Continuous minimax theorems

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Abstract

In matrix theory, there exist useful extremal characterizations of eigenvalues and their sums for Hermitian matrices (due to Ky Fan, Courant-Fischer-Weyl and Wielandt) and some consequences such as the majorisation assertion in Lidskii's theorem. In this paper, we extend these results to the context of self-adjoint elements of finite von Neumann algebras, and their distribution and quantile functions. This work was motivated by a lemma in [BV93] that described such an extremal characterization of the distribution of a self-adjoint operator affiliated to a finite von Neumann algebra - suggesting a possible analogue of the Courant-Fischer-Weyl minimax theorem for Hermitian matrices, for a self-adjoint operator in a finite von Neumann algebra[1].

1 Introduction

This paper is arranged as follows: in Section 2, we prove an extension of the 'classical' minimax theorem of Ky Fan's ([Fan49]) in a von-Neumann algebraic setting for self-adjoint operators having no atoms in their distributions, and then, give a few applications in Section 3. First we state and prove an exact analogue of the Courant-Fischer-Weyl minimax theorem ([CH89]) for operators in non-commutative probability spaces satisfying a *continuity condition*. (Specifically we shall say a finite von Neumann algebra (M, τ) is continuous if $\{\tau(q) : q \in \mathcal{P}(M), q \leq p\} = [0, \tau(p)]\ \forall p \in \mathcal{P}(M)$.)

It is interesting to note that for matrices, Courant-Fischer-Weyl minimax theorem preceded Ky Fan's theorem as is seen from the title of [Fan49], whereas the order of events is reversed in our proofs. Then, as an application of our version of the Courant-Fischer-Weyl minimax theorem, we prove that that for self-adjoint operators without eigenvalues in a 'continuous' finite von Neumann algebra (M, τ), the association of quantile functions to self-adjoint operators is an order-preserving one. Finally we discuss a continuous analogue of Lidskii's majorization relation between the eigenvalue-lists of two Hermitian matrices and their sum. Discussions and proofs of the finite dimensional version can be found in [Lid50], [Lid82], [Wie55]. In Section 4, we state and prove an analogue of Wielandt's minimax theorem ([Wie55]), for $a = a^* \in M$, with both M and $A = W^*(a)$ being in the

[1]The only von Neumann algebras considered here have separable pre-duals.

'continuous case' in our sense. The matricial (and not 'continuous' in out sense) version of it yields an extremal characterization for arbitrary sums of eigenvalues of Hermitian matrices.

These and other continuous analogues of minimax-type results have been worked out earlier, for example in [FK86], [Hia87] and [HN87], at the level of generality of unbounded operators affiliated to semi-finite von Neumann algebras equipped with a semi-finite trace. However in those papers, the emphasis has been on positive operators and the von Neumann algebraic versions of minimax-type results corresponded to singular values of Hermitian matrices. On the other hand, our proofs are simple, independent of the approach of these papers, deal explicitly with self-adjoint (as against positive) operators in certain von Neumann algebras and correspond to eigenvalues (as against singular values) of Hermitian matrices in the finite dimensional case. Moreover as far as we know, unlike former works on this topic, our formulations, for the particular case of fintie dimensional matrix algebras, give the exact statements of Ky Fan's, Courant-Fischer-Weyl's and Wielandt's theorems for matrices. However in the continuous case, our results are restricted to the case when both M and A (as above) are continuous.

In order to describe our results, which are continuous analogues of certain inequalities that appear among the set of inequalities mentioned in Horn's conjecture ([Hor62]), it will be convenient to re-prove the well-known fact that any monotonic function with appropriate one-sided continuity is the distribution function of a random variable X - which can in fact be assumed to be defined on the familiar Lebesgue space $[0,1)$ equipped with the Borel σ-algebra and Lebesgue measure. (We adopt the convention of [BV93] that the *distribution function* F_μ of a compactly supported probability measure[2] μ defined on the σ-algebra $\mathcal{B}_{\mathbb{R}}$ of Borel sets in $\mathbb{R}$, is *left-continuous*; thus $F_\mu(x) = \mu((-\infty, x))$.

Proposition 1.1. *If $F : \mathbb{R} \to [0,1]$ is monotonically non-decreasing and left continuous and if there exists $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that*

$$F(t) = 0, \text{ for } t \leq \alpha \text{ and } F(t) = 1 \text{ for } t \geq \beta, \tag{1.1}$$

then there exists a monotonically non-decreasing right-continuous function $X : [0,1) \to \mathbb{R}$ such that F is the distribution function of X, i.e., $F(t) = m(\{s : X(s) < t\})$, where m denotes the Lebesgue measure on $[0,1)$. Moreover range$(X) \subset [\alpha, \beta]$.

Proof. Define $X : [0,1) \to \mathbb{R}$ by

$$\begin{aligned} X(s) &= \inf\{t : F(t) > s\} \\ &= \inf\{t : t \in E_s\}\ , \end{aligned} \tag{1.2}$$

where $E_s = \{t \in \mathbb{R} : F(t) > s\}\ \forall s \in [0,1)$. (The hypothesis (1.1) is needed to ensure that E_s is a non-empty bounded set for every $s \in [0,1)$ so that, indeed $X(s) \in \mathbb{R}$.)

First deduce from the monotonicity of F that

$$\begin{aligned} s_1 \leq s_2 &\Rightarrow E_{s_2} \subset E_{s_1} \\ &\Rightarrow X(s_1) \leq X(s_2) \end{aligned}$$

[2] Actually Bercovici and Voiculescu considered possibly unbounded self-adjoint operators affiliated to M, so as to also be able to handle probability measures which are not necessarily compactly supported, but we shall be content with the case of bounded $a \in M$, having a compactly supported probability measure as its distribution.

and hence X is indeed monotonically non-decreasing.

The definition of X and the fact that F is monotonically non-increasing and left continuous are easily seen to imply that $E_s = (X(s), \infty)$, and hence, it is seen that

$$\begin{aligned} X(s) < t &\Leftrightarrow \exists t_0 < t \text{ such that } F(t_0) > s \\ &\Leftrightarrow F(t) > s \text{ (since } F \text{ is left-continuous)} \end{aligned} \tag{1.3}$$

Hence, if $t \in \mathbb{R}$

$$m(\{s \in [0,1) : X(s) < t\}) = m([0, F(t)) = F(t), \text{ proving the required statement.} \tag{1.4}$$

Moreover, if for any $s \in [0,1)$, $X(s) < \alpha$, then by definition of X, $\exists\ t' < \alpha$ such that $F(t') > s \geq 0$, a contradiction to the first hypothesis in equation (1.1). On the other hand, if for any $s \in [0,1)$, $X(s) > \beta$, then by (1.3), $s \geq F(\beta) = 1$ (by the second hypothesis in (1.1)), a contradiction. Hence indeed $range(X) \subset [\alpha, \beta]$. □

This function X is known as *quantile function*[3] of the distribution F. If $F = F_\mu$ for a probability measure μ on $\mathbb{R}$, then X is denoted as X_μ. The function X can also be thought of as an element of $L^\infty(\mathbb{R}, \mu)$, where μ is a compactly supported probability measure on $\mathbb{R}$ such that $\mu = m \circ X^{-1}$ and $supp\ \mu \subset [\alpha, \beta]$. It should be observed that the quantile function $X(s)$ (corresponding to the self-adjoint operator a) here is the non-decreasing version of the generalized s-numbers $\mu_s(a)$ in [FK86] as well as the spectral scale $\lambda_s(a)$ in [Pet85]. We will elaborate further on this function later in Proposition 2.1.

Given a self-adjoint element a in a von Neumann algebra M and a (usually faithful normal) tracial state τ on M, define

$$\mu_a(E) := \tau(1_E(a)) \tag{1.5}$$

(for the associated scalar spectral measure) to be the *distribution of* a. Since τ is positivity preserving, μ_a indeed turns out to be a probability measure on $\mathbb{R}$.

For simplicity we write F_a, X_a instead of F_{μ_a}, X_{μ_a} (to be pedantic, one should also indicate the dependence on (M, τ), but the trace τ and the M containing a will usually be clear.) Note that only the abelian von Neumann subalgebra A generated by a and $\tau|_A$ are relevant for the definition of F_a and X_a.

For M, a, τ as above, it was shown in [BV93] that

$$1 - F_{\mu_a}(t) = \max\{\tau(p) : p \in \mathcal{P}(M), pap \geq tp\}. \tag{1.6}$$

Example 1.2. *Let $M = M_n(\mathbb{C})$ with τ as the tracial state on this M. If $a = a^* \in M$ has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then $F_a(t) = \frac{1}{n}|\{j : \lambda_j < t\}| = \sum_{j=1}^n \frac{j}{n} 1_{(\lambda_j, \lambda_{j+1}]}$. We see that the distinct numbers less than 1 in the range of F_a are attained at the n distinct*

[3] This function acts as the inverse of the distribution function at every point that is not an atom of the probability measure μ.

eigenvalues of a, and further that equation (1.6) for $t = \lambda_j$ says that $n - j + 1$ is the largest possible dimension of a subspace W of $\mathbb{C}^n$ such that $\langle a\xi, \xi\rangle \geq \lambda_j$ for every unit vector $\xi \in W$. In other words equation (1.6) suggests a possible extension of the matricial Courant-Fischer minimax theorem for a self-adjoint operator in a von Neumann algebra, involving its distribution.

It is also true and not hard to see that the right side of equation (1.6) is indeed a maximum (and not just a supremum), and is in fact attained at a spectral projection of a; i.e., the two sides of equation (1.6) are also equal to $\max\{\tau(p) : p \in \mathcal{P}(A), pap \geq ta\}$, where $A = \{a\}''$.

2 Our version of Ky Fan's theorem

In this section we wish to proceed towards obtaining non-commutative counterparts of the matricial Ky Fan's minimax theorem formulated for appropriate self-adjoint elements of appropriate finite von Neumann algebras. This result (Theorem 2.3) is not new - Lemma 4.1 of [FK86] but we give its detailed proof with our language in order to make the exposition of the paper self-contained.

Proposition 2.1. *Let $(\Omega, \mathcal{B}, P)$ be a probability measure space, and suppose $Y : \Omega \to \mathbb{R}$ is an essentially bounded random variable. Let $\sigma(Y) = \{Y^{-1}(E) : E \in \mathcal{B}_{\mathbb{R}}\}$ and let $\mu = P \circ Y^{-1}$ be the distribution of Y. Then, for any $s_0 \in F_\mu(\mathbb{R})$, we have*

$$
\begin{aligned}
&\inf\{\int_{\Omega_0} Y dP : \Omega_0 \in \sigma(Y), P(\Omega_0) \geq s_0\}\\
&= \inf\{\int_E f_0 d\mu : E \in \mathcal{B}_{\mathbb{R}}, \mu(E) \geq s_0\}\\
&= \inf\{\int_G X_\mu dm : G \in \sigma(X_\mu), m(G) \geq s_0\}\\
&= \int_0^{s_0} X_\mu dm,
\end{aligned}
\tag{2.1}
$$

where $f_0 = id_{\mathbb{R}}$ and m denotes Lebesgue measure on [0,1).

Proof. The version of the change of variable theorem we need says that if $(\Omega_i, \mathcal{B}_i, P_i), i = 1, 2$ are probability spaces and $T : \Omega_1 \to \Omega_2$ is a measurable function such that $P_2 = P_1 \circ T^{-1}$, then

$$
\int_{\Omega_2} g dP_2 = \int_{\Omega_1} g \circ T dP_1 \ , \tag{2.2}
$$

for every bounded measurable function $g : \Omega_2 \to \mathbb{R}$.

For every $\Omega_0 \in \sigma(Y)$, which is of the form $Y^{-1}(E)$ for some $E \in \mathcal{B}_{\mathbb{R}}$, set $G = X_\mu^{-1}(E)$. Notice, from equations (1.3) and (1.4) that

$$
\begin{aligned}
m \circ X_\mu^{-1}(-\infty, t) &= m(\{s \in [0, 1) : X_\mu(s) < t\})\\
&= m(\{s \in [0, 1) : s < F_\mu(t)\})\\
&= F_\mu(t)\\
&= \mu(-\infty, t) \ ;
\end{aligned}
$$

i.e. $m \circ X_\mu^{-1} = \mu = P \circ Y^{-1}$. Now, set $g = 1_E \cdot f_0$. Since $g \circ Y = 1_E \circ Y \cdot Y = 1_{Y^{-1}(E)} Y = 1_{\Omega_0} Y$, and (similarly) $g \circ X_\mu = 1_G X_\mu$, we see that the first two equalities in (2.1) are immediate consequences of two applications of the version stated in equation (2.2) above, of the 'change of variable' theorem.

As for the last, if $G \in \mathcal{B}_{[0,1)}$ with $m(G) \geq s_0$, then write $I = G \cap [0, s_0)$, $J = [0, s_0) \setminus I$, $K = G \setminus I$ and note that $G = I \coprod K$, $[0, s_0) = I \coprod J$ (where $\coprod$ denotes disjoint union, and $K = G \setminus [0, s_0) \subset [s_0, 1)$. So we may deduce that

$$\begin{aligned} \int_G X_\mu dm - \int_0^{s_0} X_\mu dm &= \int_K X_\mu dm - \int_J X_\mu dm \\ &\geq X_\mu(s_0) m(K) - X_\mu(s_0) m(J) \\ &\geq 0 \ , \end{aligned}$$

since $s_1 \in J, s_2 \in K \Rightarrow s_1 \leq s_0 \leq s_2 \Rightarrow X_\mu(s_1) \leq X_\mu(s_0) \leq X_\mu(s_2)$ (by the monotonicity of X_μ), and $m(K) \geq m(J)$. Thus, we see that

$$\inf\{\int_G X_\mu dm : G \in \sigma(X_\mu), m(G) \geq s_0\} \geq \int_0^{s_0} X_\mu dm \ ,$$

while conversely,

$$\inf\{\int_G X_\mu dm : G \in \sigma(X_\mu), m(G) \geq s_0\} \leq \int_{[0,s_0)} X_\mu dm = \int_0^{s_0} X_\mu dm \ ,$$

thereby establishing the last equality in (2.1). □

Remark 2.2. *With the same notations as in the above proposition, a change of variables gives us the following simple but useful equation that would be applied many times in this paper :*

$$\int_0^{F(t)} X_\mu \ dm = \int_{-\infty}^t f_0 \ d\mu_a = \tau(a 1_{(-\infty,t]}(a)),$$

where μ is the distribution of a self-adjoint element a in a von Neumann algebra equipped with a faithful normal tracial state τ.

Theorem 2.3. *Let a be a self-adjoint element of a von Neumann algebra M equipped with a faithful normal tracial state τ. Let A be the von Neumann subalgebra generated by a in M and $\mathcal{P}(M)$ be the set of projections in M. Then, for all $s \in F_a(\mathbb{R})$,*

$$\begin{aligned} &\inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\} \\ &= \inf\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \\ &= \int_0^s X_a dm \end{aligned} \tag{2.3}$$

(hence the infima are attained and are actually minima),

if either

1. *('continuous case')* μ_a *has no atoms, or*

2. *('finite case')* $M = M_n(\mathbb{C})$ *for some* $n \in \mathbb{N}$ *and* a *has spectrum* $\{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$.

Proof. We begin by noting that in both the cases, the last equality in equation (2.3) is an immediate consequence of Proposition 2.1. Moreover the set $\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\}$ being contained in $\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\}$, it is clear that

$$\inf\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \geq \inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\}.$$

So we just need to prove that

$$\inf\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\} \leq \inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\}. \tag{2.4}$$

1. (the continuous case) Due to the assumption of μ_a being compactly supported and having no atoms, it is clear that F_a is continuous and that $F_a(\mathbb{R}) = [0, 1]$.

   Under the standing assumption of separability of pre-duals of our von Neumann algebras, the hypothesis of this case implies the existence of a probability space $(\Omega, \mathcal{B}, P)$ and a map $\pi : A \to L^\infty(\Omega, \mathcal{B}, P)$ such that $\int \pi(x)dP = \tau(x)\ \forall x \in A$, $Y := \pi(a)$ is a random variable and π is an isomorphism onto $L^\infty(\Omega, \sigma(Y), P)$.

   We shall establish the first equality of (2.3) by showing that if $p' \in \mathcal{P}(M)$ and $\tau(p') = s$, then $\tau(ap') \geq \min\{\tau(ap) : p \in \mathcal{P}(A), \tau(p) \geq s\}$. For this, first note that since τ is a faithful normal tracial state on M, there exists a τ-preserving conditional expectation $\mathcal{E} : M \to A$. Then

$$\tau(ap') = \tau(a\mathcal{E}(p')) = \int YZdP,$$

   where $Z = \pi(\mathcal{E}(p'))$. Since $\mathcal{E}$ is linear and positive, it is clear that $0 \leq Z \leq 1$ $P - a.e.$ So it is enough to prove that

$$\begin{aligned}&\inf\{\int_\Omega YZdP : 0 \leq Z \leq 1, \int ZdP \geq s\}\\ &= \inf\{\int_E YdP : E \in \sigma(Y), P(E) \geq s\}.\end{aligned}$$

   For this, it is enough, thanks to the Krein-Milman theorem (see, e.g. [KM40]), to note that $K = \{Z \in L^\infty(\Omega, \mathcal{B}, P) : 0 \leq Z \leq 1, \int ZdP \geq s\}$ is a convex set which is compact in the weak* topology inherited from $L^1(\Omega, \mathcal{B}, P)$, and prove that the set $\partial_e(K)$ of its extreme points is $\{1_E : P(E) \geq s\}$.

   For this, suppose $Z \in K$ is not a projection, Clearly then $P(\{Z \in (0, 1)\}) > 0$, so there exists $\epsilon > 0$ such that $P(\{\epsilon < Z < 1-\epsilon\}) > 0$. Since μ_a, and hence P has no atoms, we may find disjoint Borel subsets $E_1, E_2 \subset \{Z \in (\epsilon, 1-\epsilon)\}$ such that $P(E_1) = P(E_2) > 0$. If we now set $Z_1 = Z + \epsilon(1_{E_1} - 1_{E_2})$ and $Z_2 = Z + \epsilon(1_{E_2} - 1_{E_1})$, it is not hard to see that $Z_1, Z_2 \in K, Z_1 \neq Z_2$ and $Z = \frac{1}{2}(Z_1 + Z_2)$ showing that $Z \notin \partial_e(K)$, thereby proving equation (2.4).

2. (the finite case) Since a has distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, A is a maximal abelian self-adjoint subalgebra of $M_n(\mathbb{C})$. Recall that in this case, $F_a(t) = \frac{1}{n}|\{j : \lambda_j < t\}| = \sum_{j=1}^n \frac{j}{n} 1_{(\lambda_j, \lambda_{j+1}]}$. It then follows that $F_a(\mathbb{R}) = \{\frac{j}{n} : 0 \leq j \leq n\}$ and that $X_a = \sum_{j=1}^n \lambda_j 1_{[\frac{j-1}{n}, \frac{j}{n})}$ and equation (2.3) is then (after multiplying by n) precisely the statement of Ky Fan's theorem (in the case of self-adjoint matrices with distinct eigenvalues):

   For $1 \leq j \leq n$,

$$\begin{aligned}&\inf\{\tau(ap) : p \in \mathcal{P}(M_n(\mathbb{C})), rank(p) \geq j\} \\ &= \inf\{\tau(ap) : p \in \mathcal{P}(A), rank(p) \geq j\} = \frac{1}{n}\sum_{i=1}^{j} \lambda_i = \int_0^{\frac{j}{n}} X_a(s)ds.\end{aligned}$$

   It suffices to prove the following:

$$\inf\{\tau(ap) : p \in \mathcal{P}(A), rank(p) \geq j\} \leq \inf\{\tau(ap) : p \in \mathcal{P}(M_n(\mathbb{C})), rank(p) \geq j\}.$$

   For this, begin by deducing from the compactness of $\mathcal{P}(M_n(\mathbb{C}))$ that there exists a $p_0 \in \mathcal{P}(M_n(\mathbb{C}))$ with $rank(p_0) \geq j$ such that $\tau(ap_0) \leq \tau(ap)$ $\forall p \in \mathcal{P}(M_n(\mathbb{C}))$ with $rank(p) \geq j$. We assert that any such minimizing p_0 must belong to A. The assumption that A is a masa means we only need to prove that $p_0 a = ap_0$. For this pick any self-adjoint $x \in M_n(\mathbb{C})$, and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \tau(e^{itx} p_0 e^{-itx} a)$. Since clearly $e^{itx} p_0 e^{-itx} \in \mathcal{P}(M)$ and $rank(e^{itx} p_0 e^{-itx}) = rank(p_0) \geq j$, for all $t \in \mathbb{R}$, we find that $f(t) \geq f(0)$ $\forall t$. As f is clearly differentiable, we may conclude that $f'(0) = 0$. Hence,

$$0 = \tau(ixp_0 a - ip_0 xa) = i(\tau(xp_0 a) - \tau(p_0 xa)) = i(\tau(xp_0 a) - \tau(xap_0)),$$

   so that $\tau(x(p_0 a - ap_0)) = 0$ for all $x = x^* \in M$, and indeed $ap_0 = p_0 a$ as desired.

□

Case 1 of Theorem 2.3 is our continuous formulation of Ky Fan's result while Case 2 only captures the classical Ky Fan's theorem for the case of distinct eigenvalues. However the general case of non-distinct eigenvalues can also be deduced from our proof, as we show in the following corollary:

Corollary 2.4. *Let a be a Hermitian matrix in $M_n(\mathbb{C})$ with spectrum $\{\lambda_1 \leq \cdots \leq \lambda_n\}$, where not all λ_js are necessarily distinct. Then for all $j \in \{1, \cdots, n\}$,*

$$\min\{\tau(ap) : p \in \mathcal{P}(M_n(\mathbb{C})),\ rank(p) \geq j\} = \frac{1}{n}\sum_{i=1}^{j} \lambda_i.$$

Proof. We may assume that a is diagonal. Let A_1 be the set of all diagonal matrices, so that $A \subsetneq A_1$. Pick $a^{(m)} = diag(\lambda_1^{(m)}, \lambda_2^{(m)}, \cdots, \lambda_n^{(m)}) \in A_1$ such that $\lambda_j^{(m)}$s are all distinct and $\lim_{m\to\infty} \lambda_j^{(m)} = \lambda_j \ \forall 1 \leq j \leq n$. Then the already established case of Theorem 2.3 in the case of distinct eigenvalues shows that for all $p \in \mathcal{P}(M_n(\mathbb{C}))$ with $rank(p) \geq j$, we have

$$\begin{aligned}\tau(ap) &= \lim_{m\to\infty} \tau(a^{(m)}p) \\ &\geq \lim_{m\to\infty} \frac{1}{n} \sum_{i=1}^{j} \lambda_i^{(m)} \\ &= \frac{1}{n} \sum_{i=1}^{j} \lambda_i\end{aligned}$$

The above, along with the fact that $\tau(ap_j) = \frac{1}{n}\sum_{i=1}^{j} \lambda_i$, where p_j is the obvious diagonal projection, completes our proof of Ky Fan's theorem for Hermitian matrices in full generality. □

Remark 2.5. *It is not difficult to see that equation (2.3) holds even if we replace the inequality $\tau(p) \geq s$ with equality.*

Remark 2.6. *Notice that the hypothesis and hence the conclusion, of the 'continuous case' of Theorem 2.3 are satisfied by any self-adjoint generator of a masa in a II_1 factor.*

3 Applications of our version of Ky Fan's theorem

In this section we discuss three applications[4] of our version of Ky Fan's theorem, the first one being a generalization of the classical Courant Fischer-Weyl minimax theorem:

Theorem 3.1. *Let a be a self-adjoint element of a von Neumann algebra M equipped with a faithful normal tracial state τ. Let t_0 and $t_1 \in \mathbb{R}$ such that $t_0 < t_1$ and $F_a(t_1) - F_a(t_0) =: \delta > 0$. Then*

$$\int_{F_a(t_0)}^{F_a(t_1)} X_a(s)\, ds = \sup_{\substack{r\in\mathcal{P}(M) \\ \tau(r)\geq 1-F_a(t_0)}} \ \inf_{\substack{q\in\mathcal{P}(M) \\ q\leq r \\ \tau(q)=\delta}} \tau(aq), \tag{3.1}$$

if either

1. *('continuous case') if $B \in \{M, A\}$ (with A the von Neumann subalgebra generated by a in M as before) and $p \in \mathcal{P}(B)$, then $\{\tau(r) : r \in \mathcal{P}(B), r \leq p\} = [0, \tau(p)]$ (this assumption for $B = A$ amounts to requiring that μ_a has no atoms; or*

[4] In the rest of the paper we would frequently make use of the equation in Remark 2.2 without mentioning it.

2. *('finite case') M is a type I_n factor for some $n \in \mathbb{N}$ and a has spectrum $\{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$.*

   Moreover there exists $r_0 \in \mathcal{P}(A) \subset \mathcal{P}(M)$ with $\tau(r_0) \geq 1 - F(t_0)$ such that

$$\int_{F_a(t_0)}^{F_a(t_1)} X_a(s)\ ds = \min_{\substack{q \in \mathcal{P}(M) \\ q \leq r_0 \\ \tau(q)=\delta}} \tau(aq),$$

   so that the supremum is actually maximum.

Proof. For simplicity we write F and X for F_a and X_a respectively.

1. (the continuous case) For proving "$\leq$", let $r_0 = 1_{[t_0,\infty)}(a)$ and $q_0 = 1_{[t_0,t_1)}(a)$.

   Then $\tau(r_0) = 1 - F(t_0), \tau(q_0) = \delta$ and $q_0 \leq r_0$.

   If we consider any other $q \in A, q \leq r_0$ with $\tau(q) = \delta$, then q is of the form $1_E(a)$, such that $E \subset [t_0, \infty)$, $\mu_a(E) = \delta$. Arguing as in the proof of Proposition 2.1,

$$\begin{aligned} &\int_{F(t_0)}^{F(t_1)} X(s)\ ds \leq \int_{F(E)} X(s)\ ds \\ \Rightarrow &\int_{t_0}^{t_1} t\ d\mu_a(t) \leq \int_E t\ d\mu_a(t) \\ \Rightarrow &\tau(aq_0) \leq \tau(aq). \end{aligned}$$

   To prove the same for any $q \leq r_0$, first we note that since $r_0 \in W^*(\{a\})$, $(M_0, \tau_0) := \left(r_0 M r_0, \frac{\tau(\cdot)}{\tau(r_0)}\right)$ is also a von Neumann algebra (satisfying the same 'continuity hypotheses as M and A) equipped with a faithful normal traical state and $a_0 := r_0 a r_0$ is a self-adjoint element with a continuous distribution μ_0 (with respect to τ_0) in it.

   Let the von Neumann subalgebra generated by a_0 in M_0 be A_0.

   Any $q \leq r_0$ with $\tau(q) = \delta$ can be thought of as $q \in \mathcal{P}(M_0)$ with $\tau_0(q) = \delta_0 := \frac{\delta}{\tau(r_0)}$, and conversely.

   Now as in the proof of the continuous case of Theorem 2.3 we can assume that there exists a non-atomic probability space $(\Omega_0, \mathcal{B}_0, P_0)$ and a map $\pi_0 : A_0 \to L^\infty(\Omega_0, \mathcal{B}_0, P_0)$ such that $\int \pi_0(x) dP_0 = \tau_0(x)\ \forall x \in A_0$, $Y_0 := \pi_0(a_0)$ and π_0 is an isomorphism onto $L^\infty(\Omega_0, \sigma(Y_0), P_0)$.

   It follows from Theorem 2.3 - applied to $a_0, A_0, M_0, \tau_0, Y_0, P_0, \delta_0$ - that there exists $E \in \sigma(Y_0)$ with $P_0(E) = \frac{\delta}{\tau(r_0)}$ such that

$$\min\{\int Y_0 Z_0\ dP_0 : Z_0 \in L^\infty(\Omega_0, \mathcal{B}_0, P_0), 0 \leq Z_0 \leq 1, \int Z_0 dP_0 = \frac{\delta}{\tau(r_0)}\} = \int_E Y_0 dP_0.$$

Thus if $\pi_0(q_0) = 1_E$, we have

$$\begin{aligned}
&\tau_0(a_0q_0) = \min_{\substack{q\in\mathcal{P}(M_0)\\ \tau_0(q)=\delta_0}} \tau_0(a_0q)\\
\Rightarrow &\frac{\tau(a_0q_0)}{\tau(r_0)} = \min_{\substack{q\in\mathcal{P}(M)\\ q\leq r_0\\ \frac{\tau(q)}{\tau(r_0)}=\frac{\delta}{\tau(r_0)}}} \frac{\tau(a_0q)}{\tau(r_0)}\\
\Rightarrow &\tau(aq_0) = \min_{\substack{q\in\mathcal{P}(M)\\ q\leq r_0\\ \tau(q)=\delta}} \tau(aq), \text{ since } r_0 \text{ commutes with } a \text{ and any } q \leq r_0,\\
\Rightarrow &\int_{F(t_0)}^{F(t_1)} X(s)\, ds \leq \sup_{\substack{r\in\mathcal{P}(M)\\ \tau(r)\geq 1-F(t_0)}} \inf_{\substack{q\in\mathcal{P}(M)\\ q\leq r\\ \tau(q)=\delta}} \tau(aq). \qquad (3.2)
\end{aligned}$$

For "$\geq$", let us choose any projection r with $\tau(r) \geq 1 - F(t_0)$.

Let $r_1 = 1_{(-\infty,t_1)}(a)$. Then $\tau(r_1) = F(t_1) \Rightarrow \tau(r_1 \wedge r) \geq F(t_1) - F(t_0) = \delta$.

Hence, by the hypothesis in this continuous case, $\exists\ q_1 \leq r \wedge r_1$ with $\tau(q_1) = \delta$.

Now consider the II_1 factor $(M_1, \tau_1) := \left(r_1Mr_1, \frac{\tau(\cdot)}{\tau(r_1)}\right)$, where τ_1 is a faithful normal tracial state on M_1. Then q_1 can be thought of as a projection in $\mathcal{P}(M_1)$ with $\tau_1(q_1) = \frac{\delta}{\tau(r_1)}$.

Note that $q_0 = 1_{[t_0,t_1)}(a) \leq r_1$.

As above $a_1 := r_1ar_1$ is a self-adjoint element with continuous distribution in M_1. So we can consider our version of Ky Fan's theorem in M_1 (Theorem 2.3) (also see Remark 2.5):

$$\frac{\int_0^{F(t_0)} X(s)ds}{\tau(r_1)} = \tau_1(a(r_1 - q_0)) = \min_{\substack{q\in\mathcal{P}(M_1)\\ \tau_1(q)=\frac{F(t_0)}{\tau(r_1)}}} \tau_1(aq).$$

(using the fact that a, q_0 and $q \in \mathcal{P}(M_1)$ commute with r_1.)

Subtracting both sides from $\tau_1(a_1)$ and writing q' for $r_1 - q$ in the index, we can rewrite it as:

$$\frac{\int_{F(t_0)}^{F(t_1)} X(s)\, ds}{\tau(r_1)} = \max_{\substack{q'\in\mathcal{P}(M_1)\\ \tau_1(q')=\frac{F(t_1)-F(t_0)}{\tau(r_1)}=\frac{\delta}{\tau(r_1)}}} \tau_1(aq'),$$

or equivalently,

$$\int_{F(t_0)}^{F(t_1)} X(s)\ ds = \max_{\substack{q' \in \mathcal{P}(M) \\ q' \leq r_1 \\ \tau(q')=\delta}} \tau(aq').$$

Now using the fact that $q_1 \leq r \wedge r_1$, we have:

$$\int_{F(t_0)}^{F(t_1)} X(s)\ ds = \max_{\substack{q' \in \mathcal{P}(M) \\ q' \leq r_1 \\ \tau(q')=\delta}} \tau(aq') \geq \tau(aq_1) \geq \inf_{\substack{q \in \mathcal{P}(M) \\ q \leq r \\ \tau(q)=\delta}} \tau(aq),$$

thus, and using the fact that our choice of r was arbitrary with $\tau(r) \geq 1 - F(t_0)$, we have:

$$\int_{F(t_0)}^{F(t_1)} X(s)\ ds \geq \sup_{\substack{r \in \mathcal{P}(M) \\ \tau(r) \geq 1-F(t_0)}} \inf_{\substack{q \in \mathcal{P}(M) \\ q \leq r \\ \tau(q)=\delta}} \tau(aq). \tag{3.3}$$

Equations (3.2) and (3.3) together give us the required equality.

2. (the finite case) Notice that if we set $t_0 = \lambda_i, t_1 = \lambda_{i+j}, \delta = \frac{j}{n}$, where $i, j \in \{1, \cdots, n\}$ such that $i + j - 1 \leq n$, equation (3.1) translates to:

$$\lambda_i + \lambda_{i+1} + \cdots + \lambda_{i+j-1} = \sup_{\substack{r \in \mathcal{P}(M_n(\mathbb{C})) \\ Tr(r) \geq n-i+1}} \inf_{\substack{q \in \mathcal{P}(M_n(\mathbb{C})) \\ q \leq r \\ Tr(q)=j}} Tr(aq),$$

where Tr is the sum of the diagonal entries of matrices.

For the inequality "$\leq$" we prove,

$$\lambda_i + \lambda_{i+1} + \cdots + \lambda_{i+j-1} = Tr(aq_0) = \min_{\substack{q \in \mathcal{P}(M_n(\mathbb{C})) \\ q \leq r_0 \\ Tr(q)=j}} Tr(aq),$$

where $r_0 = 1_{\{\lambda_i, \lambda_{i+1}, \cdots, \lambda_n\}}(a)$ and $q_0 = 1_{\{\lambda_i, \lambda_{i+1}, \cdots, \lambda_{i+j-1}\}}(a)$,

by first showing that any minimizing projection below r_0 has to commute with $r_0 a r_0$, and then using the fact that with distinct eigenvalues $r_0 a r_0$ generates a masa in $r_0 M_n(\mathbb{C}) r_0$, concluding that minimizing projections have to be spectral projections (see the exactly similar proof of the finite case of Theorem 2.3).

For proving "$\geq$", we start with an arbitrary projection r with $Tr(r) \geq n - i + 1$ and note that if we define $r_1 := 1_{\{\lambda_1, \cdots, \lambda_{i+j-1}\}}(a)$, then $\exists\ q_1 \leq r \wedge r_1$ such that $Tr(q_1) = j$. Now we proceed using Ky Fan's theorem for finite dimensional Hermitian matrix $r_1 a r_1$ in $r_1 M_n(\mathbb{C}) r_1$, exactly as in the above proof of the continuous case of this theorem.

$\square$

Remark 3.2. *Theorem 3.1 can equivalently be stated as:*

$$\int_{F(t_0)}^{F(t_1)} X(s)\ ds = \inf_{\substack{p\in\mathcal{P}(M)\\ \tau(p)\geq F(t_1)}} \sup_{\substack{q\leq p\\ \tau(q)=\delta}} \tau(aq),$$

Moreover we can get the classical Courant-Fischer-Weyl minimax theorem for Hermitian matrices in full generality (i.e. involving non-distinct eigenvalues as well) from the above theorem in exactly similar manner as in Corollary 2.4.

The classical Courant-Fischer-Weyl minimax theorem has a natural corollary that says if a, b are Hermitian matrices in $M_n(\mathbb{C})$ such that $a \leq b$ (i.e. $b - a$ is positive semi-definite), and if $\{\alpha_1 \leq \cdots \leq \alpha_n\}$ and $\{\beta_1 \leq \cdots \leq \beta_n\}$ are their spectra respectively, then $\alpha_j \leq \beta_j$ for all $j \in \{1, \cdots, n\}$. As expected, Theorem 3.1 leads us to the same corollary for the 'continuous case':

Corollary 3.3. *Let M be a II_1 factor equipped with faithful normal tracial state τ. If $a, b \in M$ such that $a = a^*, b = b^*$ and μ_a, μ_b have no atoms. Then*

$$a \leq b \Rightarrow X_a \leq X_b. \tag{3.4}$$

Proof. Notice that since $a \leq b$ and τ is positivity preserving, we have

$$\tau(xax^*) \leq \tau(xbx^*). \tag{3.5}$$

for all $x \in M$.

Fix $0 \leq s_0 < s_1 < 1$.

By our assumptions on a and b, μ_a, μ_b are compactly supported probability measures with no atoms. Hence F_a and F_b are continuous functions with $range(F_a) = range(F_b) = [0, 1]$. Thus $\exists\ t_0^a, t_1^a, t_0^b, t_1^b \in \mathbb{R}$ such that $s_0 = F_a(t_0^a) = F_b(t_0^b)$ and $s_1 = F_a(t_1^a) = F_b(t_1^b)$.

Now using Theorem 3.1

$$\begin{aligned}
\int_{s_0}^{s_1} X_a\ dm &= \sup_{\substack{r\in\mathcal{P}(M)\\ \tau(r)\geq 1-F_a(t_0^a)}} \inf_{\substack{q\in\mathcal{P}(M)\\ q\leq r\\ \tau(r)=s_1-s_0}} \tau(aq) \\
&= \sup_{\substack{r\in\mathcal{P}(M)\\ \tau(r)\geq 1-F_a(t_0^a)}} \inf_{\substack{q\in\mathcal{P}(M)\\ q\leq r\\ \tau(r)=s_1-s_0}} \tau(qaq) \\
&\leq \sup_{\substack{r\in\mathcal{P}(M)\\ \tau(r)\geq 1-F_b(t_0^b)}} \inf_{\substack{q\in\mathcal{P}(M)\\ q\leq r\\ \tau(r)=s_1-s_0}} \tau(qbq) \qquad \text{by the inequality (3.5)} \\
&= \sup_{\substack{r\in\mathcal{P}(M)\\ \tau(r)\geq 1-F_b(t_0^b)}} \inf_{\substack{q\in\mathcal{P}(M)\\ q\leq r\\ \tau(r)=s_1-s_0}} \tau(bq) \\
&= \int_{s_0}^{s_1} X_b\ dm.
\end{aligned}$$

This proves that

$$\int_I X_a \ dm \leq \int_I X_b \ dm \tag{3.6}$$

for any interval $I = [s_0, s_1) \subset [0,1)$, and in fact for any $I \in \mathcal{A} := \{\sqcup_{j=1}^k [s_0^j, s_1^j) : 0 \leq s_0^j < s_1^j < 1, k \in \mathbb{N}\}$.

But $\mathcal{A}$ is an algebra of sets which generates the σ-algebra $\mathcal{B}_{[0,1)}$. Thus for any Borel $E \subset [0,1)$, there exists a sequence $\{I_n : n \in \mathbb{N}\} \subset \mathcal{A}$ such that $\mu(I_n \Delta E) \to 0$.

Recall from Proposition 1.1 that our quantile functions of self-adjoint elements of von Neumann algebras are elements of $L^\infty([0,1), \mathcal{B}_{[0,1)}, m)$. We may hence deduce from the sentence following equation (3.6) that if E, I_n are the previous paragraph, we have:

$$\begin{aligned}
\int_E X_a dm &= \lim_{n\to\infty} \int_{I_n} X_a dm \\
&\leq \lim_{n\to\infty} \int_{I_n} X_a dm \\
&= \int_E X_b dm.
\end{aligned}$$

As $E \in \mathcal{B}_{[0,1)}$ was arbitrary, this shows that, $X_a \leq X_B \ m - a.e.$; as X_a, X_b are continuous by our hypotheses, this shows that indeed $X_a \leq X_b$. ☐

The following application of continuous version of Ky Fan's theorem gives a continuous analogue of a majorization result, which can be seen as a special case of of Lidskii-Mirsky-Wielandt's theorem, or more popularly known as Lidskii's theorem. We will discuss this theorem in section 5 as an application (Theorem 5.1) of Wielandt's theorem.

By Theorem 2.3, we have the following lemma:

Lemma 3.4. *If M is a von Neumann algebra with a faithful normal tracial state τ on it, then for $a = a^*, b = b^* \in M$ with μ_a, μ_b non-atomic and for all $s \in [0,1)$,*

$$\int_0^s X_{a+b} \ dm \geq \int_0^s (X_a + X_b) \ dm.$$

Moreover,

$$\int_0^1 X_{a+b} \ dm = \int_0^1 (X_a + X_b) \ dm.$$

Proof. Recall from our proof of Theorem 2.3 that there exists a projection $q \in \mathcal{P}(M)$ (in fact in the von Neumann algebra generated by $a+b$) such that $\tau(q) \geq s$ and

$$\begin{aligned}
\int_0^s X_{a+b}\, dm &= \tau((a+b)q) \\
&= \tau(aq) + \tau(bq) \\
&\geq \inf\{\tau(ap) : p \in \mathcal{P}(M), \tau(p) \geq s\} + \inf\{\tau(bp) : p \in \mathcal{P}(M), \tau(p) \geq s\} \\
&= \int_0^s X_a\, dm + \int_0^s X_b\, dm \\
&= \int_0^s (X_a + X_b)\, dm.
\end{aligned}$$

Finally, it is clear (from our change-of-variable argument in Proposition 2.1 for instance) that for any $c = c^* \in M$, we have $\int_0^1 X_c dm = \tau(c)$ and hence

$$\int_0^1 X_{a+b}\, dm = \tau(a+b) = \tau(a) + \tau(b) = \int_0^1 X_a\, dm + \int_0^1 X_b\, dm = \int_0^1 (X_a + X_b)\, dm.$$

□

The above is an analogue of the fact that for $n \times n$ Hermitian matrices a, b, with their eigenvalues $\lambda_1(a) \leq \cdots \leq \lambda_n(a)$ and $\lambda_1(b) \leq \cdots \leq \lambda_n(b)$, for all $k \in \{1, \cdots, n-1\}$,

$$\sum_{j=1}^k \lambda_j(a+b) \geq \sum_{j=1}^k \lambda_j(a) + \sum_{j=1}^k \lambda_j(b),$$

and

$$\sum_{j=1}^n \lambda_j(a+b) = \sum_{j=1}^n \lambda_j(a) + \sum_{j=1}^n \lambda_j(b),$$

i.e. $\lambda(a) + \lambda(b)$ is majorized by $\lambda(a+b)$ in the sense of [HLP29].

We consider the definition of majorization in the continuous context (see for example, [Sak85]) as follows:

Definition 3.5. *For $a = a^*, b = b^*$ in a von Neumann algebra M with a faithful normal tracial state τ on it, a is said to be majorized by b if $\int_0^s X_a\, dm \geq \int_0^s X_b\, dm$ for all $s \in [0,1)$ and $\int_0^1 X_a\, dm = \int_0^1 X_b\, dm$. When this happens, we simply write $X_a \prec X_b$.*

Then, Lemma 3.4 can be written as:

$$X_{a+b} \prec X_a + X_b.$$

Majorization is a weaker concept of comparing self-adjoint operators in von Neumann algebras, for example, Corollary 3.3 together with Lemma 3.4 proves that for $a = a^*, b = b^*$ with $\tau(a) = \tau(b)$,

$$a \leq b \Rightarrow \sigma(a) \prec \sigma(b),$$

but the converse is easily seen to be not true. Similarly, it can be seen that analogue of Lidskii's result does not imply that $X_{a+b} \leq X_a + X_b$. The study of majorization and its von Neumann algebraic analogue is vast (see for example, [Kam83], [Kam85], [Kam86], [Hia87]) and closely related to the minimax-type results but we will not discuss it further within this paper.

4 Continuous version of Wielandt's minimax principle

In this section we state and prove a continuous analogue of Wielandt's minimax theorem. As in the case of Theorem 3.1, our proof for the finite dimensional version of Ky Fan's theorem would give a new proof for Wielandt's original result for Hermitian matrices too. But in order to avoid repetitions, we shall be content with the continuous case here. *We make the standing 'continuity assumption' throughout this section that:* (M, τ) is a von Neumann algebra with a faithful normal tracial state on it, $a = a^* \in M$ and $A = W^*(a)$ the generated commutative von Neumann subalgebra, and that: if $B \in \{M, A\}$, $r \in \mathcal{P}(B)$, then $\forall \epsilon \in [0, \tau(r)]$, $\exists r' \leq r$ in $\mathcal{P}(B)$, with $\tau(r') = \epsilon$. Thus our results are valid for any von Neumann algebra that admits of a faithful normal tracial state and has the above-mentioned property.

Our version of Wielandt's theorem is as follows:

Theorem 4.1. *Let F, X be the distribution and quantile function of a. Let $\delta_j \in \mathbb{R}_+$ and t_0^j, t_1^j, $j = 1, \cdots, k$, be points in the spectrum of a such that $t_0^1 < t_1^1 \leq t_0^2 < t_1^2 \leq \cdots \leq t_0^{k-1} < t_1^{k-1} \leq t_0^k < t_1^k$ and $F(t_1^j) - F(t_0^j) = \delta_j$, for all j. Then*

$$\sum_{j=1}^{k} \int_{[F(t_0^j), F(t_1^j))} X(s)\, ds = \inf_{\substack{p_j \in \mathcal{P}(M) \\ p_1 \leq \cdots \leq p_k \\ \tau(p_j) \geq F(t_1^j)}} \sup_{\substack{\hat{q}_j \in \mathcal{P}(M) \\ \hat{q}_j \leq p_j \\ \tau(\hat{q}_j) = \delta_j \\ \hat{q}_j \perp \hat{q}_i \text{ for } j \neq i}} \sum_{j=1}^{k} \tau(a\hat{q}_j).$$

Moreover, $\exists p_1 \leq \cdots \leq p_k$ with $p_j \in \mathcal{P}(A) \subset \mathcal{P}(M)$, for which there exist mutually orthogonal projections $\hat{q}_j \leq p_j, \tau(\hat{q}_j) = \delta_j$, $\forall j$ such that

$$\sum_{j=1}^{k} \int_{[F(t_0^j), F(t_1^j))} X(s)\, ds = \max_{\substack{\hat{q}_j \leq p_j \\ \tau(\hat{q}_j) = \delta_j \\ \hat{q}_j \perp \hat{q}_i}} \sum_{j=1}^{k} \tau(a\hat{q}_j);$$

The following lemmas lead to the proof of the theorem above:

Lemma 4.2. *Let (M, τ) be as above. Consider, for any $k \geq 2$,*

$$\begin{aligned}
&\{r_1, r_2, \cdots r_k; q'_1, \cdots, q'_{k-1}\} \subset \mathcal{P}(M), \\
&r_1 \geq \cdots \geq r_k, \\
&\tau(r_j) \geq \delta_k + \cdots + \delta_j \ \forall\ 1 \leq j \leq k, \\
&q'_j \leq r_j \ \forall 1 \leq j \leq k-1, \\
&q'_s q'_t = 0 \ \forall 1 \leq s < t \leq k-1, \\
&\tau(q'_j) = \delta_j \ \forall 1 \leq j < k-1.
\end{aligned}$$

Then there exist mutually orthogonal projections $q_j \leq r_j$ $\forall 1 \leq j \leq k$ in M, such that $\sum_{j=1}^{k} q_j \geq \sum_{j=1}^{k-1} q'_j$, and $\tau(q_j) = \delta_j$ $\forall 1 \leq j \leq k$.

Proof. The proof follows by induction. For $k = 2$, choose $q_2 \leq r_2$ such that $\tau(q_2) = \delta_2$.

Let $e = q_2 \vee q'_1$.

Then $\tau(e) \leq \tau(q_2) + \tau(q'_1) = \delta_2 + \delta_1$ and $e \leq r_1$.

But by the hypothesis for $k = 2$, $\tau(r_1) \geq \delta_2 + \delta_1$.

Hence by the 'standing continuity assumption', there exists $f \in \mathcal{P}(M)$ such that $e \leq f \leq r_1$ and $\tau(f) = \delta_2 + \delta_1$. In particular $q_2 \leq e \leq f$; thus $f - q_2 \in \mathcal{P}(M)$ with trace δ_1.

Choose $q_1 = f - q_2$. Then $q_j \leq r_j$ with trace δ_j for $j = 1, 2$ and $q_1 + q_2 = f \geq e \geq q'_1$, as required.

Suppose now, for the inductive step, that this result holds with k replaced by $k - 1$, and that $r_1, \cdots, r_k, q_1, \cdots, q_{k-1}$ are as in the statement of the Lemma.

By induction hypothesis - applied to $\{r_2, \cdots, r_k; q'_2, \cdots, q'_{k-1}\} \subset \mathcal{P}(M)$ - there exist mutually orthogonal projections $q_2, \cdots, q_k$ in M such that $q_j \leq r_j$ and $\tau(q_j) = \delta_j, \forall 2 \leq j \leq k$ and

$$\sum_{j=2}^{k} q_j \geq \sum_{j=2}^{k-1} q'_j \tag{4.1}$$

.

Let $e_2 = q_2 + \cdots + q_k$ and $e = e_2 \vee q'_1$.

Then $\tau(e) \leq \tau(e_2) + \tau(q'_1) = (\delta_k + \cdots + \delta_2) + \delta_1$ and $e \leq r_1$.

But $\tau(r_1) \geq \delta_k + \cdots + \delta_1$; thus (by the 'standing continuity assumption') there exists $f \in \mathcal{P}(M)$ such that $e \leq f \leq r_1$ and $\tau(f) = \delta_k + \cdots + \delta_1$. In particular $e_2 \leq e \leq f$; thus $f - e_2 \in \mathcal{P}(M)$ with trace δ_1.

Choose $q_1 = f - e_2$. Then $q_1 \leq r_1$ and $q_1 \perp q_j$ for $2 \leq j \leq k$.

Moreover,

$$\begin{aligned}
q_1 + q_2 + \cdots + q_k = f \geq e &= e_2 \vee q'_1 \\
&= (\sum_{j=2}^{k} q_j) \vee q'_1 \\
&\geq (\sum_{2}^{k-1} q'_j) \vee q'_1 \quad \text{by equation (4.1)} \\
&= \sum_{1}^{k-1} q'_j,
\end{aligned}$$

thus completing the proof of the inductive step. $\square$

Lemma 4.2 can be rewritten as:

Lemma 4.3. *Let (M, τ) be as above. Suppose $\delta_j \in \mathbb{R}_+$, and $\{r_1 \geq \cdots \geq r_k\} \subset \mathcal{P}(M)$ such that $\tau(r_j) \geq \delta_k + \cdots + \delta_j, \forall j = 1, \cdots, k$ and suppose we are given $(k-1)$ mutually orthogonal projections q'_j such that $q'_j \leq r_j$ and $\tau(q'_j) = \delta_j \ \forall \ j = 1, \cdots, k-1$. Let*

$$e' = q'_1 + \cdots + q'_{k-1} \leq r_1.$$

Then there exist projections $q \leq r_1 - e', q_j \leq r_j \ \forall \ j = 1, \cdots, k$, such that $\tau(q) = \delta_k$ and $\tau(q_j) = \delta_j \ \forall \ j$, $\{q_j : 1 \leq j \leq k\}$ pairwise mutually orthogonal and

$$q + e' = q_1 + \cdots + q_k,$$

which is also a projection below r_1.

Proof. Use Lemma 4.2 and choose $q = (q_1 + \cdots + q_k) - e'$. □

Before proceeding further, we state a short but useful result:

Lemma 4.4. *For (M, τ) as above and $r, e \in \mathcal{P}(M)$,*

$$\tau(r \wedge e^\perp) \geq \tau(r) - \tau(e)$$

where, of course, $e^\perp = 1 - e$.

Proof.

$$\begin{aligned} 1 + \tau(r \wedge e^\perp) &\geq \tau(r \vee e^\perp) + \tau(r \wedge e^\perp) \\ &= \tau(r) + 1 - \tau(e) \end{aligned}$$

as required. □

The above results lead to the following lemma:

Lemma 4.5. *Let (M, τ), t_0^j, t_1^j, δ_j be as in Wielandt's theorem. Let $\{r_1 \geq \cdots \geq r_k\}$ and $\{p_1 \leq \cdots \leq p_k\}$ be sets of projections in M such that $\tau(p_j) \geq F(t_1^j)$, $\tau(r_j) \geq 1 - F(t_0^j)$ for all $1 \leq j \leq k$. Then there exist mutually orthogonal projections $q_j \leq r_j$ and mutually orthogonal projections $\tilde{q}_j \leq p_j$ such that $\tau(q_j) = \tau(\tilde{q}_j) = \delta_j \ \forall \ j$ and $q_1 + \cdots + q_k = \tilde{q}_1 + \cdots + \tilde{q}_k$.*

Proof. The proof is by induction.

For $k = 1$, deduce from Lemma 4.4 that

$$\begin{aligned} \tau(p_1 \wedge r_1) &\geq \tau(p_1) - \tau(r_1^\perp) \\ &= \tau(p_1) - 1 + \tau(r_1) \\ &\geq F(t_1^1) - 1 + 1 - F(t_0^1) \\ &= F(t_1^1) - F(t_0^1) \\ &= \delta_1, \end{aligned}$$

and thus (by our standing 'continuity assumption) there exists a projection $q_1 = \tilde{q}_1 \leq p_1 \wedge r_1$ of trace δ_1.

For the inductive step, assume $p_1 \le \cdots \le p_k, r_1 \ge \cdots \ge r_k$ are as in the lemma and that the lemma is valid with k replaced by $k-1$. By the induction hypothesis applied to $p_1 \le \cdots \le p_{k-1}, r_1 \ge \cdots \ge r_{k-1}$, there are mutually orthogonal projections $q'_j \le r_j$ and mutually orthogonal projections $\tilde{q}_j \le p_j$ such that $\tau(q'_j) = \tau(\tilde{q}_j) = \delta_j$ for all $j = 1, \cdots, k-1$ and $\sum_{j=1}^{k-1} q'_j = \sum_{j=1}^{k-1} \tilde{q}_j =: e'$, say.

Then $e' \le p_{k-1} \le p_k$.

Let $\ell_j = r_j \wedge p_k, \ \forall\, j = 1, \cdots, k$.

Then $\ell_k \le \cdots \le \ell_1$. An application of Lemma 4.4, as seen above in the $k = 1$ case, gives:

$$\begin{aligned}\tau(\ell_j) &\ge F(t_1^k) - F(t_0^j) \\ &\ge F(t_1^k) - F(t_0^k) + F(t_1^{k-1}) - F(t_0^{k-1}) + \cdots + F(t_1^j) - F(t_0^j) \\ &= \delta_k + \cdots + \delta_j \ \forall\, j = 1, \cdots, k.\end{aligned}$$

Now by Lemma 4.3 - applied with ℓ_j in place of r_j - we may conclude that $\exists\, q \le \ell_1 - e', q_j \le \ell_j \ (\le r_j)$ with $\tau(q) = \delta_k$, $\tau(q_j) = \delta_j \ \forall\, j$ and $q_j \perp q_i \ \forall\, j \ne i$, such that $q + e' = q_1 + \cdots + q_k$.

But $q + e' = q + \tilde{q}_1 + \cdots + \tilde{q}_{k-1}$, where $\tilde{q}_j \le p_j \ \forall\, j = 1, \cdots, k-1$ and $q \le \ell_1 - e' \le \ell_1 = r_1 \wedge p_k$.

Choosing $\tilde{q}_k = q$, the proof of the inductive step is complete.

□

Now we are ready to prove Theorem 4.1.

Proof. For "$\ge$" , we take $p_j := 1_{(-\infty, t_1^j)}(a)$ and $\tilde{q}_j := 1_{[t_0^j, t_1^j)}(a) \le p_j$.

For proving "$\le$" here, let us choose any $p_1 \le \cdots \le p_k$ such that $p_j \in \mathcal{P}(M)$ and $\tau(p_j) \ge F(t_1^j)$.

Let $r_j = 1_{[t_0^j, \infty)}(a) \ \forall\, j = 1, \cdots, k$. Then $r_1 \ge \cdots \ge r_k$ with $\tau(r_j) = 1 - F(t_0^j)$.

Now by Lemma 4.5, there exist mutually orthogonal projections $q_j \le r_j$ and mutually orthogonal projections $\tilde{q}_j \le p_j$ with $\tau(q_j) = \tau(\tilde{q}_j) = \delta_j$ such that $q_1 + \cdots + q_k = \tilde{q}_1 + \cdots + \tilde{q}_k$.

Notice that by our version of Ky Fan's theorem,

$$\tau(aq_j) \ge \inf_{\substack{q \in \mathcal{P}(M) \\ q \le r_j \\ \tau(q) = \delta_j}} \tau(aq) = \int_{F(t_0^j)}^{F(t_1^j)} X(s)\ ds.$$

Hence,

$$\sum_{j=1}^{k} \int_{F(t_0^j)}^{F(t_1^j)} X(s)\ ds \le \sum_{j=1}^{k} \tau(aq_j) = \sum_{j=1}^{k} \tau(a\tilde{q}_j)$$

(since $q_1 + \cdots + q_k = \tilde{q}_1 + \cdots + \tilde{q}_k$), where $\tilde{q}_j \in \mathcal{P}(M), \tilde{q}_j \le p_j$ with $\tau(\tilde{q}_j) = \delta_j$ and $\tilde{q}_j \perp \tilde{q}_i$. Hence,

$$\sum_{j=1}^{k} \int_{F(t_0^j)}^{F(t_1^j)} X(s)\ ds \le \sup_{\substack{\hat{q}_j \in \mathcal{P}(M) \\ \hat{q}_j \le p_j \\ \tau(\hat{q}_j) = \delta_j \\ \hat{q}_j \perp \hat{q}_i}} \sum_{j=1}^{k} \tau(a\hat{q}_j).$$

Now the theorem follows from the fact that $p_1 \leq \cdots \leq p_k$ were chosen arbitrarily.

$\square$

Remark 4.6. *For $\delta_1 = \cdots = \delta_k = \delta$, the theorem can be written as:*

$$\sum_{j=1}^{k} \int_{[F(t_0^j), F(t_1^j))} X(s)\; ds = \min_{\substack{p_j \in \mathcal{P}(M) \\ p_1 \leq \cdots \leq p_k \\ \tau(p_j) \geq F(t_1^j)}} \sup_{\substack{q \in \mathcal{P}(M) \\ q \leq p_k \\ \tau(q) = k\delta \\ \tau(q \wedge p_j) \geq j\delta}} \tau(aq).$$

5 Continuous version of Lidskiil's theorem

The continuous analogue of Lidskii's majorization theorem is a majorization result similar to Lemma 3.4 above, but a strictly stronger one. The matricial version of this result states that given $1 \leq i_1 < \cdots < i_k \leq n$, for $n \times n$ Hermitian matrices a and b with eigenvalues given as $\lambda_1(a) \leq \cdots \leq \lambda_n(a)$ and $\lambda_1(b) \leq \cdots \leq \lambda_n(b)$,

$$\sum_{j=1}^{k} \lambda_{i_j}(a+b) \geq \sum_{j=1}^{k} \lambda_{i_j}(a) + \sum_{j=1}^{k} \lambda_j(b).$$

In this section we state and prove a continuous version of the above. However, we would like to mention here that continuous versions of Lidskii's result have been discussed and proved in several other places, e. g. in [HN87]. But it is a natural application of Theorem 4.1, so we would like to present it for the sake of completion of our article.

Theorem 5.1. *Let $a = a^*, b = b^* \in M$ be such that μ_a, μ_b, μ_{a+b} are non-atomic. Let F_a, F_b, F_{a+b} and X_a, X_b, X_{a+b} be the distribution and quantile functions of a, b and $(a+b)$ respectively. Let $\delta_j \in \mathbb{R}_+$. Let us choose points $\{t_0^j, t_1^j,\ j = 1, \cdots, k\}$ and $\{u_0^j, u_1^j,\ j = 1, \cdots, k\}$ in the spectra of $(a+b)$ and a respectively such that $t_0^1 < t_1^1 \leq t_0^2 < t_1^2 \leq \cdots \leq t_0^{k-1} < t_1^{k-1} \leq t_0^k < t_1^k$, $u_0^1 < u_1^1 \leq u_0^2 < u_1^2 \leq \cdots \leq u_0^{k-1} < u_1^{k-1} \leq u_0^k < u_1^k$ and $F_{a+b}(t_1^j) - F_{a+b}(t_0^j) = F_a(u_1^j) - F_a(u_0^j) = \delta_j$ for all j. Then*

$$\sum_{j=1}^{k} \int_{F_{a+b}(t_0^j)}^{F_{a+b}(t_1^j)} X_{a+b}\; dm \geq \sum_{j=1}^{k} \int_{F_a(u_0^j)}^{F_a(u_1^j)} X_a\; dm + \int_0^{\sum_{i=1}^{k} \delta_i} X_b\; dm$$

Proof. We know by Theorem 4.1 that M contains projections p_j^{a+b} with $\tau(p_j^{a+b}) = F_{a+b}(t_1^j)$ for all $j = 1, \cdots, k$ such that

$$\sum_{j=1}^{k} \int_{F_{a+b}(t_0^j)}^{F_{a+b}(t_1^j)} X_{a+b}\; dm = \sup_{\substack{\hat{q}_j \in \mathcal{P}(M) \\ \hat{q}_j \leq p_j^{a+b} \\ \tau(\hat{q}_j) = \delta_j \\ \hat{q}_j \perp \hat{q}_i \text{ for } j \neq i}} \sum_{j=1}^{k} \tau((a+b)\hat{q}_j).$$

Note that if $\{\alpha_j, \beta_j : j \in \mathcal{J}\} \subset \mathbb{R}$ for some index set $\mathcal{J}$, then $\sup_{\{j\in\mathcal{J}\}}\{\alpha_j + \beta_j\} \geq \sup_{\{j\in\mathcal{J}\}}\{\alpha_j\} + \inf_{\{j\in\mathcal{J}\}}\{\beta_j\}$, since we clearly have $\alpha_{j'} + \beta_{j'} \geq \alpha_{j'} + \inf_{\{j\in\mathcal{J}\}}\{\beta_j\}$ for all $j' \in \mathcal{J}$, and we may now take $\sup_{j'\in\mathcal{J}}$ of both sides.

$$
\begin{aligned}
&\text{Thus, } \sum_{j=1}^{k} \int_{F_{a+b}(t_0^j)}^{F_{a+b}(t_1^j)} X_{a+b}\, dm \\
&= \sup_{\substack{\hat{q}_j \in \mathcal{P}(M) \\ \hat{q}_j \leq p_j^{a+b} \\ \tau(\hat{q}_j)=\delta_j \\ \hat{q}_j \perp \hat{q}_i \text{ for } j \neq i}} \sum_{j=1}^{k} \tau((a+b)\hat{q}_j) \\
&\geq \sup_{\substack{\hat{q}_j \in \mathcal{P}(M) \\ \hat{q}_j \leq p_j^{a+b} \\ \tau(\hat{q}_j)=\delta_j \\ \hat{q}_j \perp \hat{q}_i \text{ for } j \neq i}} \sum_{j=1}^{k} \tau(a\hat{q}_j) + \inf_{\substack{\hat{q}_j \in \mathcal{P}(M) \\ \hat{q}_j \leq p_j^{a+b} \\ \tau(\hat{q}_j)=\delta_j \\ \hat{q}_j \perp \hat{q}_i \text{ for } j \neq i}} \sum_{j=1}^{k} \tau(b\hat{q}_j) \\
&\geq \sum_{j=1}^{k} \int_{F_a(u_0^j)}^{F_a(u_1^j)} X_a\, dm + \int_0^{\sum_{i=1}^{k}\delta_i} X_b\, dm \qquad \text{by Theorem 4.1 and Theorem 2.3,}
\end{aligned}
$$

proving the continuous analogue of Lidskii's theorem. □

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