Fusion rule algebras and walks on graphs

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Abstract

We begin by discussing the relation between fusion algebras and Cartan algebras in $M_n(\mathbb{R})$; as a result of our analysis, we arrive at a very simple prescription for starting from 'most' real symmetric matrices and arriving at a 'signed Hermitian fusion algebra', which turns out, in many cases to have non-negative structure constants and consequently defines a genuine fusion algebra.

We next show that the fusion rule algebras $su(2)_n$ introduced by Geppner and Witten have structure constants with a simple geometric meaning involving quadrilaterals in the plane. This is then reinterpreted into graph theoretic terms involving certain walks on trees (or more general graphs) called procrastinations and allows us to associate to any graph a (generally non associative) algebra $\mathcal{P}(X)$ which we call the procrastination algebra of X. Apart from associativity, this is seen to be a Hermitian fusion rule algebra which contains the adjacency matrix of the graph as the matrix of multiplication of a distinguished vertex. Remarkably, for many examples this procrastination algebra coincides with the fusion rule algebra of the graph $\mathcal{K}(X)$ defined earlier and thus provides an alternate explicit interpretation of the fusion rules.

1 Fusion algebras and Cartan algebras

We begin by getting various definitions and notational conventions out of the way.

DEFINITION 1 *A (finite-dimensional)* **fusion algebra** *is an associative unital *-algebra $\mathcal{A}$ with a distinguished basis $\{x_0, x_1, \cdots, x_n\}$ such that the 'structure constants' N_{ij}^k defined by the equation*

$$x_i x_j = \sum_{k=0}^{n} N_{ij}^k x_k$$

satisfy the following conditions:

(i) $N_{i0}^k = N_{0i}^k = \delta_i^k$; in other words, x_0 is the identity of the algebra $\mathcal{A}$;

(ii) $N_{ij}^k \geq 0$;

(iii) there exists an involution $i \mapsto i^*$ of the index set $\{0, 1, \cdots, n\}$ such that:

(a) $x_i^* = x_{i^*}$, and

(b)

$$N_{ij}^0 = \begin{cases} 1 & if \ i = j^* \\ 0 & if \ i \neq j^* \end{cases}$$

We shall call $\mathcal{A}$ a **signed fusion algebra** *if it satisfies conditions (i) and $(iii)(a), (b)$ above and the requirement that $N_{ij}^k \in \mathbb{R}$ (rather than (ii) above).*

Finally, we shall call a (signed) fusion algebra **Hermitian** *if it is the case that $x_i^* = x_i \ \forall i$.*

Before proceeding further, we pause to make a few remarks about this definition.

REMARK 2 *(1) Suppose we are given a *-algebra $\mathcal{A}$ with basis $\{x_i\}$ as above, and with associated structure constants N_{ij}^k. By a 're-normalisation', let us mean the choice of another basis $\{y_i\}$ for $\mathcal{A}$ of the form $y_i = a_i x_i$, for some constants $a_i > 0$ such that $a_0 = 1$ and $a_{i^*} = a_i$. It should be clear that the y_i's also yield a basis which contains the identity and is *-closed; further, the basis $\{y_i\}$ will satisfy the positivity condition (ii) of the above definition precisely when the basis $\{x_i\}$ does.*

*(2) Suppose that we have a *-algebra $\mathcal{A}$ with a basis $\{x_i\}$ satisfying all the conditions of the above definition, except that, instead of condition (iii)(b), we now require that $N_{ij}^0 \neq 0 \Leftrightarrow \ i = j^*$. (Thus, we have just dropped the normalisation condition $N_{ii^*}^0 = 1$.) It is then true that there exists a re-normalisation - in the sense of (1) above - such that the new basis satisfies the normalisation condition (iii)(b) of the Definition.*

It is also true that there exists yet another re-normalisation, such that the structure constants (call them n_{ij}^k) for the new basis (call it $\{c_0, c_1, \cdots, c_n\}$) satisfy the slightly different sort of normalisation condition given by $\sum_{k=0}^n n_{ij}^k = 1$. It is customary to call such a collection $\{c_i\}$ a hypergroup.

The proofs of the two assertions in this remark - concerning the existence of the two kinds of re-normalisations - may be found in [6], for instance, as also a proof of the fact that these two re-normalisations are uniquely determined by their 'normalising requirements'.

*(3) If x, y are self-adjoint elements of a *-algebra, the following conditions are clearly equivalent: (a) $xy = yx$; (b) xy is also self-adjoint. (Reason: If $x = x^*, y = y^*$, then $(xy)^* = yx$.) It follows that a Hermitian (signed) fusion algebra is necessarily commutative. (Note that a real linear combination of self-adjoint elements is self-adjoint.)*

We will be concerned primarily with Hermitian signed fusion algebras in this paper. We pause to record a simple fact which we shall have cause to use.

LEMMA 3 *Suppose $\mathcal{A}$ is a *-algebra with basis $\{x_0, x_1, \cdots, x_n\}$, such that the structure constants N_{ij}^k satisfy all the requirements of a signed fusion algebra, except that we replace the condition (iii)(b) of Definition 1 by the requirement that*

$$N_{ij}^0 \neq 0 \ \Leftrightarrow \ i = j^* \ .$$

Then the following conditions are equivalent:

(i) $\mathcal{A}$ is a Hermitian signed fusion algebra;

(ii) $\mathcal{A}$ is commutative, and $N_{ii}^0 = 1$;

(iii) N_{ij}^k is a symmetric function - call it N_{ijk} - of i,j,k.

Proof: The implication $(i) \Rightarrow (ii)$ follows from Remark 2(3).

$(ii) \Rightarrow (iii)$: Note that

$$\begin{aligned} x_i x_j x_k &= \sum_{l=0}^{n} N_{ij}^{l} x_l x_k \\ &= \sum_{l,m=0}^{n} N_{ij}^{l} N_{lk}^{m} x_m \ , \end{aligned}$$

whence the coefficient of x_0 in $(x_i x_j x_k)$ is seen to be equal to $\sum_{l=0}^{n} N_{ij}^{l} N_{lk}^{0} = N_{ij}^{k}$, since our hypothesis is that $N_{pq}^{0} = \delta_{pq}$. Since $\mathcal{A}$ is assumed to be commutative, we thus find that N_{ij}^{k} is indeed invariant under arbitrary permutation of the indices i, j, k.

$(iii) \Rightarrow (i)$: The assumed symmetry shows that

$$N_{ii}^{0} = N_{i0}^{i} = 1 \ ,$$

which, under the hypothesis of this lemma, shows that $i = i^*$, as desired. □

We now pause to describe one manner in which we will think of fusion algebras. Given $\mathcal{A}$ and x_i's as in the definition, we shall write L_i (or L_{x_i}, if it is necessary to avoid possible ambiguity) to denote the matrix, with respect to the ordered basis $\{x_j\}$, of left-multiplication by x_i; thus, L_i is the $(n+1) \times (n+1)$ matrix - with rows and columns indexed by the set $\{0, 1, \cdots, n\}$ - defined by $x_i x_j = \sum_k L_i(k, j) x_k$.

It is a fact - see [6] - that the mapping $x_i \rightarrow L_i$ extends to a *-algebra homomorphism from $\mathcal{A}$ into $M_{n+1}(\mathbb{C})$, and consequently $\mathcal{A}$ has the structure of a finite-dimensional C^*-algebra. Thus, we shall think of $\mathcal{A}$ as a finite-dimensional inner product space for which $\{x_i\}$ is an orthonormal basis. When we wish to distuingish the algebra structure and this inner-product structure, we shall denote the inner-product space as $L^2(\mathcal{A})$.

It is further true - see [4] or [3], for instance - that the equation

$$\tau(f) = \langle f, x_0 \rangle \tag{1.1}$$

defines a faithful tracial state on the C^*-algebra $\mathcal{A}$. In fact, we may identify $L^2(\mathcal{A})$ with the Hilbert space underlying the GNS representation of $\mathcal{A}$ associated with τ.

Conversely, if $\mathcal{A}$ is a finite-dimensional C^*-algebra, and if τ is a faithful tracial state on $\mathcal{A}$, it is possible to find a basis for $\mathcal{A}$ - call it $\{x_0, x_1, \cdots, x_n\}$

- which contains the identity, is closed under formation of adjoints, and is orthonormal (with respect to the inner product of $L^2(\mathcal{A})$). (*Reason:* If $\mathcal{A} = M_k(\mathbb{C})$, then τ is necessarily the normalised trace - $\tau(a) = \frac{1}{k}\sum_i a_{ii}$; then a candidate for the desired basis is given by $\{x_{ij} : 1 \leq i, j \leq k\}$, where

$$\begin{aligned} x_{11} &= 1 \;\; \text{(the identity matrix)} \\ x_{jj} &= \sqrt{\frac{k}{j(j-1)}}(\sum_{i<j} e_{ii} - (j-1)e_{jj}), \; 1 < i \leq k \\ x_{ij} &= \sqrt{k}\; e_{ij}, \; 1 \leq i \neq j \leq k \;, \end{aligned}$$

where $\{e_{ij} : 1 \leq i, j \leq k\}$ denotes the usual system of matrix units. A general finite-dimensional C^*-algebra is isomorphic to a finite direct sum of full matrix algebras, and the desired assertion is seen to follow.) Thus, we see that every finite-dimensional C^*-algebra has the structure of a 'complex' fusion algebra (which is defined just like a signed fusion algebra, except that the structure constants are permitted to be complex).

However, the positivity requirement in a fusion algebra is quite restrictive; for instance, it is true - see [6] - that any fusion algebra admits a unique algebra homomorphism into $\mathbb{C}$ which attains strictly positive values on the basis; in particular, such a C^*-algebra necessarily has an ideal of co-dimension 1. The following question is natural.

Question : Which finite-dimensional C^*-algebras admit bases which endow them with the structure of a fusion algebra?

Some of our investigations are motivated by an attempt to understand the reason for the validity of the following known result - see [5], for instance:

THEOREM 4 *Let $A = Adj(G)$ denote the (vertex-) adjacency matrix of the graph G, where G is one of the following Coxeter diagrams: A_n, D_{2n}, E_6, E_8. Then, there exists a unique fusion algebra such that $L_{x_1} = A$ (in the notation of the paragraph immediately after the proof of Lemma 3).*

Furthermore, we have the following:

(i) this fusion algebra is Hermitian in all the above examples except for the cases where $G = D_{4n}$; and

(ii) when $G \in \{D_{2n+1}, E_7\}$, there exists no fusion algebra such that $A = L_{x_1}$ (although there exists a signed fusion algebra with this property).

Note that if $A \in M_{n+1}(\mathbb{R})$, and if there exists a signed fusion algebra such that $A = L_{x_1}$, then A is necessarily a symmetric matrix of the following form:

$$A = \begin{bmatrix} 0 & * & * & \cdots & * \\ 1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix} \tag{1.2}$$

The next result is a sort of complement to the last theorem.

THEOREM 5 *Let $A \in M_{n+1}(\mathbb{R})$ be a real symmetric matrix which has the form displayed in equation 1.2. Let us write $\{v_0, v_1, \cdots, v_n\}$ for the standard basis for $\mathbb{R}^{n+1}$ (with the use of which we think of A as a linear transformation of $\mathbb{R}^{n+1}$).*

(a) Then the following conditions are equivalent:

(i) There exists a Hermitian signed fusion algebra $\mathcal{A}$ such that $A = L_{x_1}$;

(ii) v_0 is a cyclic vector for the commutant $\{A\}' = \{T \in M_{n+1}(\mathbb{R}) : AT = TA\}$ of A.

(b) The following conditions are equivalent:

(i) There exists a unique Hermitian signed fusion algebra $\mathcal{A}$ such that $A = L_{x_1}$;

(ii) v_0 is a cyclic vector for the commutant $\{A\}'$ of A, and further, the matrix A is 'regular', meaning that A has distinct eigenvalues.

We digress briefly before getting to the proof of the theorem. We need a definition. For this, note that the following conditions on a subalgebra $\mathcal{C}$ of $M_{n+1}(\mathbb{R})$ are equivalent:

(a) $\dim \mathcal{C} = n+1$, and $\mathcal{C}$ consists entirely of symmetric matrices;

(b) $\mathcal{C}$ is a maximal commutative subalgebra of $M_{n+1}(\mathbb{R})$ which consists entirely of symmetric matrices;

(c) there exists a regular symmetric matrix $A \in M_{n+1}(\mathbb{R})$ such that $\mathcal{C}$ is the subalgebra generated by A and the identity matrix.

Let us agree to use the phrase **symmetric Cartan algebra** to describe any subalgebra $\mathcal{C}$ of $M_{n+1}(I\!R)$, which satisfies the foregoing equivalent conditions (a) - (c). Furthermore, we shall say that $\mathcal{C}$ is a **cyclic** symmetric Cartan subalgebra if it is the case that v_0 is a cyclic vector for $\mathcal{C}$.

PROPOSITION 6 *There exists a 1-1 correspondence between the following collections:*

(a) the collection of all cyclic symmetric Cartan subalgebras of $M_{n+1}(I\!R)$;

(b) the collection of all Hermitian signed fusion algebras.

Proof: Given a cyclic Cartan subalgebra $\mathcal{C}$, the map $\mathcal{C} \ni L \mapsto Lv_0$ is a linear map of $\mathcal{C}$ onto $I\!R^{n+1}$, and consequently an isomorphism of vector spaces (since these two spaces have the same dimension). Hence, we can find unique matrices $L_i \in \mathcal{C}$ such that $L_i v_0 = v_i, 0 \leq i \leq n$. It follows that $\{L_0, L_1, \cdots, L_n\}$ is a basis for $\mathcal{C}$ and that L_0 is the identity matrix. Note now that, for fixed $0 \leq i, j \leq n$, we have

$$L_i L_j \ = \ \sum_{k=0}^{n} L_i(k,j) L_k \ . \tag{1.3}$$

(This is because

$$\begin{aligned} L_i L_j v_0 &= L_i v_j \\ &= \sum_{k=0}^{n} L_i(k,j) v_k \\ &= (\sum_{k=0}^{n} L_i(k,j) L_k) v_0 \ . \ ) \end{aligned}$$

On the other hand, the requirement $L_i v_0 = v_i$ translates into $L_i(j,0) = \delta_{ij}$; since L_i is symmetric, we deduce that $L_i(0,j) = \delta_{ij}$. In other words, the L_i's form a basis of the algebra $\mathcal{C}$, so that the structure constants are given by $N_{ij}^k \ = \ L_i(k,j)$ (by equation 1.3); and the first sentence of this paragraph says that $N_{ij}^0 \ = \ \delta_{ij}$. It follows at once - see Lemma 3 - that the L_i's equip $\mathcal{C}$ with the structure of a Hermitian fusion algebra.

Conversely, if $\mathcal{A}$ is a Hermitian signed fusion algebra with distinguished basis $\{v_i : 0 \leq i \leq n\}$, and if L_i denotes the matrix of left-multiplication by

x_i (with respect to the basis $\{x_j\}$), then the L_i's clearly span a symmetric Cartan algebra with v_0 as a cyclic vector. Finally, it should be clear that the two associations we defined (from cyclic symmetric Cartan algebras to Hermitian signed fusion algebras, and vice versa) are inverse to one another. $\square$

Proof of Theorem 5:

(a) It is seen from the proof of Proposition 6 that condition (i) is equivalent to the following condition: $(i)'$ there exists a cyclic symmetric Cartan subalgebra $\mathcal{C} \subset M_{n+1}(\mathbb{R})$ such that $A \in \mathcal{C}$.

If A satisfies $(i)'$, the commutativity of $\mathcal{C}$ implies that $\mathcal{C} \subset \{A\}'$, and consequently v_0 is necessarily a cyclic vector for $\{A\}'$; thus $(i)' \Rightarrow (ii)$.

Conversely, suppose (ii) is satisfied. Suppose $\{\alpha_1, \cdots, \alpha_m\}$ is an enumeration of the distinct eigenvalules of A and $V_i \ = \ ker\ (A - \alpha_i)$. The condition (ii) is seen to be equivalent to the requirement that $v_0 \notin V_i^\perp\ \forall i$. We may consequently find, for each $i = 1, \cdots, m$, an orthonormal basis $\{e_p^{(i)} : 1 \leq p \leq \ \dim\ V_i\}$ for V_i such that $\langle v_0, e_p^{(i)} \rangle \neq 0\ \forall p, i$. Let $\{e_k : 0 \leq k \leq n\}$ be the orthonormal basis for $\mathbb{R}^{(n+1)}$ obtained by putting together all these bases for the V_i's. Thus, $\langle v_0, e_i \rangle \neq 0\ \forall i$. It follows that if P_i is defined to be the projection onto $\mathbb{R}e_i$, then the linear span of $\{P_0, \cdots P_n\}$ is a symmetric Cartan algebra $\mathcal{C}$ such that v_0 is a cyclic vector for $\mathcal{C}$, and such that $A \in \mathcal{C}$; thus A satisfies condition $(i)'$; and the proof of (a) is complete.

In view of Lemma 6 and the already proved (a) of this theorem, it is easy to see that in order to prove (b), it suffices to show that the regularity of A is equivalent to the existence of a unique symmetric Cartan subalgebra containing A; this latter statement clearly follows from the definitions, and the proof is complete. $\square$

We now wish to reap some consequences of Theorem 5(b). So, assume $A \in M_{n+1}(\mathbb{R})$ is a symmetric matrix with the form displayed in equation 1.2. A moment's thought shows that A will satisfy the condition (ii) of Theorem 5(b) if and only if $\{v_0, Av_0, A^2v_0, \cdots, A^nv_0\}$ is a spanning set of vectors for $\mathbb{R}^{n+1}$; since this is a set of $n+1$ vectors in an $(n+1)$-dimensional space, the spanning condition is equivalent to the requirement that these vectors are linearly independent.

Hence, we form an auxiliary matrix B whose j-th column is the vector $A^j v_0$, for $0 \leq j \leq n$; thus, we define

$$b_{ij} \ = \ \langle A^j v_0, v_i \rangle \ ; \tag{1.4}$$

and the conclusion of the preceding paragraph is that A will satisfy the equivalent conditions of Theorem 5(b) if and only if the matrix B is invertible; thus we just have to verify the non-vanishing of the determinant of B in order to conclude that there exists a unique signed fusion algebra 'containing' A (in the sense of Theorem 5(b).

More information about the fusion algebra can be milked from the matrix B; indeed, suppose B is indeed invertible; let $C = B^{-1}$. Then, notice that, for $0 \leq i, k \leq n$, we have:

$$\begin{aligned} \langle v_k, v_i \rangle &= \delta_{ki} \\ &= (BC)_{ik} \\ &= \sum_{j=0}^{k} b_{ij} c_{jk} \\ &= \sum_{j=0}^{k} c_{jk} \ \langle A^j v_0, v_i \rangle \\ &= \langle (\sum_{j=0}^{k} c_{jk} A^j) \ v_0, v_i \rangle \ ; \end{aligned}$$

we conclude that the matrices L_i of the associated unique signed fusion algebra are given by the simple formula:

$$L_k \ = \ \sum_{j=0}^{n} c_{jk} A^j \ . \tag{1.5}$$

We summarise the preceding discussion thus:

PROPOSITION 7 *Let $A \in M_{n+1}(\mathbb{R})$ be a real symmetric matrix, with the form displayed in equation 1.2. Define the matrix $B \in M_{n+1}(\mathbb{R})$ by equation 1.4. Suppose B is invertible. Let $C = B^{-1}$. For $0 \leq k \leq n$, define $L_k \in M_{n+1}(\mathbb{R})$ by equation 1.5. Then, there exists a unique signed fusion algebra such that $A = L_{x_1}$; the structure constants for this fusion algebra are given by*

$$n_{ij}^k \ = \ L_i(k, j) \ .$$

We say nothing more about the proof. Although this proposition is an immediate consequence of Theorem 5(b), we have chosen to single it out as a separate proposition, because of its usefulness as a device to construct numerous (signed) fusion algebras. In fact, it is easy to write a simple computer programme, which, upon being fed the data of an arbitrary A as in the proposition, will proceed as follows: (i) construct the matrix B; (ii) check if B is non-singular; (iii) if B is non-singular, compute the inverse matrix, call it C; (iv) compute the matrices L_k as defined by 1.5.

It is interesting to see what some of these matrices mean when A is the adjacency matrix of a graph G. Let us denote by * (or $*_G$, if it is necessary to emphasise the role of G) the vertex which corresponds to the 0-th column of A. Then the matrix B has the following combinatorial interpretation: b_{ij} is the number of paths of length j from * to the vertex corresponding to the i-th row of A.

The case of A_{n+1} : In case $G = A_{n+1}$, then the matrix C is also a familiar object; indeed, the entries of the j-th column of C are the coefficients of the j-th Chebyshev polynomial of the second kind (appropriately normalised); more explicitly, if we define

$$P_j(t) \;=\; \sum_{i=0}^{n} c_{ij}\; t^i \;=\; \sum_{i=0}^{j} c_{ij}\; t^i \;,$$

then $P_0(t) = 1,\; P_1(t) = t$ and $P_{j+1}(t) = tP_j(t) - P_{j-1}(t)$ for $0 \leq j < n$. Thus, in this case, it follows from the invertibility of B that there is a unique Hermitian signed fusion algebra with $A = L_{x_1}$. In fact, it turns out that the matrices L_i are all (entry-wise) non-negative and consequently, we have a *bona fide* (meaning positive, and not just signed) fusion algebra. In general, establishing that a given signed fusion algebra (constructed as in the above proposition) does exhibit such positivity, is a difficult and combinatorial problem. Later in this paper, we shall give one such combinatorial proof of positivity in the case of the A_{n+1} signed fusion algebra.

Notice that in the above analysis, we took the vertex * to be an end-vertex of the graph. For a general connected graph, it is a consequence of the Perron-Frobenius theorem that if the associated matrix B is invertible, then a necessary condition for the resulting signed fusion algebra to be a (positive) fusion algebra is that the vertex * should satisfy the following requirement: the Perror-Frobenius eigenvector must assume its minimum value at *.

The case of E_n, $n \geq 6$: By E_n, of course, we mean the grpah with n vertices obtained by lengthening the long arm of E_6. Since we hope to get genuine fusion algebras, we place the vertex * at the end of the long arm. (This is because of the last comment in the previous paragraph). It turns out that the associated matrix B is invertible and so Proposition 7 applies; and we find that we do have positivity of the resulting fusion algebra *except* for the cases $n = 7, 10$. (A proof of this positivity assertion may be found in [8], for instance.)

The case of D_n : If we choose the vertex * to be at the end of the long arm (as the Perron-Frobenius eigenvector requirement would demand), we find that the corresponding matrix B is singular, so the above Proposition would not seem to apply. However, in case n is an *odd* integer, if we choose * to be one of the end-vertices adjacent to the triple point, we find that the corresponding matrix B is non-singular; the consequence is: there exists a unique signed fusion algebra 'containing the adjacency matrix' of D_n; this signed fusion algebra necessarily contains negative structure constants because the Perron-Frobenius requirement was violated. Thus, we conclude that there is no fusion algebra containing the adjacency matrix of D_{2n+1}.

We raise the natural next question:

Question : Suppose A is the adjacency matrix of graph and B is defined as above. What is a combinatorial interpretation of:
(a) the requirement that B is invertible? and
(b) the matrix C (in case B happens to be invertible)?

Some new examples of fusion algebras: Let us use the notation $T(p, q, r)$ to denote the '3-star' in Figure 1.

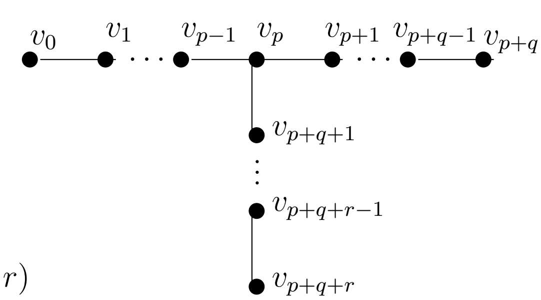


Figure 1 : $T(p, q, r)$

We have the following proposition:

PROPOSITION 8 *Assume (without loss of generality) that $p \geq q \geq r$, and let A denote the adjacency matrix of $T(p,q,r)$, with vertices numbered as indicated in the above diagram. Then A has v_0 as cyclic vector if and only if:*

(i) r is even and $(q+1)$ is not divisible by $(r+1)$; or

(ii) r is odd, and q is even.

Proof: Let $W = \{p(A)v_0 : p \text{ a polynomial}\}$.

Then, it should be clear that $v_i \in W \ \forall\ 0 \leq i \leq p$ (since $v_0 \in W$, $v_1 = Av_0 \in W$, and for $1 < i \leq p$, we have $v_i = Av_{i-1} - v_{i-2}$); similarly, we also see that $v_{p+i} + v_{p+q+i} \in W$ for $1 \leq i \leq r$ (since $v_{p+1} + v_{p+q+1} = Av_p - v_{p-1} \in W$, $v_{p+2} + v_{p+q+2} = A(v_{p+1} + v_{p+q+1}) - v_p \in W$, etc.).

Rather than give a detailed proof of the general assertion, we shall content ourselves with an outline of the argument for some specific values of the parameters.

(ia) Suppose $p = 3, q = r = 2$. The above analysis shows that $\{v_0, v_1, v_2, v_3, (v_4 + v_7), (v_5 + v_8)\} \subset W$. Notice now that $A(v_5 + v_8) = v_4 + v_7 + v_6$, and conclude that $v_6 \in W$; deduce from $Av_6 = v_5$ that $v_5 \in W$ and hence also $v_8 = (v_5 + v_8) - v_5 \in W$; similar reasoning shows that also $v_4, v_7 \in W$; thus $W = I\!R^9$ and v_0 is cyclic for A, as desired.

(ib) Suppose $p = q = 3$ and $r = 3$. We see as above that $S = \{v_0, v_1, v_2, v_3, (v_4 + v_7), (v_5 + v_8), (v_6 + v_9)\} \subset W$. In fact, since $A(v_6 + v_9) = (v_5 + v_8)$, we find that the linear span of the above set S is a proper subspace of $I\!R^9$ which is invariant under A. Since $v_0 \in S$, we conclude that v_0 is not cyclic for A.

The proof of (ii) is similar, and we omit it. □

In particular, there is hope that the the procedure outlined by Proposition 7 is applicable to the matrix A of Proposition 8. Our interest is in situations where the signed fusion algebra resulting from such an A by this method is actually a positive fusion algebra. As a result of using the simple computer programe discussed earlier, we are able to observe the following facts.

The above procedure yields a *bona fide* fusion algebra, if we apply the above procedure to $T(p,q,r)$, in the following cases:

(i) $q = 2, r = 1,\ p \in \{2, 4, 5, 7, 8, 9, \cdots\}$;

(ii) $q = 4, r = 1,\ p \in \{4, 6, 7, 10, 12, 14, 15, 16, 17, 18, 19, 20\}$;

(ii) $q = 4, r = 3,\ p \in \{4, 8, 9, 13, 14, 15, 16, 17\}$.

Further, it is known - see [8] - that the graph $T(p, 2, 1),\ p \geq 2$, which might be called E_{p+4}, yields a *bona fide* fusion algebra if and only if $p \neq 3, 6$.

On the basis of the above empirical evidence, albeit meager, we make the following slightly vague conjecture:

Conjecture: Let T be any tree. Then, if G is the tree obtained by 'attaching an A_n to T, and if n is large enough, then there exists a *bona fide* fusion algebra containing the adjacency matrix of the graph G.

We now turn our attention to more general (not necessarily symmetric) Cartan subalgebras of $M_{n+1}(I\!R)$ - i.e., (necessarily maximal) commutative $n+1$-dimensional *-subalgebras of $M_{n+1}(I\!R)$.

Given a general fusion algebra $\mathcal{A}$ with basis $\{x_0, x_1, \cdots, x_n\}$, we write $\mathcal{C} = \{L_x : x \in \mathcal{A}\}$, where, of course, $L_x \in M_{n+1}(I\!R)$ is defined by $L_x(i, j) =$ the coefficient of x_j in xx_i.

In the remainder of this section, we assume that $\mathcal{A}$ is a commutative fusion algebra. Then $\mathcal{C}$ is a *Cartan subalgebra* of $M_{n+1}(I\!R)$.

We list some facts concerning Cartan subalgebras as a lemma, primarily to fix notation for subsequent use. (We omit the simple linear algebraic proof.)

LEMMA 9 *Let $\mathcal{C}$ be a Cartan subalgebra of $M_{n+1}(I\!R)$, associated with a fusion algebra $\mathcal{A}$, as above. Then there exist (i) a uniquely determined pair of non-negative integers p, q, (ii) projections $P_0, P_1, \cdots, P_p$ of rank 1 in $\mathcal{C}$, and (iii) projections $Q_0, Q_1, \cdots, Q_q$ of rank 2 in $\mathcal{C}$, such that $\{P_0, \cdots, P_p, Q_1, \cdots, Q_q\}$ is precisely the set of minimal projections of $\mathcal{C}$.*

In particular, $\sum_{i=0}^{p} P_p + \sum_{j=1}^{q} Q_q = I$ (= the identity matrix of size $n+1$), and $p + 2q = 1$.

REMARK 10 *(1) Since every fusion algebra admits a unique 'dimension function', it is seen that $\mathcal{C}$ always contains a rank 1 projection; this is why our numbering is such that there are $p+1$ projections of rank 1 and q projections of rank 2. It should be noted that the above lemma is valid even if the Cartan subalgebra does not arise from a fusion algebra, provided that we index the rank one projections as $P_1, \cdots, P_p$ (so that the possibility of there being no rank one projections in $\mathcal{C}$ is also covered.)*

(2) Any pair of non-negative integers p, q for which $p + 2q = 1$ will so arise for some Cartan subalgebra; viz., simply take

$$\mathcal{C} = \left\{ \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_1 R_{\theta_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_2 R_{\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_q R_{\theta_q} \end{bmatrix} : \lambda_i, a_i \in I\!R, \theta_i \in [0, 2\pi] \right\} ,$$

where we write

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} .$$

We now wish to determine precisely which Cartan subalgebras of $M_{n+1}(I\!R)$ arise in this fashion from (signed) fusion algebras. Suppose we have a Cartan algebra $\mathcal{C}$, with associated P_i, Q_j as in Lemma 9. Then it is not hard to see that there exists an orthogonal matrix $U \in O(n+1, I\!R)$ such that $U\mathcal{C}U^{-1}$ is the algebra described in Remark 10(2); hence, it follows that q is precisely the dimension of the skew-symmetric matrices in $\mathcal{C}$. (Note that R_θ is skew-symmetric precisely when $\theta = \pm\frac{\pi}{2}$.) It also follows that: (a) for each $1 \leq j \leq q$, there exists a skew-symmetric matrix $S_j \in \mathcal{C}$ of (operator) norm 1 such that $S_j = Q_j S_j Q_j$; (b) if T_j is any skew-symmetric matrix in $\mathcal{C}$ with operator norm 1, such that $T_j = Q_j T_j Q_j$, then $T_j = \pm S_j$; and (c) $\{S_j : 1 \leq j \leq q\}$ is a basis for the real vector space of skew-symmetric matrices in $\mathcal{C}$.

On the other hand, if $\mathcal{A}$ is a fusion algebra with basis $\{x_i : 0 \leq i \leq n\}$, define $I = \{i : i = i^*\}$; then there exists a subset $J \subset \{1, \cdots, n\}$ such that $|I| + 2|J| = n + 1$, and $j \in J \Rightarrow j^* \notin J$; it follows that $\{L_{(x_j - x_{j^*})} : j \in J\}$ is a linearly independent set of $|J|$ skew-symmetric elements in the associated Cartan algebra $\mathcal{C}$, while $\{L_{x_i} : i \in I\} \cup \{L_{(x_j + x_{j^*})} : j \in J\}$ is a linearly independent set of $|I| + |J|$ symmetric elements of $\mathcal{C}$.

In other words, if $\mathcal{A}$ is an $(n+1)$-dimensional fusion algebra whose basis contains precisely p self-adjoint elements and $2q$ non-self-adjoint elements, then it is this pair p, q which corresponds to $\mathcal{C}$ as in Lemma 9.

In the sequel, we write $v_0, \cdots, v_n$ for the standard basis of $\mathbb{R}^{n+1}$. (Thus, v_i has the $(i+1)$-th co-ordinate equal to 1 and all other co-ordinates equal to 0.)

LEMMA 11 *Suppose $\mathcal{C}$ is a Cartan subalgebra of $M_{n+1}(\mathbb{R})$ associated with a commutative fusion algebra $\mathcal{A}$ as above. Let $\mathcal{C}_{sym}$ (resp., $\mathcal{C}_{ss}$) denote the set of symmetric (resp., skew-symmetric) elements of $\mathcal{C}$. Then,*

(i) v_0 is a cyclic vector for $\mathcal{C}$, so that the equation $\phi(T) = Tv_0$ defines a linear isomorphism of $\mathcal{C}$ onto $\mathbb{R}^{n+1}$;

*(ii) if we use the isomorphism ϕ to 'transport structures' from its domain to range and conversely, we may (and do) regard both $\mathcal{C}$ and $\mathbb{R}^{n+1}$ as commutative *-algebras with an inner-product (with respect to which $\{\phi^{-1}(v_i)\}$ (resp., $\{v_i\}$) is an orthonormal basis;*

(iii) with respect to the inner-product defined in (ii), $\mathcal{C}_{sym}$ is precisely the orthogonal complement of $\mathcal{C}_{ss}$.

Proof: (i) is an immediate consequence of the fact that x_0 is an identity for $\mathcal{A}$, while there is nothing to prove in (ii).

As for (iii), suppose $S \in \mathcal{C}_{sym}$ and $A \in \mathcal{C}_{ss}$; then, since $\mathcal{C}$ is commutative and since the inner product on $\mathbb{R}^{n+1}$ is a symmetric bilinear form, we see that

$$\begin{aligned} \langle SAv_0, v_0 \rangle &= \langle Av_0, Sv_0 \rangle \\ &= \langle v_0, -ASv_0 \rangle \\ &= \langle -ASv_0, v_0 \rangle \\ &= -\langle SAv_0, v_0 \rangle , \end{aligned}$$

and hence, indeed $\langle A, S \rangle = 0$. □

PROPOSITION 12 *(a) Suppose $\mathcal{C}$ is a Cartan subalgebra of $M_{n+1}(\mathbb{R})$ associated with a commutative fusion algebra $\mathcal{A}$ as above; let the symbols S_j, p, q, I, J be as in the discussion preceding Lemma 11, and let $P_0, \cdots P_p, Q_1, \cdots, Q_q$ be the minimal projections in $\mathcal{C}$, as in Lemma 9. Then,*

(i) v_0 is a cyclic vector for $\mathcal{C}$; and

(ii) $\{S_j v_0 : 1 \leq j \leq q\}$ and $\{v_j - v_{j^*} : j \in J\}$ are both orthogonal bases of the same subspace of $\mathbb{R}^{n+1}$.

(b) Conversely, suppose $\mathcal{C}$ is a Cartan subalgebra of $M_{n+1}(I\!R)$, with minimal projections $P_0, \cdots P_p, Q_1, \cdots, Q_q$ as in Lemma 9. Suppose there exists a $2q$ element set $J \cup \{j^* : j \in J\} \subset \{1, 2, \cdots, n\}$ such that conditions (i) and (ii) of (a) are satisfied. Then there exists a signed fusion algebra $\mathcal{A}$ with precisely p self-adjoint basis elements such that $\mathcal{C}$ is the Cartan subalgebra associated with $\mathcal{A}$.

(c) Suppose $\mathcal{C}$ is as in (b) above; let $\{A_j : 0 \leq j \leq n\}$ be any basis for $\mathcal{C}$. (For instance, we may take the basis as $P_0, P_1, \cdots, P_p, Q_1, S_1, Q_2, S_2, \cdots, Q_q, S_q$.) Define the matrix $B \in M_{n+1}(I\!R)$ by $b_{ij} = \langle A_j v_0, v_i \rangle$. (Thus, the j-th column of the matrix B is just $A_j v_0$.) Then B is a non-singular matrix. Let $C = B^{-1}$. Define

$$L_k = \sum_{j=0}^{n} c_{jk} A_k \ .$$

Then these are preisely the matrices L_{v_k} obtained from the fusion algebra corresponding to $\mathcal{C}$ as in (b) above.

Proof: We continue to use the notations and conventions of Lemma 11.

(a) The validity of assertion (i) has already been noted in Lemma 11; as for (ii), observe, to start with, that $\{S_j v_0 : 1 \leq j \leq q\}$ is a basis for the real subspace $\phi(\mathcal{C}_{ss})$ of $I\!R^{n+1}$, since $\{S_j : 1 \leq j \leq q\}$ is an orthogonal basis for $\mathcal{C}_{ss}$ (as has already been noticed before). On the other hand, $\{\frac{1}{\sqrt{2}}(v_j - v_{j^*}) : j \in J\}$ is clearly an orthonormal set of q skew-adjoint elements of $I\!R^{n+1}$, and the desired assertion follows.

(b) Suppose $\mathcal{C}, P_i, Q_j, J$ are as in (b) of the proposition. Since v_0 is assumed to be a cyclic vector for $\mathcal{C}$, we use the linear isomorphism $\phi : \mathcal{C} \to I\!R^{n+1}$ (defined by $\phi(T) = T v_0$) to equip $I\!R^{n+1}$ with the structure of a commutative *-algebra; further, it is seen, as in the proof of Lemma 11 that $\phi(\mathbb{C}_{sym}) = \phi(\mathcal{C}_{ss})^{\perp}$.

By hypothesis, we see that $\{\frac{1}{\sqrt{2}}(v_j - v_{j^*}) : j \in J\}$ is an orthonormal basis for $\phi(\mathcal{C}_{ss})$. It follows that $\{v_i : i \notin J \cup \{j^* : j \in J\}\} \cup \{\frac{1}{\sqrt{2}}(v_j + v_{j^*}) : j \in J\}$ is an orthonormal basis for $\phi(\mathcal{C}_{ss})^{\perp} = \phi(\mathcal{C}_{sym})$. In particular, if we write $I = \{0, 1, \cdots, n\} \setminus (J \cup \{j^* : j \in J\})$, we see that $v_i = v_i^*$ and that for each $j \in J$,

$$v_j^* = \frac{1}{2}[(v_j + v_{j^*}) + (v_j - v_{j^*})]^*$$

$$
\begin{aligned}
&= \frac{1}{2}[(v_j + v_{j^*}) - (v_j - v_{j^*})] \\
&= v_{j^*} \ .
\end{aligned}
$$

In other words, if we define

$$
k^* \ = \ \begin{cases} i & if \ k = i \in I \\ j^* & if \ k = j \in J \\ j & if \ k = j^*, \ j \in J \end{cases} \qquad ,
$$

then we find that $k \mapsto k^*$ is an involution of $\{0, 1, \cdots, n\}$ such that $v_k^* = v_{k^*} \ \forall k$.

Further, we find that, for arbitrary $0 \leq i, j \leq n$,

$$
\begin{aligned}
\langle v_i v_j^*, v_0 \rangle &= \langle \phi(\phi^{-1}(v_i)\phi^{-1}(v_j^*)), v_0 \rangle \\
&= \langle (\phi^{-1}(v_i)\phi^{-1}(v_j^*))v_0, v_0 \rangle \\
&= \langle (\phi^{-1}(v_j^*)\phi^{-1}(v_i))v_0, v_0 \rangle \\
&= \langle (\phi^{-1}(v_i))v_0, (\phi^{-1}(v_j))v_0 \\
&= \langle v_i, v_j \rangle \\
&= \delta_{ij} \ .
\end{aligned}
$$

Finally, note that $Iv_0 = v_0 \ \Rightarrow \ \phi^{-1}(v_0) = I$, and consequently v_0 is the identity element of $I\!R^{n+1}$. It follows that $I\!R^{n+1}$ is a signed fusion algebra. It is trivially verified that the Cartan algebra associated with this fusion algebra is precisely $\mathcal{C}$, and the proof of the proposition is complete.

(c) Since v_0 is a cyclic vector for $\mathcal{C}$ and $\{A_0, \cdots, A_n\}$ is a basis for $\mathcal{C}$, it follows that the matrix B is indeed non-singular. Computing exactly as in the verification of Proposition 7, we find that for $0 \leq i, k \leq n$,

$$
\begin{aligned}
\langle v_k, v_i \rangle &= \delta_{ki} \\
&= (BC)_{ik} \\
&= \sum_{j=0}^{k} b_{ij} c_{jk} \\
&= \sum_{j=0}^{k} c_{jk} \ \langle A_j v_0, v_i \rangle \\
&= \langle (\sum_{j=0}^{k} c_{jk} A_j) \ v_0, v_i \rangle \\
&= \langle L_k v_0, v_i \rangle \ ,
\end{aligned}
$$

and the desired conclusion is seen to follow at once. □

2 The quadrilateral lemma and $su(2)_n$

The simplest and most intensively studied class of fusion rule algebras which arise in conformal field theory are the $su(2)_n$ algebras associated to the WZW minimal model for $su(2)$ at level n. These have the form $\mathcal{K} = \{C_0, \ldots, C_n\}$ with structure equations

$$C_i C_j = \sum_{k=0}^{n} N_{ij}^k C_k$$

where

$$N_{ij}^k = \begin{cases} 1 & \text{if } |i-j| \leq k \leq \min(i+j, 2n-(i+j)) \\ & \text{and } k \equiv i+j \ (mod\ 2) \\ 0 & \text{otherwise.} \end{cases}$$

If we denote by L_i the matrix of multiplication of C_i with respect to the basis $\{C_0, \ldots, C_n\}$ then L_0 is the identity and L_1 is the adjacency matrix of the Dynkin diagram A_{n+1}.

It is not clear that the multiplication in $\mathcal{K}$ is associative. We now give an alternate interpretation of the structure constants which will explain associativity directly and will generalise to other Dynkin diagrams, and in fact more general trees and graphs.

A set $\{i, j, k\}$ of non-negative real numbers is called a *triangle set* if there exists a triangle ABC in the plane with these as side lengths. Define $s = (i+j+k)/2$, the half perimeter. We will say that the triangle set $\{i, j, k\}$ is *integral* if each of i, j, k and s is an integer.

LEMMA 13 *Let $\{i, j, k\}$ be a set of non-negative real numbers. Then the following are equivalent. (a) $\{i, j, k\}$ is a triangle set (b) $\{i, j, k\}$ satisfies the inequalities $i+j \geq k$, $j+k \geq i$, $k+i \geq j$ (c) $s \geq max(i, j, k)$.*

□

A set $\{i, j, k, l\}$ of non-negative real numbers is called a *quadrilateral set* if there exists a quadrilateral $ABCD$ in the plane, not necessarily convex, with these as side lengths. We say that the quadrilateral set $\{i, j, k, l\}$ is *integral* if each of i, j, k, l and s is an integer.

Lemma 14 *Let $\{i, j, k, l\}$ be a set of non-negative real numbers. Then the following are equivalent. (a) $\{i, j, k, l\}$ is a quadrilateral set (b) $\{i, j, k, l\}$ satisfies the inequalities $i+j+k \geq l$, $j+k+l \geq i$, $k+l+i \geq j$, $l+i+j \geq k$ (c) $s \geq max(i, j, k, l)$.*

□

Note that if $\{i, j, k\}$ and $\{k, l, m\}$ are triangle sets, then $\{i, j, l, m\}$ is a quadrilateral set (just glue the triangles at their common edge). Conversely if $\{i, j, l, m\}$ is a quadrilateral set, then there exists $k \geq 0$ such that $\{i, j, k\}$ and $\{k, l, m\}$ are triangle sets – just consider an actual quadrilateral with sides i, j, l, m in that order and consider the appropriate diagonal. The order i, j, l, m can be obtained since we may always switch adjacent sides. In fact since quadrilaterals are not rigid, there exist a set of such possible k.

Lemma 15 *(The Quadrilateral Lemma) Let $S = \{i, j, l, m\}$ be a set of non-negative real numbers and choose some ordering $\{i_1, i_2, i_3, i_4\}$ of S. Consider all quadrilaterals $ABCD$ in the plane, not necessarily convex, whose sides AB,BC,CD,DA have lengths i_1, i_2, i_3, i_4 respectively. Let I denote the set of all possible lengths of the diagonals AC (as we vary over all such quadrilaterals). Then $|I|$ depends only on the set S and not on the ordering.*

Proof: Let $s = (i+j+l+m)/2$, $M_1 = \min(i, j, l, m)$ and $M_2 = \max(i, j, l, m)$. We will show that for any choice of ordering,

$$|I| \ = \ 2 \min(M_1, s - M_2).$$

Suppose without loss of generality that the ordering is i, j, l, m. The triangle ABC has fixed side lengths i and j so that its third side length AC may lie in $[\ |i-j|, i+j]$. Similarly triangle CDA has fixed side lengths l and m so that its third side length lies in $[\ |l-m|, l+m]$. The allowed range of values of AC is thus the intersection I of these two intervals, whose length is

$$\begin{aligned} & \min(i+j-|i-j|, l+m-|i-j|, i+j-|l-m|, l+m-|l-m|) \\ = \ & \min(2 \min(i, j), 2s - \max(i, j), 2s - \max(l, m), 2 \min(l, m)) \\ = \ & 2 \min(M_1, s - M_2) \end{aligned}$$

where we have adopted the convention that a negative length denotes an empty interval and used the fact that for any real numbers a, b

$$2 \min(a,b) = a+b-|a-b|$$

and

$$2 \max(a,b) = a+b+|a-b|.$$

Corollary 16 *Suppose that $\{i,j,l,m\}$ is an integral quadrilateral set. Let $\{i_1,i_2,i_3,i_4\}$ be an ordering of S and consider the set of all possible quadrilaterals $ABCD$ in the plane with side lengths $\{i_1,i_2,i_3,i_4\}$ in that order. Then the number of possible values of AC such that both ABC and CDA are integral triangles is $min(M_1, s-M_2)+1$ and so is independent of the given ordering of S.*

Proof: We know that the range of values of AC is an interval with integer endpoints with length $2d = 2\min(M_1, s-M_2)$ This interval thus contains $2d+1$ integer points of which $d+1$ result in integral triangle sets for the lengths of ABC and CDA. □

Now for $i,j,k \in \mathbb{N}$, define

$$N_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is an integral triangle set} \\ 0 & \text{otherwise} \end{cases}$$

and introduce a multiplication on $\mathcal{K} = \{C_0, C_1, \cdots\}$ by

$$C_i C_j = \sum_k N_{ij}^k C_k$$

for all $i,j \in \mathbb{N}$.

Proposition 17 *The above multiplication makes $\mathcal{K}$ into an (associative) Hermitian positive generalised hypergroup.*

Proof: It is clear that C_0 acts as an identity and that the structure is commutative. Associativity amounts to showing that

$$\sum_k N_{ijk} N_{klm}$$

is unchanged under permutations of i, j, l, m. But this is just the content of the above Corollary. That we get a Hermitian positive generalised hypergroup is clear. $\square$

It is worth pointing out that the fusion rule algebra $\mathcal{K}$ above is nothing but the dual of the group $SU(2)$ and that the N_{ijk} are just the usual Clebsch - Gordon coefficients in this situation.

We will also need a 'level n' version of the above.

LEMMA 18 *(The Quadrilateral Lemma – level n) Let $S = \{i, j, l, m\}$ be a set of non-negative real numbers and choose some ordering $\{i_1, i_2, i_3, i_4\}$ of S. Consider all quadrilaterals $ABCD$ in the plane, not necessarily convex, such that the sides AB, BC, CD, DA have lengths i_1, i_2, i_3, i_4 respectively and such that the triangles ABC and CDA both have perimeter less than n. Let I denote the set of all possible lengths of the diagonal AC (as we vary over all such quadrilaterals). Then $|I|$ depends only on the set S and not on the ordering.*

Proof: We will show that, with the notation of the previous proof, $|I| = 2\min(M_1, n - s + M_1, s - M_2, n - M_2)$ which depends only on the set S.

The triangle ABC has fixed side lengths i and j and its perimeter is constrained to be less than or equal to n. Thus the third side AC has length in the interval $[\,|i - j|, \min(i + j, 2n - (i + j))]$. By considering the triangle CDA we similarly find that AC must lie in $[\,|l-m|, \min(l+m, 2n-(l+m))]$. The intersection of these intervals has size

$$\begin{aligned}
& \min\{x - y, 2n - x - y \;:\; x \in \{i + j, l + m\},\; y \in \{|i - j|, |l - m|\}\;\} \\
= \;& \min((2\mathrm{min}(i, j), 2s - \max(i, j), 2s - \max(l, m), 2\mathrm{min}(l, m), \\
& 2n - 2\mathrm{max}(i, j), 2n - 2s - 2\mathrm{min}(l, m), 2n - 2s - 2\mathrm{min}(i, j), 2n - 2\mathrm{max}(l, m)) \\
= \;& 2\mathrm{min}(\min(i, j, l, m), 2s - \max(i, j, l, m), n - \max(i, j, l, m), n - s - \min(i, j, l, m)) \\
= \;& 2\mathrm{min}(M_1, n - s + M_1, s - M_2, n - M_2)
\end{aligned}$$

$\square$

We introduce a relative version of $\mathcal{K}$ above by setting $\mathcal{K}_n = \{C_0, \cdots, C_n\}$ with multiplication given by the following structure constants.

$$N_{ijk}^{(n)} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is an integral triangle set and } s \leq n \\ 0 & \text{otherwise} \end{cases}$$

Proposition 19 *The above multiplication makes $\mathcal{K}_n$ into an (associative) Hermitian positive generalised hypergroup.*

Proof: The proof follows from the previous result in the same way that 17 followed from the quadrilateral lemma. $\square$

In the literature the fusion rule algebra $\mathcal{K}_n$ is given the name $su(2)_n$.

3 The procrastination algebra of a graph

We now generalise the discussion to an arbitrary simple graph X with distinguished vertex 0 which we assume to have degree 1. To such a pointed graph we will associate an algebra $\mathcal{P}(X)$ with basis indexed by the vertices and whose structure constants count numbers of special closed walks from 0 to 0 called procrastinations. It must be emphasised that these algebras are by construction not necessarily associative. However it turns out that, for reasons as yet unknown to us, these 'procrastination algebras' are in fact often associative and furthermore often coincide with the fusion algebras $\mathcal{K}(X)$ when the latter are defined.

Let us begin by establishing some terminology. A walk in X is an alternating sequence $w = x_0, e_1, x_1, \cdots, x_{k-1}, e_k, x_k$ of vertices x_i, not necessarily distinct, and edges e_i such that e_k joins x_{k-1} and x_k. Such a walk has *length* k. It is a *geodesic* if it has minimal length amongst all walks from x_0 to x_k. For a vertex x let $|x|$ denote the length of a geodesic from 0 to x. The walk w is called a *path* if the edges $e_1, \cdots, e_k$ are distinct.

We will denote by $\overline{w}$ the walk from x_k to x_0 obtained by reversing the sequence of w. If u is a walk from x to y and v a walk from y to z then $w = u*v$ will be the combined (or concatenated) walk from x to z. Its length is thus the sum of the lengths of u and v. We will also say that u can be *extended* to

w. A path w from x to y will be called *returning* if it can be extended to a geodesic from x to 0 and *non returning* otherwise. A closed walk w from 0 to 0 will be called a *procrastination* if it is of the form $w = w_1 * w_2 * \cdots * w_{k-1} * w_k$ where w_1 and w_k are geodesics (necessarily from and to 0 respectively) and where w_s is a non returning path for $s = 2, \cdots, k-1$.

As an example, let X be the graph A_{n+1} with vertices labelled 0 to n. This is a tree with base point 0; thus for any vertex i there is a unique geodesic w_i from 0 to i. The procrastinations are exactly those walks of the form $w_i * \overline{w}_i$ for some i.

Returning to the general case, suppose that the graph X has vertices labelled $\{0, 1, \cdots, n\}$ and that 1 is the unique neighbour of 0. For an ordered triple of vertices $[i, j, k]$ in X consider all procrastinations w which have the form

$$w \;=\; w_i * h * \overline{w_k}$$

where w_i and w_k are geodesics from 0 to i and k respectively and where h is a walk of length $|j|$ from i to k. Note that h need not be a path. We see that such a walk w has length $|i| + |j| + |k|$. Let $\langle i, j, k\rangle$ denote the number of all such walks w. Now define

$$M_{ijk} \;=\; \min\{\langle \sigma(i), \sigma(j), \sigma(k)\rangle \;:\; \sigma \in S_3\}$$

the minimum over all permutations σ in the symmetric group S_3. Let us define a (possibly non-associative) algebra structure on the space spanned by $\{C_0, \cdots, C_n\}$ via

$$C_i C_j \;=\; \sum_k M_{ijk} C_k$$

We will call this (in general non associative structure) the *procrastination algebra* of X and denote it by $\mathcal{P}(X)$. Note that the structure is however commutative and that C_0 is an identity. Furthermore $M_{ij0} = M_{i0j}$ is non zero if and only if $i = j$ in which case $M_{ij0} = 1$. Thus $\mathcal{P}(X)$ is, apart from associativity, a Hermitian fusion rule algebra.

Lemma 20 *Let L_1 denote the matrix of multiplication by C_1 in the algebra $\mathcal{P}(X)$ with respect to the basis $\{C_0, \cdots, C_n\}$. Then $L_1 = A$, the adjacency matrix of the graph X.*

Proof: This follows from the fact that $M_{ij1} = M_{i1j}$ is non zero exactly when i and j are neighbours, in which case $M_{ij1} = 1$. $\square$

This suggests that if $\mathcal{P}(X)$ is associative, it will likely coincide with the fusion rule algebra $\mathcal{K}(X)$ of the graph X defined earlier using only the adjacency matrix and the cyclical condition. It is somewhat remarkable that this is indeed exactly what happens for a large class of cyclical pointed graphs. We begin with the following result for trees of Dynkin type.

THEOREM 21 *Let $(X, 0)$ be a pointed tree and let $\mathcal{P}(X, 0)$ be the procrastination algebra based at* 0*. Then $\mathcal{P}(X, 0)$ is isomorphic to $\mathcal{K}(X, 0)$ in the cases when X is one of the following: A_k, D_{2k}, E_6, E_8. In particular, $\mathcal{P}(X)$ is a Hermitian fusion rule algebra in these cases.*

Proof: This follows by explicit calculations and a direct verification that the two structures coincide. In the case of D_{2k} one must establish the general structure constants by induction but this is not difficult. The explicit structures of the algebras in the cases E_6 and E_8 are given in the next section. $\square$

4 Examples of Fusion Rule algebras of graphs and Procrastination algebras of graphs

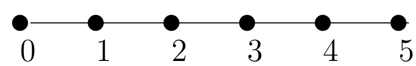
We now list the concrete fusion rule algebras $\mathcal{K}(X)$ and the procrastination algebras $\mathcal{P}(X)$ for the cases when X is one of A_n, D_n, E_6, or E_8. We include also the generalised fusion algebra $\mathcal{K}(E_7)$, first of all to show that this is 'almost' a fusion rule algebra with the all except one N_{ijk} non-negative (a phenomenon that appears in fact to be not so unusual), and second to compare it with $\mathcal{P}(E_7)$ which is *not* associative in this case. Throughout we adopt the convention that vertices are labelled by their distances to 0 with primes used to distinguish vertices of the same distance. This simplifies the presentation of the structure constants.

Some further experimentation with the computer programs that allow us to calculate the generalised fusion algebras of graphs reveal that in many other cases besides the above do the two notions of $\mathcal{P}(X, 0)$ and $\mathcal{K}(X, 0)$ coincide. At this point the reason and extent of this phenomenon is a mystery to us. In fact we suspect that our definition of a procrastination may not be precisely what is needed in the general case, in other words that is there is possibly another notion which coincides in the cases we have looked at with ours but allows the mysterious equality between $\mathcal{P}(X,0)$ and $\mathcal{K}(X,0)$ to be extended yet further.

For this reason we suggest that the notion of a procrastination be considered as temporary. In any case we end by showing some other more complicated trees which are not of Dynkin type but for which the fusion rule algebras and procrastination algebras coincide and include also an example where the same phenomenon occurs for a graph which is not a tree – an example we call the 'Witches Hat'.

Type A_6: $\mathcal{K}(X,0) = \mathcal{P}(X,0)$

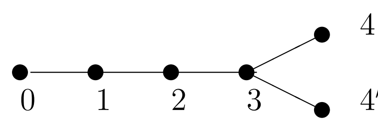
A_6



$1\underset{\sim}{1} = 0 + 2$	$2\underset{\sim}{2} = 0 + 2 + 4$	$3\underset{\sim}{3} = 0 + 2 + 4$	$4\underset{\sim}{4} = 0 + 2$
$1\underset{\sim}{2} = 1 + 3$	$2\underset{\sim}{3} = 1 + 3 + 5$	$3\underset{\sim}{4} = 1 + 3$	$4\underset{\sim}{5} = 1$
$1\underset{\sim}{3} = 2 + 4$	$2\underset{\sim}{4} = 2 + 4$	$3\underset{\sim}{5} = 2$	$5\underset{\sim}{5} = 0$
$1\underset{\sim}{4} = 3 + 5$	$2\underset{\sim}{5} = 3$		
$1\underset{\sim}{5} = 4$			

Type D_6: $\mathcal{K}(X,0) = \mathcal{P}(X,0)$

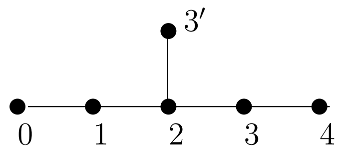
D_6



$1\underset{\sim}{1} = 0 + 2$	$2\underset{\sim}{2} = 0 + 2 + 4 + 4'$	$3\underset{\sim}{3} = 0 + 2 + 4 + 4'$	$4\underset{\sim}{4} = 0 + 4$
$1\underset{\sim}{2} = 1 + 3$	$2\underset{\sim}{3} = 1 + 2(3)$	$3\underset{\sim}{4} = 1 + 3$	$4\underset{\sim}{4}' = 2$
$1\underset{\sim}{3} = 2 + 4 + 4'$	$2\underset{\sim}{4} = 2 + 4'$	$3\underset{\sim}{4}' = 1 + 3$	$4'\underset{\sim}{4}' = 0 + 4'$
$1\underset{\sim}{4} = 3$	$2\underset{\sim}{4}' = 2 + 4$		
$1\underset{\sim}{4}' = 3$			

Type E_6: $\mathcal{K}(X,0) = \mathcal{P}(X,0)$

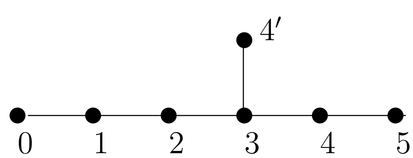
E_6



$1\underset{\sim}{1} = 0 + 2$	$2\underset{\sim}{2} = 0 + 2(2)$	$3\underset{\sim}{3} = 0 + 2(2)$	$4\underset{\sim}{4} = 0$
$1\underset{\sim}{2} = 1 + 3 + 3'$	$2\underset{\sim}{3} = 1 + 3 + 3'$	$3\underset{\sim}{4} = 1$	$4\underset{\sim}{3}' = 3'$
$1\underset{\sim}{3} = 2 + 4$	$2\underset{\sim}{4} = 2$	$3\underset{\sim}{3}' = 2$	$3'\underset{\sim}{3}' = 0 + 2 + 4$
$1\underset{\sim}{4} = 3$	$2\underset{\sim}{3}' = 1 + 3$		
$1\underset{\sim}{3}' = 2$			

Type E_7: $\mathcal{K}(X,0) \neq \mathcal{P}(X,0)$

E_7



$\mathcal{K}(E_7)$

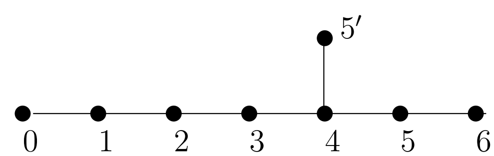
$1\underset{\sim}{1} = 0 + 2$	$2\underset{\sim}{2} = 0 + 2 + 4 + 4'$	$3\underset{\sim}{3} = 0 + 2(2) + 2(4) + 4'$	$4\underset{\sim}{4} = 0 + 2 + 4$
$1\underset{\sim}{2} = 1 + 3$	$2\underset{\sim}{3} = 1 + 2(3) + 5$	$3\underset{\sim}{4} = 1 + 2(3)$	$4\underset{\sim}{5} = 1 + 5$
$1\underset{\sim}{3} = 2 + 4 + 4'$	$2\underset{\sim}{4} = 2 + 4 + 4'$	$3\underset{\sim}{5} = 2 + 4'$	$4\underset{\sim}{4}' = 2 + 4'$
$1\underset{\sim}{4} = 3 + 5$	$2\underset{\sim}{5} = 3$	$3\underset{\sim}{4}' = 1 + 3 + 5$	$5\underset{\sim}{5} = 0 + 4 - 4'$
$1\underset{\sim}{5} = 4$	$2\underset{\sim}{4}' = 2 + 4$	$5\underset{\sim}{4}' = 3 - 5$	$4'\underset{\sim}{4}' = 0 + 4$
$1\underset{\sim}{4}' = 3$			

$\mathcal{P}(E_7)$: Not associative

$1\underset{\sim}{1} = 0 + 2$	$2\underset{\sim}{2} = 0 + 2 + 4 + 4'$	$3\underset{\sim}{3} = 0 + 2(2) + 2(4) + 4'$	$4\underset{\sim}{4} = 0 + 2 + 4 + 4'$
$1\underset{\sim}{2} = 1 + 3$	$2\underset{\sim}{3} = 1 + 2(3) + 5$	$3\underset{\sim}{4} = 1 + 2(3)$	$4\underset{\sim}{5} = 1$
$1\underset{\sim}{3} = 2 + 4 + 4'$	$2\underset{\sim}{4} = 2 + 4 + 4'$	$3\underset{\sim}{5} = 2 + 4'$	$4\underset{\sim}{4}' = 2 + 4 + 4'$
$1\underset{\sim}{4} = 3 + 5$	$2\underset{\sim}{5} = 3$	$3\underset{\sim}{4}' = 1 + 3 + 5$	$5\underset{\sim}{5} = 0$
$1\underset{\sim}{5} = 4$	$2\underset{\sim}{4}' = 2 + 4$	$5\underset{\sim}{4}' = 3$	$4'\underset{\sim}{4}' = 0 + 4$
$1\underset{\sim}{4}' = 3$			

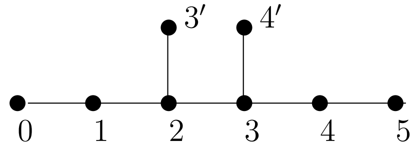
Type E_8: $\mathcal{K}(X, 0) = \mathcal{P}(X, 0)$

E_8



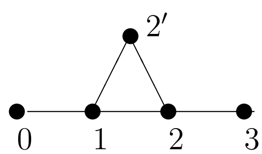
$1\underset{\sim}{1} = 0 + 2$	$2\underset{\sim}{2} = 0 + 2 + 4$	$3\underset{\sim}{3} = 0 + 2 + 2(4) + 6$	$4\underset{\sim}{4} = 0 + 2(2) + 3(4) + 6$
$1\underset{\sim}{2} = 1 + 3$	$2\underset{\sim}{3} = 1 + 3 + 5 + 5'$	$3\underset{\sim}{4} = 1 + 2(3) + 2(5) + 5'$	$4\underset{\sim}{5} = 1 + 2(3) + 5 + 5'$
$1\underset{\sim}{3} = 2 + 4$	$2\underset{\sim}{4} = 2 + 2(4) + 6$	$3\underset{\sim}{5} = 2 + 2(4)$	$4\underset{\sim}{6} = 2 + 4$
$1\underset{\sim}{4} = 3 + 5 + 5'$	$2\underset{\sim}{5} = 3 + 5 + 5'$	$3\underset{\sim}{6} = 3 + 5'$	$4\underset{\sim}{5}' = 1 + 3 + 5 + 5$
$1\underset{\sim}{5} = 4 + 6$	$2\underset{\sim}{6} = 4$	$3\underset{\sim}{5}' = 2 + 4 + 6$	$5\underset{\sim}{5} = 0 + 2 + 4 + 6$
$1\underset{\sim}{6} = 5$	$2\underset{\sim}{5}' = 3 + 5$	$5\underset{\sim}{6} = 1 + 5$	$5\underset{\sim}{5}' = 2 + 4$
$1\underset{\sim}{5}' = 4$	$6\underset{\sim}{6} = 0 + 6$	$6\underset{\sim}{5}' = 3$	$5'\underset{\sim}{5}' = 0 + 4$

Another tree for which $\mathcal{K}(X,0) = \mathcal{P}(X,0)$



$1\underset{\varsigma}{1} = 0 + 2$	$2\underset{\varsigma}{2} = 0 + 2(2) + 4 + 4'$	$3\underset{\varsigma}{3} = 0 + 2(2) + 4 + 4'$	$4\underset{\varsigma}{4} = 0 + 2$
$1\underset{\varsigma}{2} = 1 + 3 + 3'$	$2\underset{\varsigma}{3} = 1 + 2(3) + 5 + 3'$	$3\underset{\varsigma}{4} = 1 + 3 + 3'$	$4\underset{\varsigma}{5} = 1$
$1\underset{\varsigma}{3} = 2 + 4 + 4'$	$2\underset{\varsigma}{4} = 2 + 4 + 4'$	$3\underset{\varsigma}{5} = 2$	$4\underset{\varsigma}{3}' = 3$
$1\underset{\varsigma}{4} = 3 + 5$	$2\underset{\varsigma}{5} = 3$	$3\underset{\varsigma}{3}' = 2 + 4$	$4\underset{\varsigma}{4}' = 2$
$1\underset{\varsigma}{5} = 4$	$2\underset{\varsigma}{3}' = 1 + 3$	$3\underset{\varsigma}{4}' = 1 + 3$	$5\underset{\varsigma}{5} = 0$
$1\underset{\varsigma}{3}' = 2$	$2\underset{\varsigma}{4}' = 2 + 4$	$5\underset{\varsigma}{3}' = 4'$	$3'\underset{\varsigma}{3}' = 0 + 4'$
$1\underset{\varsigma}{4}' = 3$	$5\underset{\varsigma}{4}' = 3'$	$3'\underset{\varsigma}{4}' = 5 + 3'$	$4'\underset{\varsigma}{4}' = 0 + 4'$

The Witches Hat: $\mathcal{K}(X,0) = \mathcal{P}(X,0)$



$1\underset{\varsigma}{1} = 0 + 2 + 2'$	$2\underset{\varsigma}{2} = 0 + 2 + 2'$	$3\underset{\varsigma}{3} = 0$	$2'\underset{\varsigma}{2}' = 0 + 3 + 2'$
$1\underset{\varsigma}{2} = 1 + 3 + 2'$	$2\underset{\varsigma}{3} = 1$	$3\underset{\varsigma}{2}' = 2'$	
$1\underset{\varsigma}{3} = 2$	$2\underset{\varsigma}{2}' = 1 + 2$		
$1\underset{\varsigma}{2}' = 1 + 2$			

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