Free Probability - classical and free CLTs
ISI Bangalore, November 2009

V.S. Sunder
Institute of Mathematical Sciences
Chennai, India
sunder@imsc.res.in

11th November 2009

Definition

A **non-commutative probability space** - henceforth *NCPS* - is a pair (A, ϕ) of a unital algebra A equipped with a linear functional ϕ satisfying $\phi(1) = 1$

Definition

A **non-commutative probability space** - henceforth *NCPS* - is a pair (A, ϕ) of a unital algebra A equipped with a linear functional ϕ satisfying $\phi(1) = 1$

Definition

An *NCPS* (A, ϕ) can come in various flavours; it is said to be a

1. $*$ *NCPS* if A is a *-algebra and $\phi(a^*a) \geq 0$ for all $a \in A$.
2. C^* *NCPS* if it is a $*$ *NCPS* and A is a C^*-algebra.
3. *von Neumann NCPS* if it is a $*$ *NCPS*, A is a von Neumann algebra and ϕ is *normal*[a].

[a] meaning appropriately continuous

Definition

A **non-commutative probability space** - henceforth *NCPS* - is a pair (A, ϕ) of a unital algebra A equipped with a linear functional ϕ satisfying $\phi(1) = 1$

Definition

An *NCPS* (A, ϕ) can come in various flavours; it is said to be a

1. $*$ *NCPS* if A is a *-algebra and $\phi(a^*a) \geq 0$ for all $a \in A$.
2. C^* *NCPS* if it is a $*$ *NCPS* and A is a C^*-algebra.
3. *von Neumann NCPS* if it is a $*$ *NCPS*, A is a von Neumann algebra and ϕ is *normal*[a].

[a] meaning appropriately continuous

We shall reserve the term *random variable* for a self-adjoint element $a = a^*$ in a $*$ *NCPS*.

1. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Then $A = L^\infty(\Omega, \mathcal{B}, \mu)$ is a von Neumann NCPS.

Examples

1. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Then $A = L^\infty(\Omega, \mathcal{B}, \mu)$ is a von Neumann NCPS.

2. With $(\Omega, \mathcal{B}, \mu)$ as above, $A = \cap_{p=1}^{\infty} L^p(\Omega, \mathcal{B}, \mu)$ is a $* \, NCPS$.

Examples

1. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Then $A = L^\infty(\Omega, \mathcal{B}, \mu)$ is a von Neumann NCPS.

2. With $(\Omega, \mathcal{B}, \mu)$ as above, $A = \cap_{p=1}^{\infty} L^p(\Omega, \mathcal{B}, \mu)$ is a $*$ *NCPS*.

3. Let $A = M_n(\mathbb{C})$ and $\phi(a) = tr\ a = \frac{1}{n} Tr\ a$ be the normalised trace. More generally, if (A, ϕ) is an *NCPS*, then we can consider its *ampliation* $(M_n(A), \phi^{(n)})$ where $\phi^{(n)}(((a_{ij}))) = \frac{1}{n} \sum_{i=1}^{n} \phi(a_{ii})$. If (A, ϕ) is as in example 2 above, the elements of $(A^{(n)}, \phi^{(n)})$ are random matrices which have long interested physicists.

Examples

1. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Then $A = L^\infty(\Omega, \mathcal{B}, \mu)$ is a von Neumann NCPS.

2. With $(\Omega, \mathcal{B}, \mu)$ as above, $A = \cap_{p=1}^{\infty} L^p(\Omega, \mathcal{B}, \mu)$ is a $*$ *NCPS*.

3. Let $A = M_n(\mathbb{C})$ and $\phi(a) = tr\ a = \frac{1}{n} Tr\ a$ be the normalised trace. More generally, if (A, ϕ) is an *NCPS*, then we can consider its *ampliation* $(M_n(A), \phi^{(n)})$ where $\phi^{(n)}(((a_{ij}))) = \frac{1}{n} \sum_{i=1}^{n} \phi(a_{ii})$. If (A, ϕ) is as in example 2 above, the elements of $(A^{(n)}, \phi^{(n)})$ are random matrices which have long interested physicists.

4. For a unital $*$ subalgebra $A \subset \mathcal{L}(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$, define $\phi_\xi(a) = \langle a\xi, \xi \rangle$. Then (A, ϕ_ξ) is a $*$ *NCPS*.

1. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Then $A = L^\infty(\Omega, \mathcal{B}, \mu)$ is a von Neumann NCPS.

2. With $(\Omega, \mathcal{B}, \mu)$ as above, $A = \cap_{p=1}^{\infty} L^p(\Omega, \mathcal{B}, \mu)$ is a $*$ *NCPS*.

3. Let $A = M_n(\mathbb{C})$ and $\phi(a) = tr\ a = \frac{1}{n} Tr\ a$ be the normalised trace. More generally, if (A, ϕ) is an *NCPS*, then we can consider its *ampliation* $(M_n(A), \phi^{(n)})$ where $\phi^{(n)}(((a_{ij}))) = \frac{1}{n} \sum_{i=1}^{n} \phi(a_{ii})$. If (A, ϕ) is as in example 2 above, the elements of $(A^{(n)}, \phi^{(n)})$ are random matrices which have long interested physicists.

4. For a unital $*$ subalgebra $A \subset \mathcal{L}(\mathcal{H})$ and a unit vector $\xi \in \mathcal{H}$, define $\phi_\xi(a) = \langle a\xi, \xi \rangle$. Then (A, ϕ_ξ) is a $*$ *NCPS*.

5. For a countable group Γ, let $\ell^2(\Gamma)$ be the Hilbert space with ortonormal basis $\{\xi_t : t \in \Gamma\}$, and let λ be the left-regular (unitary) representation of Γ on $\ell^2(\Gamma)$ given by $\lambda(s)\xi_t = \xi_{st}$. Then, $(\mathbb{C}\Gamma, \phi_{\xi_1} \circ \lambda)$ is a $*$ *NCPS*. The closure of $\lambda(\mathbb{C}\Gamma)$ in the norm topology, resp. the *strong operator topology* is denoted by $C^*_{red}(\Gamma)$, resp. $L\Gamma$; these are C^* *NCPS* resp. *von Neumann NCPS*, when endowed with with the 'vector state ϕ_{ξ_1}'.

The theory of free probability was created by Voiculescu as a means to (hopefully!) solving the still unsolved problem:

$$\textit{Does } n \neq m \Rightarrow L\mathbb{F}_n \ncong L\mathbb{F}_m?$$

Here $\mathbb{F}_n$ denotes the free group on n generators.

The theory of free probability was created by Voiculescu as a means to (hopefully!) solving the still unsolved problem:

$$Does\ n \neq m \Rightarrow L\mathbb{F}_n \ncong L\mathbb{F}_m?$$

Here $\mathbb{F}_n$ denotes the free group on n generators.

Although that problem is still open, great strides have been made in 'Free Probability' theory. Just as the standard normal distribution occupies pride of place in classical probability theory, the corresponding role is played in Free probability by the so-called standard **semi-circular distribution**, which is the compactly supported probability measure defined by

$$\mu(E) = \frac{1}{2\pi} \int_{-2}^{2} 1_E(t)\sqrt{4-t^2}dt.$$

The theory of free probability was created by Voiculescu as a means to (hopefully!) solving the still unsolved problem:

$$\textit{Does } n \neq m \Rightarrow L\mathbb{F}_n \ncong L\mathbb{F}_m?$$

Here $\mathbb{F}_n$ denotes the free group on n generators.

Although that problem is still open, great strides have been made in 'Free Probability' theory. Just as the standard normal distribution occupies pride of place in classical probability theory, the corresponding role is played in Free probability by the so-called standard **semi-circular distribution**, which is the compactly supported probability measure defined by

$$\mu(E) = \frac{1}{2\pi} \int_{-2}^{2} 1_E(t)\sqrt{4 - t^2}dt.$$

This standard semi-circular law has moments given by

$$\int t^n d\mu(t) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ C_p & \text{if } n = 2p \end{cases}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

A family of subalgebras $A_i, i \in I$ of a NCPS (A, ϕ) is said to be **free** (or freely independent) if whenever $x_j \in A_{i_j}, 1 \leq j \leq n$ satisfy $i_j \neq i_{j+1}$ $\forall 1 \leq j < n$ and $\phi(x_j) = 0 \forall j$, then necessarily also $\phi(x_1 x_2 \cdots x_n) = 0$.

A family of subalgebras $A_i, i \in I$ of a NCPS (A, ϕ) is said to be **free** (or freely independent) if whenever $x_j \in A_{i_j}, 1 \leq j \leq n$ satisfy $i_j \neq i_{j+1}\ \forall 1 \leq j < n$ and $\phi(x_j) = 0 \forall j$, then necessarily also $\phi(x_1 x_2 \cdots x_n) = 0$.

Theorem

Given a family (A_i, ϕ_i) of NCPS of the same flavour, *there exists an NCPS (A, ϕ) also with the same flavour and the following properties:*

- *there exist monomorphisms $\pi_i : A_i \to A$ such that $\phi_i = \phi \circ \pi_i\ \forall i$; and*
- *the subalgebras $\pi_i(A_i)$ are freely independent in (A, ϕ);*
- *the NCPS (A, ϕ) is unique up to isomorphism if it is required to be generated by $\cup \phi_i(A_i)$, denoted by $(A, \phi) = *_{i \in I}(A_i, \phi_i)$, and is called the free product of the family $\{(A_i, \phi_i) : i \in I\}$.*

A family of subalgebras $A_i, i \in I$ of a NCPS (A, ϕ) is said to be **free** (or freely independent) if whenever $x_j \in A_{i_j}, 1 \leq j \leq n$ satisfy $i_j \neq i_{j+1}$ $\forall 1 \leq j < n$ and $\phi(x_j) = 0 \forall j$, then necessarily also $\phi(x_1 x_2 \cdots x_n) = 0$.

Theorem

Given a family (A_i, ϕ_i) of NCPS of the same flavour, *there exists an NCPS (A, ϕ) also with the same flavour and the following properties:*

- *there exist monomorphisms $\pi_i : A_i \to A$ such that $\phi_i = \phi \circ \pi_i$ $\forall i$; and*
- *the subalgebras $\pi_i(A_i)$ are freely independent in (A, ϕ);*
- *the NCPS (A, ϕ) is unique up to isomorphism if it is required to be generated by $\cup \phi_i(A_i)$, denoted by $(A, \phi) = *_{i \in I}(A_i, \phi_i)$, and is called the free product of the family $\{(A_i, \phi_i) : i \in I\}$.*

Example

$$L\mathbb{F}_n \cong *^n L\mathbb{Z}$$

Corollary

If a family of subalgebras $A_i, i \in I$ of an NCPS (A, ϕ) is freely independent, and if A_0 is the subalgebra generated by $\cup_i A_i$, then $\phi|_{A_0}$ is determined by $\phi|_{\cup_i A_i}$.

Corollary

If a family of subalgebras $A_i, i \in I$ of an NCPS (A, ϕ) is freely independent, and if A_0 is the subalgebra generated by $\cup_i A_i$, then $\phi|_{A_0}$ is determined by $\phi|_{\cup_i A_i}$.

Proof.

Any element of A_0 may be written as $a(1)a(2)\cdots a(n)$, where $a(j) \in A_{i_j}$ and $i_j \neq i_{j+1}$ for all j. For any $x \in A$, write $x = \bar{x} + x_0$, where $\bar{x} = \phi(x)1$ and $\phi(x_0) = 0$. The proof is by induction on the number ν of j for which $a(j)_0 \neq 0$. If $\nu = 0$, then $a(j) = \overline{a(j)}\ \forall j$, and hence $\phi(a(1)\cdots a(n)) = \phi(a(1))\cdots\phi(a(n))$.

Corollary

If a family of subalgebras $A_i, i \in I$ of an NCPS (A, ϕ) is freely independent, and if A_0 is the subalgebra generated by $\cup_i A_i$, then $\phi|_{A_0}$ is determined by $\phi|_{\cup_i A_i}$.

Proof.

Any element of A_0 may be written as $a(1)a(2)\cdots a(n)$, where $a(j) \in A_{i_j}$ and $i_j \neq i_{j+1}$ for all j. For any $x \in A$, write $x = \bar{x} + x_0$, where $\bar{x} = \phi(x)1$ and $\phi(x_0) = 0$. The proof is by induction on the number ν of j for which $a(j)_0 \neq 0$. If $\nu = 0$, then $a(j) = \overline{a(j)}\ \forall j$, and hence $\phi(a(1)\cdots a(n)) = \phi(a(1))\cdots\phi(a(n))$.

Assume the lemma holds for $\nu < m$ and that $a(j)_0 \neq 0$ for exactly m j's. Suppose, for convenience, that $a(j)_0 \neq 0$. Then

$$\begin{aligned} \phi(a(1)\cdots a(n)) &= \phi(a(1)\cdots a(j-1)(\overline{a(j)} + a(j)_0)a(j+1)\cdots a(n)) \\ &= \phi(a(j))\phi(a(1)\cdots a(j-1)a(j+1)\cdots a(n)) \\ &\quad + \phi(a(1)\cdots a(j-1)a(j)_0 a(j+1)\cdots a(n)), \end{aligned}$$

and both these terms are determined by $\phi|_{\cup_i A_i}$ by induction hypothesis, and we are done. □

Definition

A probability measure μ (defined on the σ-algebra of Borel sets in $\mathbb{R}$) is said to be an analytic distribution of a random variable a in an *NCPS* (A, ϕ) if

$$\phi(a^n) = \int_{\mathbb{R}} t^n d\mu(t)$$

The numbers $\phi(x^n)$ (resp., $\int_{\mathbb{R}} t^n d\mu(t)$) are called the moments of the random variable a (rep., the measure μ).

Definition

A probability measure μ (defined on the σ-algebra of Borel sets in $\mathbb{R}$) is said to be an analytic distribution of a random variable a in an *NCPS* (A, ϕ) if

$$\phi(a^n) = \int_{\mathbb{R}} t^n d\mu(t)$$

The numbers $\phi(x^n)$ (resp., $\int_{\mathbb{R}} t^n d\mu(t)$) are called the moments of the random variable a (rep., the measure μ).

Any random variable in a C^* probability space admits a unique analytic distribution, which is compactly supported (in fact, in its spectrum).

Definition

A probability measure μ (defined on the σ-algebra of Borel sets in $\mathbb{R}$) is said to be an analytic distribution of a random variable a in an *NCPS* (A, ϕ) if

$$\phi(a^n) = \int_{\mathbb{R}} t^n d\mu(t)$$

The numbers $\phi(x^n)$ (resp., $\int_{\mathbb{R}} t^n d\mu(t)$) are called the moments of the random variable a (rep., the measure μ).

Any random variable in a C^* probability space admits a unique analytic distribution, which is compactly supported (in fact, in its spectrum).

Our example 2 permits us to include classical random variables with finite moments to be considered as our sort of random variable ($a(t) = t$) which may possess analytic distributions which are probability measures (such as the standard normal $N(0, 1)$) which are *not* compactly supported.

Definition

A sequence $\{a_n : n \in \mathbb{N}\}$ of random variables (in *NCPS* (A_n, ϕ_n)) is said to converge in distribution to a random variable a (in an *NCPS* (A, ϕ)) if we have 'moment convergence', i.e., if $\phi_n(a_n^k) \to \phi(a^k)\ \forall k$.

Definition

A sequence $\{a_n : n \in \mathbb{N}\}$ of random variables (in *NCPS* (A_n, ϕ_n)) is said to converge in distribution to a random variable a (in an *NCPS* (A, ϕ)) if we have 'moment convergence', i.e., if $\phi_n(a_n^k) \to \phi(a^k)\ \forall k$.

Call a probability measure μ on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ *moment-determined* if the only probability measure ν on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ with the same moments as μ is $\nu = \mu$.

Definition

A sequence $\{a_n : n \in \mathbb{N}\}$ of random variables (in *NCPS* (A_n, ϕ_n)) is said to converge in distribution to a random variable a (in an *NCPS* (A, ϕ)) if we have 'moment convergence', i.e., if $\phi_n(a_n^k) \to \phi(a^k)\ \forall k$.

Call a probability measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ *moment-determined* if the only probability measure ν on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with the same moments as μ is $\nu = \mu$.

Weierstrass implies that any compactly supported probability measure is moment-determined. More generally, a probability measure μ is moment-determined if its moment sequence $\{m_n : n \in \mathbb{N}\}$ satisfies the condition that $\sum_{n=1}^{\infty} |(m_n/n!)|\epsilon^n < \infty$ for some $\epsilon > 0$. (*Reason:* This will imply that $\int_{\mathbb{R}} e^{tx} d\mu(t) < \infty$)$\forall x \in (-\epsilon, \epsilon)$, and hence that also $\int_{\mathbb{R}} |e^{tz}| d\mu(t) < \infty$ whenever $|Re\ z| < \epsilon$. Some simple complex function theory now shows that that if μ and ν have the same moment sequence, then they must have identical Fourier transforms, and hence agree.)

Definition

A sequence $\{a_n : n \in \mathbb{N}\}$ of random variables (in *NCPS* (A_n, ϕ_n)) is said to converge in distribution to a random variable a (in an *NCPS* (A, ϕ)) if we have 'moment convergence', i.e., if $\phi_n(a_n^k) \to \phi(a^k) \; \forall k$.

Call a probability measure μ on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ *moment-determined* if the only probability measure ν on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ with the same moments as μ is $\nu = \mu$.

Weierstrass implies that any compactly supported probability measure is moment-determined. More generally, a probability measure μ is moment-determined if its moment sequence $\{m_n : n \in \mathbb{N}\}$ satisfies the condition that $\sum_{n=1}^\infty |(m_n/n!)|\epsilon^n < \infty$ for some $\epsilon > 0$. (*Reason:* This will imply that $\int_\mathbb{R} e^{tx} d\mu(t) < \infty$)$\forall x \in (-\epsilon, \epsilon)$, and hence that also $\int_\mathbb{R} |e^{tz}| d\mu(t) < \infty$ whenever $|Re\ z| < \epsilon$. Some simple complex function theory now shows that that if μ and ν have the same moment sequence, then they must have identical Fourier transforms, and hence agree.)

In particular, the standard normal measure is moment-determined.

Theorem (Classical CLT)

Suppose that $\{X_n : n \in \mathbb{N}\}$ *is a sequence of (stochastically) independent and identically distributed classical random variables which have moments of all orders*[a]*, and whose distribution is moment-generated. Assume for simplicity that the first moment is zero and that the variance is one. Then, the sequence* $\{\frac{X_1+\cdots+X_n}{\sqrt{n}} : n \in \mathbb{N}\}$ *converges in distribution to the standard normal distribution.*

[a] The theorem is true even if only existence of first two moments is demanded of the X_n's.

Central Limit Theorems (CLT)

Theorem (Classical CLT)

Suppose that $\{X_n : n \in \mathbb{N}\}$ is a sequence of (stochastically) independent and identically distributed classical random variables which have moments of all orders[a]*, and whose distribution is moment-generated. Assume for simplicity that the first moment is zero and that the variance is one. Then, the sequence $\{\frac{X_1+\cdots+X_n}{\sqrt{n}} : n \in \mathbb{N}\}$ converges in distribution to the standard normal distribution.*

[a] The theorem is true even if only existence of first two moments is demanded of the X_n's.

Theorem (Free CLT)

If a_n is a freely independent sequence of identically distributed centred random variables, then the sequence $\{\left(\frac{a_1+\cdots a_n}{\sqrt{n}}\right) : n = 1, 2, \cdots\}$ converges in distribution to a standard semi-circular distribution.

We shall provide proofs which are applicable to both cases up to a point.

Now

$$\phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n$$

$$= \sum_{r(1),r(2),\cdots,r(n)=1}^{N} N^{-\frac{n}{2}} \phi(a_{(r(1)} a_{(r(2)} \cdots a_{(r(n))})$$

Now

$$\phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n$$

$$= \sum_{r(1),r(2),\cdots,r(n)=1}^{N} N^{-\frac{n}{2}} \phi(a_{(r(1)} a_{(r(2)} \cdots a_{(r(n)})$$

For $\mathbf{r} \in [N]^n$, let $\pi_\mathbf{r}$ be the partition/equivalence relation on $[n]$ defined by $i \sim_{\pi_\mathbf{r}} j \Rightarrow r(i) = r(j)$. The *i.i.d.* assumption shows that $\phi(a_{(r(1)} a_{(r(2)} \cdots a_{(r(n)})$ depends only on $\pi_\mathbf{r}$. (For instance, $\phi(a_2 a_3 a_7 a_3 a_2) = \phi(a_1 a_5 a_2 a_5 a_1)$.) Call this common value $f(\pi_\mathbf{r})$. Writing $\mathcal{P}_n$ for the set of partitions of $[n]$, and letting $C_\pi = |\{\mathbf{r} : \pi_\mathbf{r} = \pi\}|$, we see that

$$\phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n = \sum_{\pi \in \mathcal{P}_n} N^{-\frac{n}{2}} C_\pi f(\pi). \qquad (1)$$

Now

$$\phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n$$

$$= \sum_{r(1),r(2),\cdots,r(n)=1}^{N} N^{-\frac{n}{2}} \phi(a_{(r(1)} a_{(r(2)} \cdots a_{(r(n)})$$

For $\mathbf{r} \in [N]^n$, let $\pi_{\mathbf{r}}$ be the partition/equivalence relation on $[n]$ defined by $i \sim_{\pi_{\mathbf{r}}} j \Rightarrow r(i) = r(j)$. The *i.i.d.* assumption shows that $\phi(a_{(r(1)} a_{(r(2)} \cdots a_{(r(n)})$ depends only on $\pi_{\mathbf{r}}$. (For instance, $\phi(a_2 a_3 a_7 a_3 a_2) = \phi(a_1 a_5 a_2 a_5 a_1)$.) Call this common value $f(\pi_{\mathbf{r}})$. Writing $\mathcal{P}_n$ for the set of partitions of $[n]$, and letting $C_\pi = |\{\mathbf{r} : \pi_{\mathbf{r}} = \pi\}|$, we see that

$$\phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n = \sum_{\pi \in \mathcal{P}_n} N^{-\frac{n}{2}} C_\pi f(\pi). \qquad (1)$$

If $\pi \in \mathcal{P}_n$ has $|\pi|$ classes, then

$$C_\pi = N(N-1)\cdots(N - |\pi| + 1)$$

for if $\pi_{\mathbf{r}} = \pi$, then r_i could be any one of N (resp., $N-1, \cdots N - |\pi| + 1$) numbers for i in the first (resp., second, $\cdots$, $|\pi|$-th) class of π.

The (stochastic or free[1]) independence and mean zero assumptions show that $f(\pi) = 0$ if π has a singleton class. So we may the sum on the RHS of equation (1) is only over π with no singleton classes; for such a π, clearly $|\pi| \leq n/2$, and as C_π is a product of $|\pi|$ terms, we see that

$$\frac{C_\pi}{N^{\frac{n}{2}}} = N^{|\pi| - \frac{n}{2}} \left(\prod_{k=1}^{|\pi|} (1 - \frac{k-1}{N}) \right) ,$$

and hence $\lim_{N \to \infty} \frac{C_\pi}{N^{\frac{n}{2}}} = 0$ unless $|\pi| = \frac{n}{2}$.

[1]The verification in this case follows from the fact - a good application of the proof of an earlier corollary -that if subalgebras A and B are free, then $\phi(a_1 b a_2) = \phi(b)\phi(a_1 a_2)$ for $a_i \in A, b \in B$.

The (stochastic or free[1]) independence and mean zero assumptions show that $f(\pi) = 0$ if π has a singleton class. So we may the sum on the RHS of equation (1) is only over π with no singleton classes; for such a π, clearly $|\pi| \leq n/2$, and as C_π is a product of $|\pi|$ terms, we see that

$$\frac{C_\pi}{N^{\frac{n}{2}}} = N^{|\pi| - \frac{n}{2}} \left(\prod_{k=1}^{|\pi|} (1 - \frac{k-1}{N}) \right) ,$$

and hence $\lim_{N\to\infty} \frac{C_\pi}{N^{\frac{n}{2}}} = 0$ unless $|\pi| = \frac{n}{2}$.

Conclude that

$$\lim_{N\to\infty} \phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}}\right)^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sum_{\mathcal{P}_{m(pair)}} f(\pi) & \text{if } n = 2m \end{cases} ,$$

where $\mathcal{P}_m(pair)$ denotes the set of 'pair-partitions' of $[2m]$ (i.e., partitions of $[2m]$ into m doublets).

[1] The verification in this case follows from the fact - a good application of the proof of an earlier corollary -that if subalgebras A and B are free, then $\phi(a_1 b a_2) = \phi(b)\phi(a_1 a_2)$ for $a_i \in A, b \in B$.

The (stochastic or free[1]) independence and mean zero assumptions show that $f(\pi) = 0$ if π has a singleton class. So we may the sum on the RHS of equation (1) is only over π with no singleton classes; for such a π, clearly $|\pi| \leq n/2$, and as C_π is a product of $|\pi|$ terms, we see that

$$\frac{C_\pi}{N^{\frac{n}{2}}} = N^{|\pi| - \frac{n}{2}} \left(\prod_{k=1}^{|\pi|} (1 - \frac{k-1}{N}) \right) ,$$

and hence $\lim_{N \to \infty} \frac{C_\pi}{N^{\frac{n}{2}}} = 0$ unless $|\pi| = \frac{n}{2}$.

Conclude that

$$\lim_{N \to \infty} \phi(\left(\frac{a_1 + \cdots + a_N}{\sqrt{N}} \right)^n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \sum_{\mathcal{P}_m(pair)} f(\pi) & \text{if } n = 2m \end{cases} ,$$

where $\mathcal{P}_m(pair)$ denotes the set of 'pair-partitions' of $[2m]$ (i.e., partitions of $[2m]$ into m doublets).

Up to now, the proof of the two cases is the same. Now we bifurcate.

[1] The verification in this case follows from the fact - a good application of the proof of an earlier corollary -that if subalgebras A and B are free, then $\phi(a_1 b a_2) = \phi(b)\phi(a_1 a_2)$ for $a_i \in A, b \in B$.

Classical case: Here, we find - under the mean 0 variance 1 and stochastic independence assumptions - that $f(\pi) = 1$ for every $\pi \in \mathcal{P}_{2m}(pair)$; further $|\mathcal{P}_{2m}(pair)| = (2m-1)(2m-3)\cdots 5.3.1$ and we find that the limit above is nothing but the n-th moment of the standard normal distribution. In view of our earlier comments about the standard normal distribution being dedtermined by its moment sequence, the proof is complete in this case.

Classical case: Here, we find - under the mean 0 variance 1 and stochastic independence assumptions - that $f(\pi) = 1$ for every $\pi \in \mathcal{P}_{2m}(pair)$; further $|\mathcal{P}_{2m}(pair)| = (2m-1)(2m-3)\cdots 5.3.1$ and we find that the limit above is nothing but the n-th moment of the standard normal distribution. In view of our earlier comments about the standard normal distribution being dedtermined by its moment sequence, the proof is complete in this case.

Free case: Suppose $\pi = \{\{k_t, l_t\} : 1 \leq t \leq m\} \in \mathcal{P}_{2m}(pair)$ and suppose $\pi_{\mathbf{r}} = \pi$. By the free indepedence and mean zero assumptions, notice that $f(\pi) = 0$ if $r_1 \neq r_2 \neq \cdots \neq r_m$. So we only need to consider the case when $r_s = r_{s+1}$ for some $s < m$, in which case, we see that

$$f(\pi) = \phi(a_{r_s}^2) f(\pi') \ ,$$

where $\pi' = \{\{k_t, l_t\} : 1 \leq t \leq m, t \neq s\} \in \mathcal{P}_{2m-2}(pair)$. Proceeding thus, we find that $f(\pi) = 0$ unless π is a *non-crossing pair partition* of $[2m]$. Since the number of such partitions is known to be the m-th Catalan number, we find that the limit we are looking at is nothing but the n-th moment of the standard semi-circular distribution, and the proof is complete.