duality).
- The group structure of G can be encoded by the morphisms:

$$\begin{aligned} \Delta : C(G) &\to C(G) \otimes C(G), & \Delta(f)(x,y) &= f(xy) \\ \epsilon : C(G) &\to \mathbb{C}, & \epsilon(f) &= f(e_G) \\ S : C(G) &\to C(G)^{op}, & S(f)(x) &= f(x^{-1}). \end{aligned}$$

- A *C^*-Hopf algebra* is a unital C^*-algebra A together with morphisms $\Delta : A \to A \otimes A$, $\epsilon : A \to \mathbb{C}$ and $S : A \to A^{op}$ (satisfying various compatibilities).
- Heuristic formula: "$A = C(G)$" where G is a *compact quantum group*.

Wang's $A_s(n)$

- View S_n as consisting of permutation matrices. $C(S_n)$ is generated by the coordinate functions f_{ij}, which satisfy the relations
  - f_{ij} is a self-adjoint projection.
  - For $1 \leq i, j \leq n$,

$$\sum_{k=1}^{n} f_{ik} = 1 = \sum_{k=1}^{n} f_{kj}.$$

  The Hopf algebra structure is determined by

$$\Delta(f_{ij}) = \sum f_{ik} \otimes f_{kj}$$
$$\epsilon(f_{ij}) = \delta_{ij}$$
$$S(f_{ij}) = f_{ji}.$$

- $A_s(n)$ is defined to be the universal C^*-algebra generated by "coordinates" $\{u_{ij} : 1 \leq i, j \leq n\}$ satisfying the relations above.
- $A_s(n)$ is a C*-Hopf algebra with morphisms defined by the above equations. We sometimes use the notation "$A_s(n) = C(S_n^+)$", where S_n^+ is the *free permutation group*.

Quantum increasing sequence spaces

- For $1 \leq k \leq n$, let $I_{kn} = \{(l_1, \dots, l_k) : 1 \leq l_1 < \cdots < l_k \leq n\}$. View as $n \times k$ matrices with (i,j) entry δ_{il_j}.
- $C(I_{kn})$ is generated by coordinates $\{f_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ which satisfy
  - f_{ij} is a self-adjoint projection.
  - For $1 \leq j \leq k$,
    $$\sum_{i=1}^{n} f_{ij} = 1.$$
  - If $j < m$ and $i \geq l$ then $f_{ij} f_{lm} = 0$.
- Define $A_i(k,n)$ to be the universal C*-algebra generated by $\{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ satisfying the above relations.
- $A_i(k,n)$ can be viewed as a "quantum family of maps" from $\{1, \dots, k\}$ to $\{1, \dots, n\}$, in the sense of (Sołtan '09).

Quantum exchangeability

- Define $\alpha_n : \mathbb{C}\langle t_1, \dots, t_n \rangle \to \mathbb{C}\langle t_1, \dots, t_n \rangle \otimes A_s(n)$ to be the unital homomorphism determined by

$$\alpha_n(t_j) = \sum_{i=1}^{n} t_i \otimes u_{ij}.$$

α_n is a *coaction* of $A_s(n)$, which can be thought of as "quantum permuting" $t_1, \dots, t_n$.

Definition

$(x_1, \dots, x_n)$ is called *quantum exchangeable* if

$$(\tau \otimes \mathrm{id})\alpha_n(p(x)) = \tau(p(x)) \cdot 1_{A_s(n)}, \qquad (p \in \mathbb{C}\langle t_1, \dots, t_n \rangle).$$

Explicitly,

$$\sum_{1 \leq i_1, \dots, i_k \leq n} \tau(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k} = \tau(x_{j_1} \cdots x_{j_k}) \cdot 1_{A_s(n)}$$

Quantum spreadability

- For $1 \leq k \leq n$ define $\beta_{k,n} : \mathbb{C}\langle t_1, \ldots, t_k\rangle \to \mathbb{C}\langle t_1, \ldots, t_n\rangle \otimes A_i(k, n)$ to be the unital homomorphism determined by

$$\beta_{k,n}(t_j) = \sum_{i=1}^{n} t_i \otimes u_{ij}.$$

Definition

$(x_1, \ldots, x_n)$ is called *quantum spreadable* if

$$(\tau \otimes \mathrm{id})\beta_{k,n}(p(x)) = \tau(p(x)) \cdot 1_{A_i(k,n)}, \qquad (1 \leq k \leq n, p \in \mathbb{C}\langle t_1, \ldots, t_k\rangle).$$

Explicitly,

$$\sum_{1 \leq i_1, \ldots, i_m \leq n} \tau(x_{i_1} \cdots x_{i_m}) u_{i_1 j_1} \cdots u_{i_m j_m} = \tau(x_{j_1} \cdots x_{j_m}) \cdot 1_{A_i(k,n)}$$

for $1 \leq k \leq n$ and $1 \leq j_1, \ldots, j_m \leq k$

Quantum exchangeability implies quantum spreadability

- Spreadability is clearly weaker than exchangeability, as it requires invariance under fewer transformations.
- Indeed any increasing sequence $1 \leq l_1 < \cdots < l_k \leq n$ can be extended to a permutation $\omega \in S_n$ with $\omega(j) = l_j$ for $1 \leq j \leq k$.
- ω is unique if one requires additionally that $\omega(k+1) < \cdots < \omega(n)$.
- For example:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Dualizing $I_{kn} \hookrightarrow S_n$ gives $C(S_n) \twoheadrightarrow C(I_{k,n})$.

Extending quantum increasing sequences to quantum permutations

Proposition

Let $\{u_{ij} : 1 \leq i, j \leq n\}$ *and* $\{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ *be the coordinates on* $A_s(n)$ *and* $A_i(k, n)$*, respectively. There is a unital* $*$*-homomorphism* $A_s(n) \to A_i(k, n)$ *which sends* u_{ij} *to* v_{ij} *for* $1 \leq i \leq n$, $1 \leq j \leq k$.

- Idea of proof: Compute $C(S_n) \to C(I_{kn})$ in terms of their coordinates, and check that this formula still works in the quantum case.
- At $k = 2$, $n = 4$ the map is given by

$$(u_{ij}) \to \begin{pmatrix} v_{11} & 0 & 1 - v_{11} & 0 \\ v_{21} & v_{22} & v_{11} - v_{22} & 1 - (v_{11} + v_{21}) \\ v_{31} & v_{32} & v_{22} & (v_{11} + v_{21}) - (v_{22} + v_{32}) \\ 0 & v_{42} & 0 & v_{22} + v_{32} \end{pmatrix}$$

Moment-cumulant formula for f.i.d. sequences

- $NC(k)$ is the collection of *noncrossing* partitions of $\{1,\dots,k\}$.
- Given $\sigma \in NC(k)$ and indices $j_1,\dots,j_k$, define

$$\delta_\sigma(\mathbf{j}) = \begin{cases} 1, & s \sim_\sigma t \Rightarrow j_s = j_t \\ 0, & \text{otherwise} \end{cases}.$$

Theorem (Speicher)

A sequence $(x_1,\dots x_n)$ in (M,τ) is f.i.d. if and only if

$$\tau(x_{j_1}\cdots x_{j_k}) = \sum_{\sigma\in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\substack{\pi\in NC(k)\\ \pi\le\sigma}} \mu(\pi,\sigma)\prod_{V\in\pi}\tau(x_1^{|V|}),$$

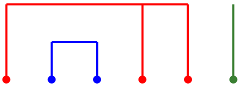
where μ is the Möbius function on $NC(k)$.

- Classical i.i.d. sequences are characterized by the same formula with $NC(k)$ replaced by $P(k)$, the set of all partitions of $\{1,\dots,k\}$.

- For $\pi \in NC(k)$ define $p_\pi \in (\mathbb{C}^n)^{\otimes k}$ by

$$p_\pi = \sum_{1 \le j_1, \dots, j_k \le n} \delta_\pi(\mathbf{j}) e_{j_1} \otimes \cdots \otimes e_{j_k}.$$

Example:

$\pi =$  , $p_\pi = \sum_{i,j,k} e_i \otimes e_j \otimes e_j \otimes e_i \otimes e_i \otimes e_k$

Theorem (Banica)

Let $u : \mathbb{C}^n \to \mathbb{C}^n \otimes A_s(n)$ be the fundamental corepresentation. Then

$$\mathrm{Fix}(u^{\otimes k}) = \mathrm{span}\{p_\pi : \pi \in NC(k)\}.$$

- For S_n the theorem holds with $NC(k)$ replaced by $P(k)$, more in Roland's talk this afternoon.

F.i.d. sequences are quantum exchangeable

- Follows from definitions that $(x_1, \ldots, x_n)$ is quantum exchangeable if and only if

$$\sum_{1 \le i_1, \ldots, i_k \le n} \tau(x_{i_1} \cdots x_{i_k}) \cdot (e_{i_1} \otimes \cdots \otimes e_{i_k}) \in \mathrm{Fix}(u^{\otimes k})$$

  for each $k \in \mathbb{N}$.

- Moment-cumulant formula for f.i.d. sequences:

$$\sum_{1 \le i_1, \ldots, i_k \le n} \tau(x_{i_1} \cdots x_{i_k}) \cdot (e_{i_1} \otimes \cdots \otimes e_{i_k})$$

$$= \sum_{\sigma \in NC(k)} \Bigg(\sum_{\substack{\pi \in NC(k) \\ \pi \le \sigma}} \mu(\pi, \sigma) \prod_{V \in \pi} \tau(x_1^{|V|}) \Bigg) \cdot p_\sigma \in \mathrm{Fix}(u^{\otimes k}).$$

Weingarten formula

- Define *Gram matrix*

$$G_{kn}(\pi,\sigma) = \langle p_\pi, p_\sigma \rangle = n^{\#\text{ of blocks of }\pi\vee\sigma}, \qquad (\pi,\sigma \in NC(k)).$$

- G_{kn} is invertible for $n \geq 4$, let W_{kn} denote it's inverse

Theorem (Banica-Collins)

For $1 \leq i_1, j_1, \ldots, i_k, j_k \leq n$,

$$\int_{S_n^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\sigma,\pi \in NC(k)} \delta_\pi(\mathbf{i})\delta_\sigma(\mathbf{j}) W_{kn}(\pi,\sigma),$$

where $\int$ denotes the Haar state on $A_s(n) = C(S_n^+)$.

Weingarten asymptotics

- Difficulty is in understanding the entries of W_{kn}.
- Partial description of leading order as $n \to \infty$ given by Möbius function on $NC(k)$ (C. '08).

Theorem

For $1 \leq i_1, j_1, \ldots, i_k, j_k \leq n$,

$$\int_{S_n^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\sigma \in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\pi \in NC(k)} \delta_\pi(\mathbf{i}) n^{-|\pi|}(\mu(\pi, \sigma) + O(n^{-1})).$$

Sketch of free de Finetti

Let $(x_i)_{i \in \mathbb{N}}$ be a quantum exchangeable sequence in (M, τ).

- Define *tail algebra*

$$B = \bigcap_{n \geq 1} W^*(x_n, x_{n+1}, \dots).$$

  For simplicity we will assume $B = \mathbb{C}$, so expectation onto B is τ.
- Ergodic theorem:

$$\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} x_i^k = \tau(x_1^k)$$

  with convergence in the strong topology. By induction on number of blocks of $\pi \in NC(k)$:

$$\lim_{n \to \infty} n^{-|\pi|} \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_\pi(\mathbf{i}) x_{i_1} \cdots x_{i_k} = \prod_{V \in \pi} \tau(x_1^{|V|})$$

  with convergence in the strong topology.

Sketch of free de Finetti

Let $(x_i)_{i\in\mathbb{N}}$ be a quantum exchangeable sequence in (M,τ).

- For $1 \leq j_1,\dots,j_k \leq n$ we have

$$\tau(x_{j_1}\cdots x_{j_k}) = \sum_{1\leq i_1,\dots,i_k\leq n} \tau(x_{i_1}\cdots x_{i_k}) \int_{S_n^+} u_{i_1j_1}\cdots u_{i_kj_k}.$$

- Apply Weingarten formula and rearrange terms:

$$\sum_{\sigma\in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\pi\in NC(k)} W_{kn}(\pi,\sigma) \sum_{1\leq i_1,\dots,i_k\leq n} \delta_\pi(\mathbf{i})\tau(x_{i_1}\cdots x_{i_k}).$$

- Apply Weingarten asymptotics and take $n\to\infty$:

$$\sum_{\sigma\in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\pi\in NC(k)} (\mu(\pi,\sigma)+O(n^{-1}))n^{-|\pi|} \sum_{1\leq i_1,\dots,i_k\leq n} \delta_\pi(\mathbf{i})\tau(x_{i_1}\cdots x_{i_k})$$

$$\xrightarrow{n\to\infty} \sum_{\sigma\in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\substack{\pi\in NC(k)\\ \pi\leq\sigma}} \mu(\pi,\sigma) \prod_{V\in\pi} \tau(x_1^{|V|}).$$

Sketch of free Ryll-Nardzewski

- Use a similar "averaging" technique, what state to average against?
- Let $\{p_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ be projections in a C*-probability space (A, φ) such that
  1. The families $(\{p_{i1} : 1 \leq i \leq n\}, \ldots, \{p_{ik} : 1 \leq i \leq n\})$ are freely independent.
  2. For $j = 1, \ldots, k$, we have
  $$\sum_{i=1}^{n} p_{ij} = 1,$$
  and $\varphi(p_{ij}) = n^{-1}$ for $1 \leq i \leq n$.
- Use representation of $A_i(k, k \cdot n)$ given by

$$\begin{pmatrix} p_{11} & \cdots & p_{n1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p_{12} & \cdots & p_{n2} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & p_{1k} & \cdots & p_{nk} \end{pmatrix}^T$$

State on $A_i(k, k \cdot n)$

- Representation on previous slide induces state $\psi_{k,n} : A_i(k, k \cdot n) \to \mathbb{C}$.
- $\psi_{k,n}$ is determined by

$$\psi_{k,n}(u_{(j_1-1)\cdot n+i_1,j_1} \cdots u_{(j_m-1)\cdot n+i_m,j_m}) = \sum_{\sigma \in NC(m)} \delta_\sigma(\mathbf{j}) \sum_{\substack{\pi \in NC(m) \\ \pi \leq \sigma}} \delta_\pi(\mathbf{i}) n^{-|\pi|} \mu(\pi, \sigma)$$

for $1 \leq j_1, \ldots, j_m \leq k$ and $1 \leq i_1, \ldots, i_m \leq n$, and all other expectations are zero.