Planar Algebras and (1+1)-dimensional TQFTs

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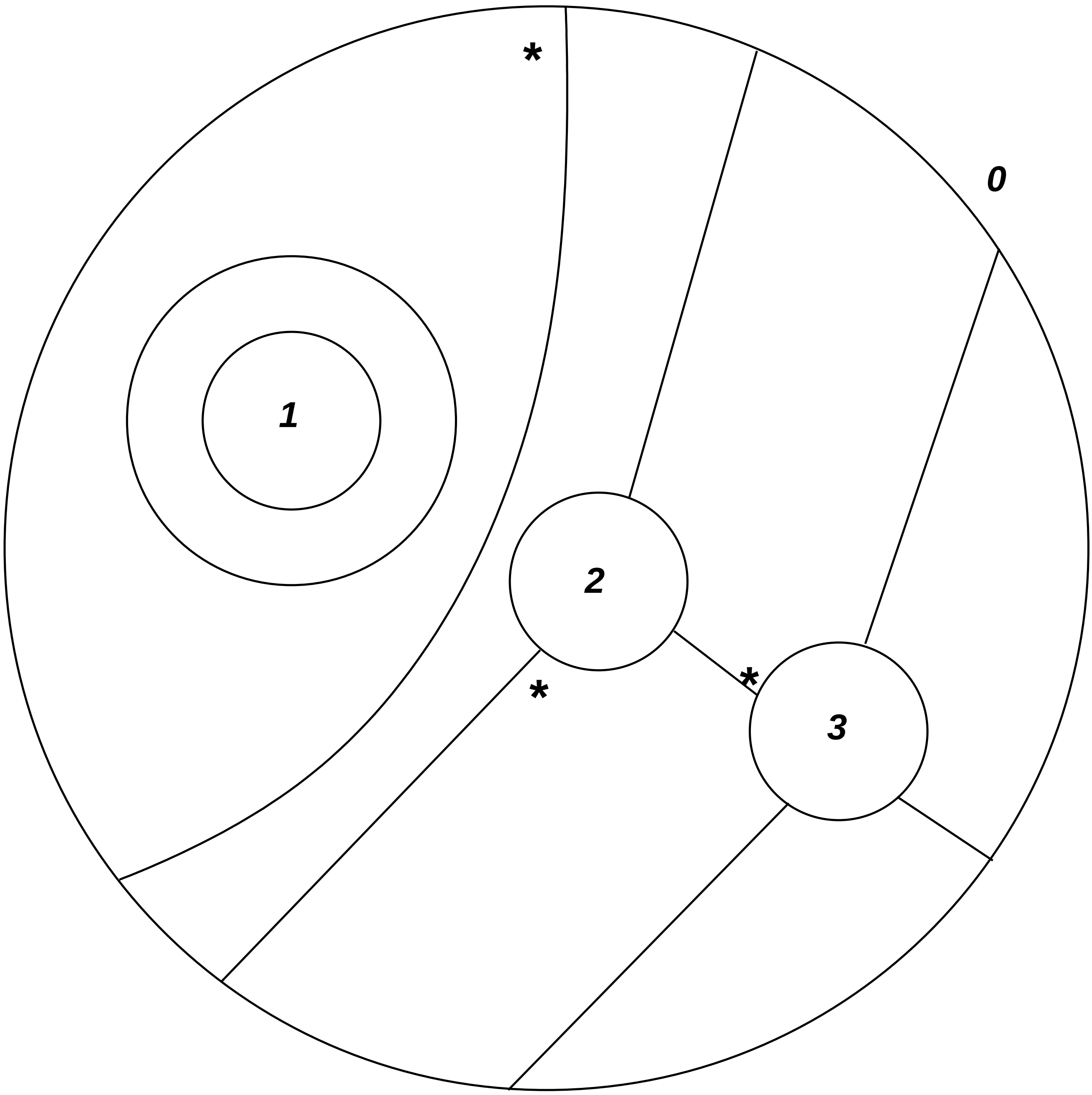
Planar Algebras are algebras over the coloured operad of planar tangles.

- Vaughan Jones

Example of planar tangle (first approximation):



Ingredients of a planar tangle T are:

- an external disc D_0

- an ordered collection $\{D_i : 1 \leq i \leq b\}$ (possibly empty; thus $b \geq 0$) of internal discs

- k_i special marked points on ∂D_i, $\forall\ 0 \leq i \leq b$ - $k_i = Col(D_i), k_0 = Col(T), Col = \{0, 1, 2, \cdots\}$

- some *strings*, with the set of all end-points (if any) of all strings equal to the set of marked points

- one distinguished point * among marked points on ∂D_i whenever $k_i \neq 0$

The *coloured operad $\mathcal{T}$ of planar tangles* is equipped with the following **operation of composition**:

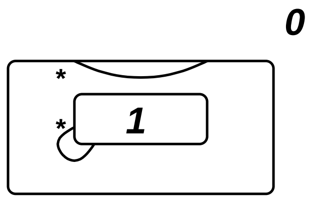
The composition $T \circ_i S$ is defined - provided $k_0(S) = k_i(T)$ - by 'sticking S into the i-th internal disc of T after aligning $*$'s and making sure the curves patch up right at $\partial D_i(T)$ - which circle is then removed, so

$$b(T \circ_i S) = b(T) + b(S) - 1 \ .$$

(Two tangles are identified if they are related by a structure-preserving - i.e. marked points and *-points - ambient isotopy.)

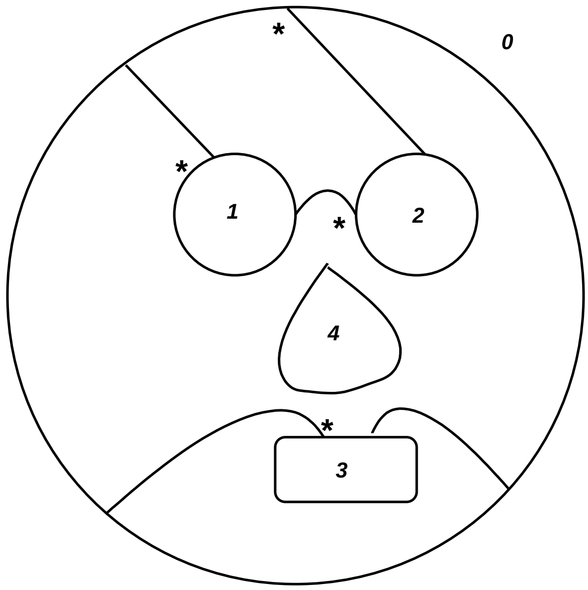
For instance, if we insert the lips given by

L =



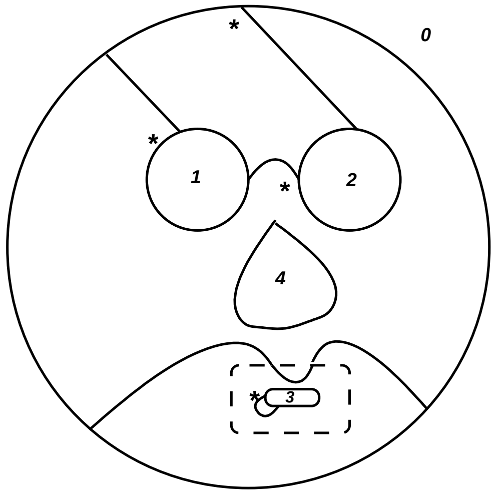
into the mouth of the following face,

F =

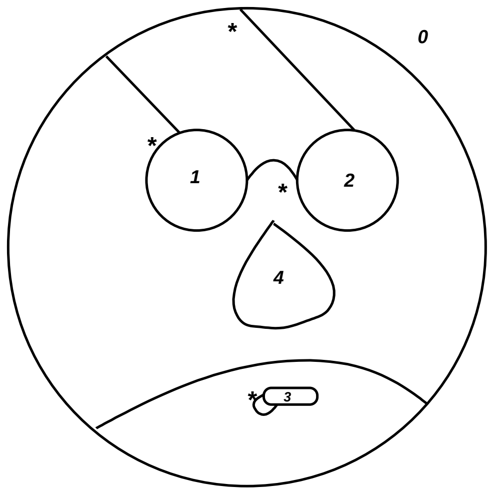


we get:

$F \circ_3 L =$



$=$



after smoothing the moustache.

Defn. (of a PA) (1st approximation):

A Planar Algebra is

- a collection $P = \{P_k : k \in \ Col \ \}$ of finite-dimensional Hilbert spaces

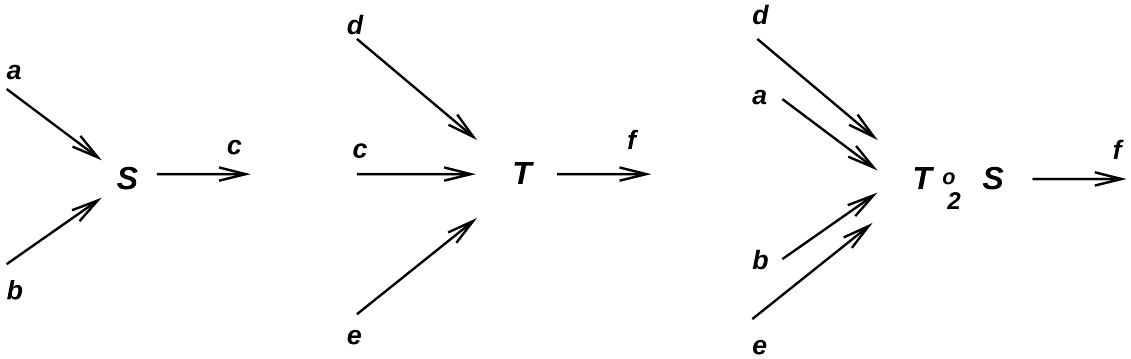
- equipped with an action of $\mathcal{T}$, thus: to each $T \in \mathcal{T}$ is associated a linear map

$$Z(T) : \otimes_{i=1}^{b(T)} P_{k_i(T)} \to P_{k_0(T)}$$

  which is *compatible with composition*, and also satisfies a few other 'reasonable requirements'.

Compatibility with composition:

If tangles S and T have colour attributes as below,



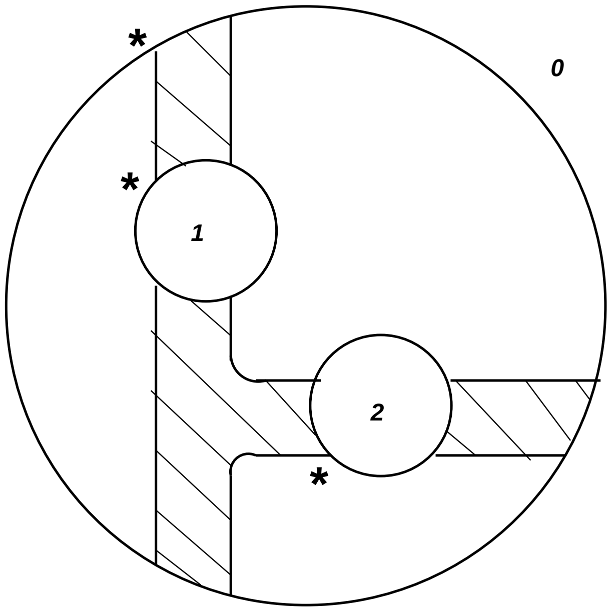
then

$$\begin{aligned} Z(S) : P_a \otimes P_b &\rightarrow P_c, \\ Z(T) : P_d \otimes P_c \otimes P_e &\rightarrow P_f, \\ Z(T \circ_2 S) : P_d \otimes P_a \otimes P_b \otimes P_e &\rightarrow P_f \ . \end{aligned}$$

and it is required that

$$Z(T \circ_2 S) = Z(T) \circ (id_{P_d} \otimes Z(S) \otimes id_{P_e})$$

Now, unlike our ‘first approximation’, here’s what an example of Jones’ planar tangles *really* looks like:



$(k_0, k_1, k_2) = (3, 2, 2)$

Note the following new features:

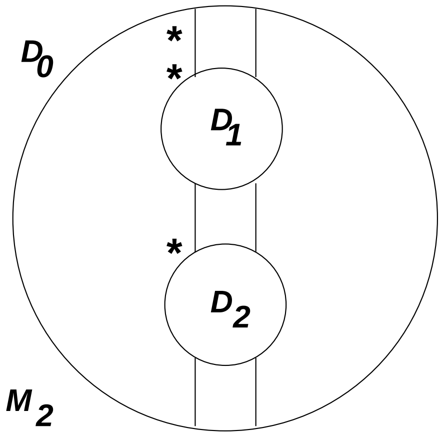
- a disc with colour k has $2k$ marked points

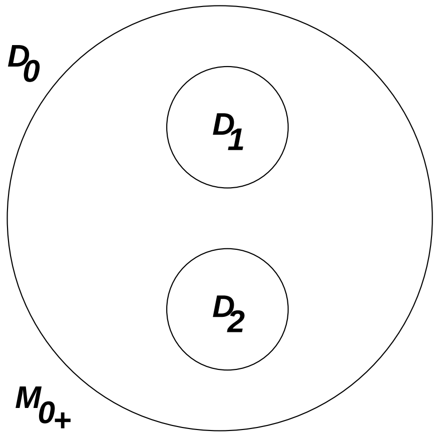
- the tangle comes equipped with a checkerboard shading

- the $*$ point can occupy only half the available slots: as you go clockwise through $*$, you must go from white to black!

Some useful Jones tangles

The multiplication tangles:

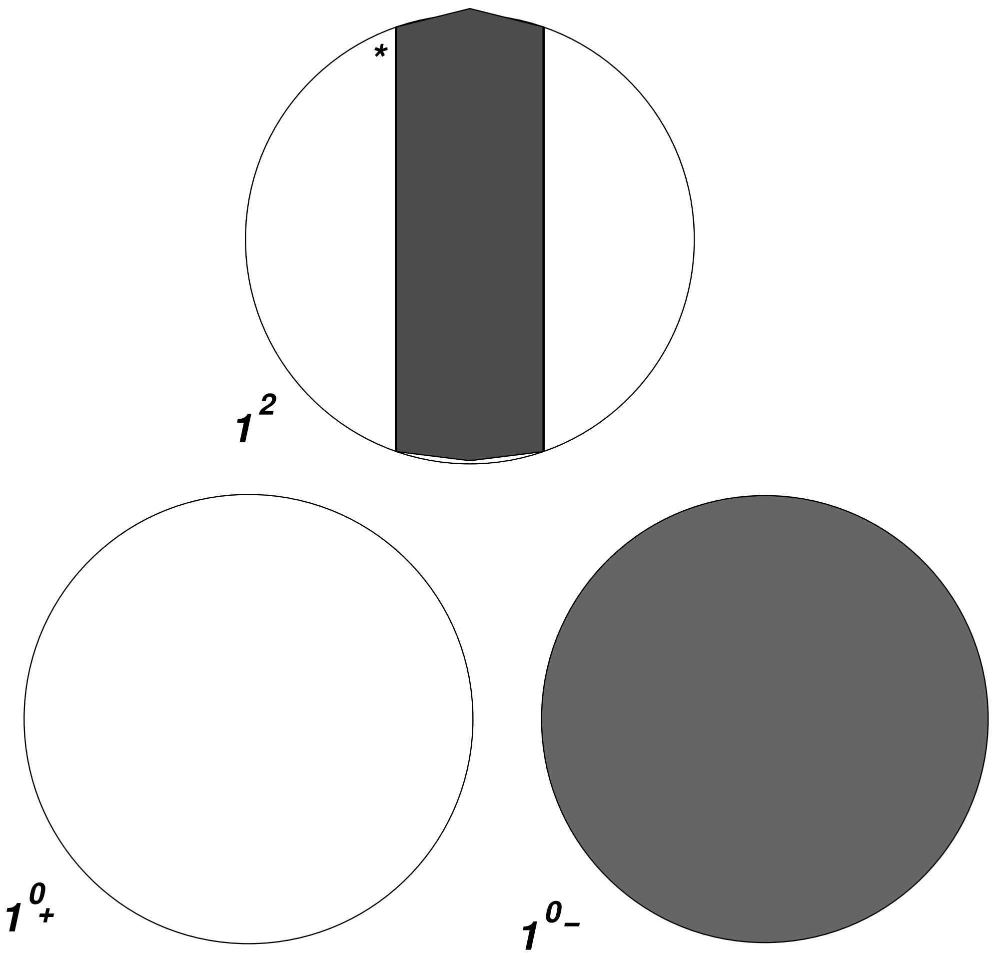




(Note: For M_k to make sense, we need an even number of marked points on the boundary!)

Each P_k is a unital associative algebra, with $P_{0_\pm}$ even commutative; we now describe the identity elements.

The identity tangles:

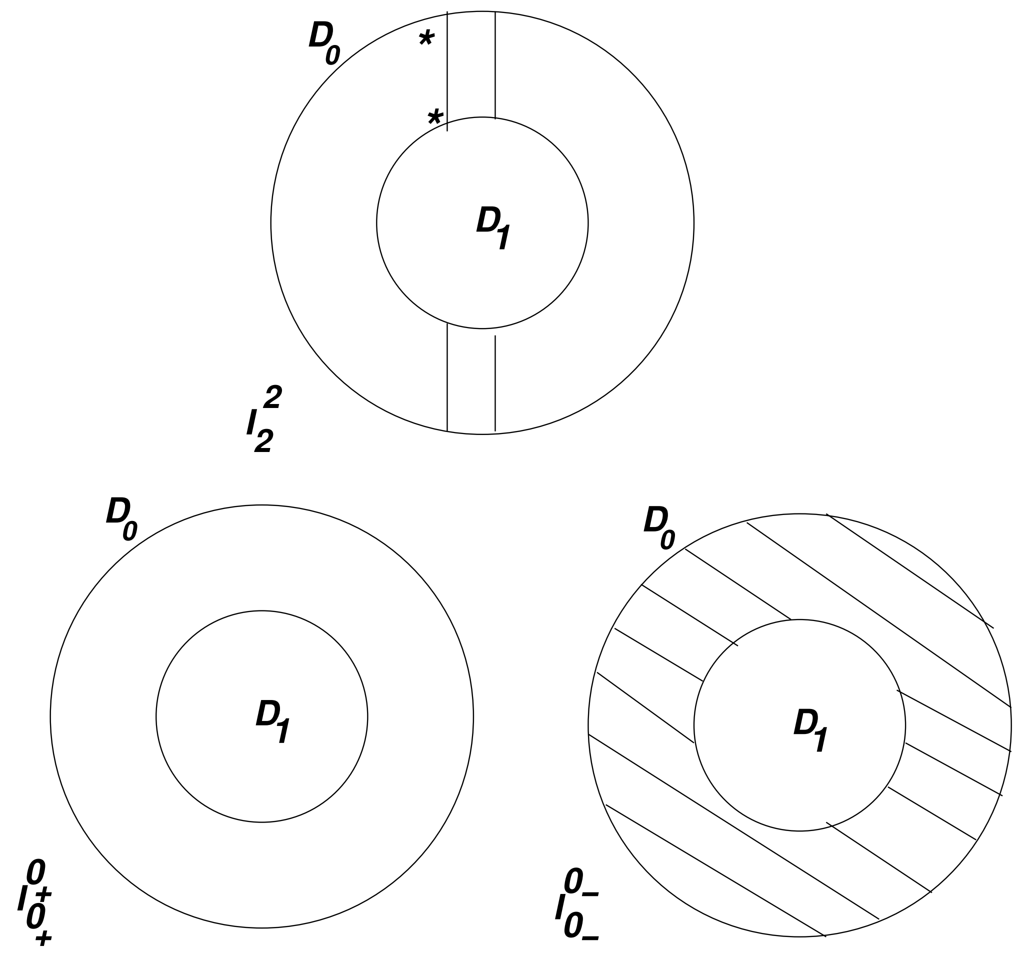


If a k-tangle T has no internal boxes (such as the 1^l's above), then $Z_T : \mathbb{C} \to P_k$. It is a fact that $1_k = Z_{1^k}(1)$ - where the '1' in parentheses denotes the 1 of $\mathbb{C}$ - is the multiplicative identity of P_k.

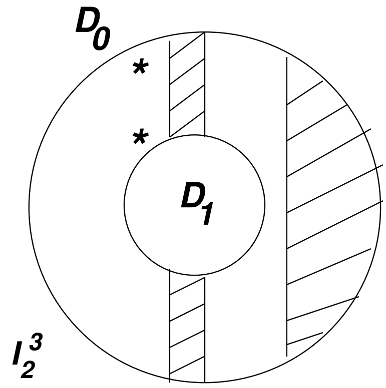
Strictly speaking, in order for 1_k to be the identity of P_k, we need to asume a 'non-degeneracy' condition which effectively demands that the ranges of Z_T's as above span P_k, or equivalently that

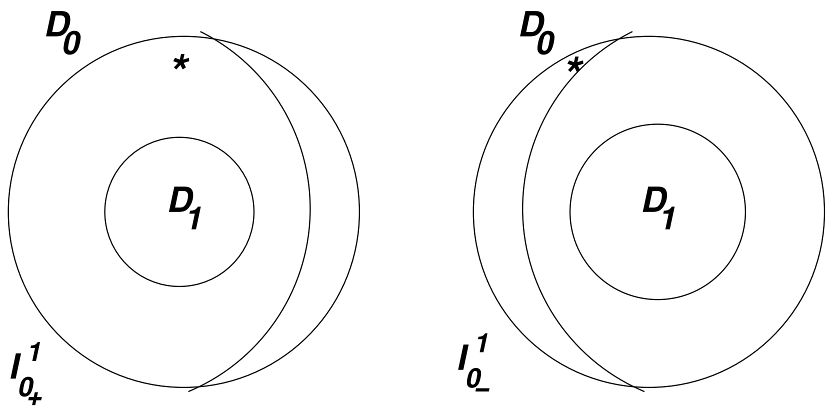
$$id_{P_k} = Z_{I_k^k} \ ,$$

where I_k^k is the *identity tangle* given by



The inclusion tangles:





(We always place the '*' points in white regions.)

The 'inclusions' are homomorphisms (of unital algebras) which are injective under mild hypotheses.

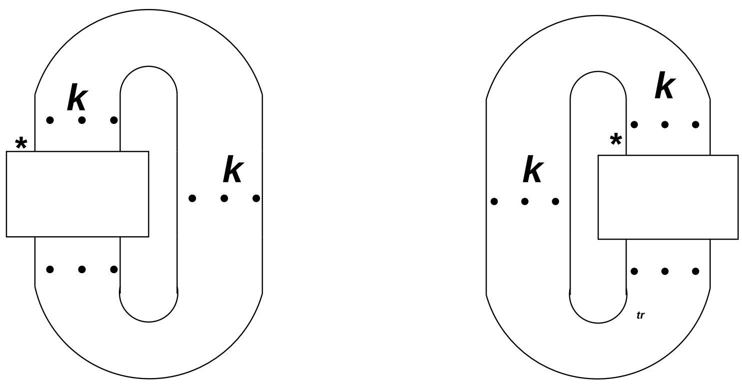
Connectedness ‘Good’ planar algebras satisfy the following requirement:

Connectedness: $P_{0_\pm}$ is 1-dimensional, or equivalently $P_{0_\pm} = \mathbb{C}$ (since a 1-dimensional algebra over $\mathbb{C}$ is canonically isomorphic to $\mathbb{C}$).

We shall assume that all our planar algebras are connected!

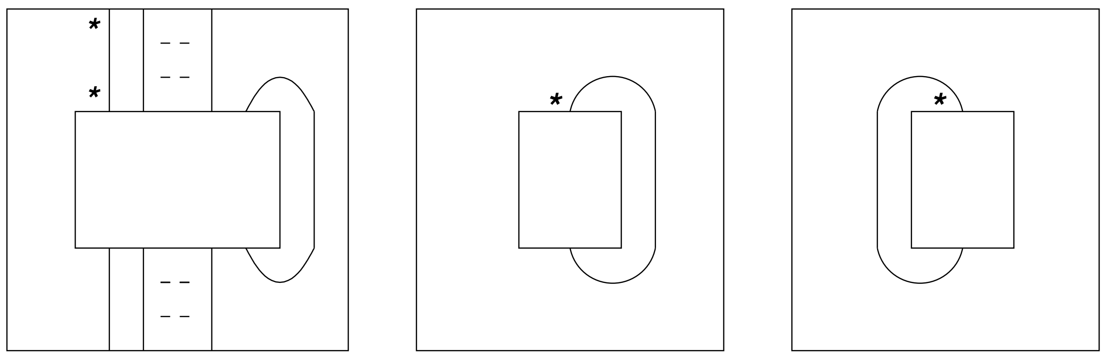
Remark: So, if T is a $0_\pm$ tangle, then we may view Z_T as a linear functional on $\otimes_{i=1}^{b} P_{k_i(T)}$.

The trace tangles: It is not hard to see that, in the sense of the previous remark, both $Z_{tr_k^{(r)}}$ and $Z_{tr_k^{(l)}}$ define traces on P_k, where $tr_k^{(r)}$ and $tr_k^{(l)}$ are the 'right- and left- trace tangles' (with internal disc of colour k) given by:



$tr_k^{(r)}$ $tr_k^{(l)}$

The **conditional expectation** tangles



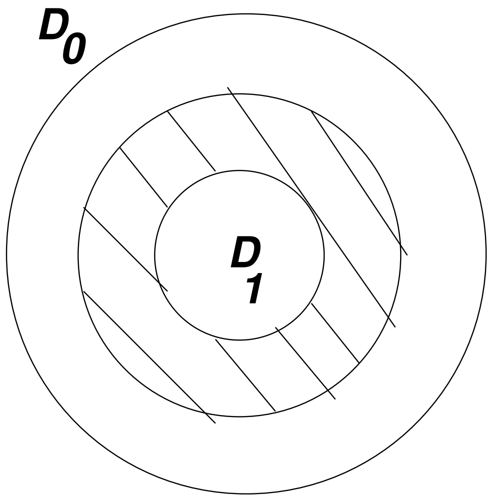
E^k_{k+1} E^{0+}_1 E^{0-}_1

induce maps $Z_{E^k_{k+1}} : P_{k+1} \to P_k$ satisfying

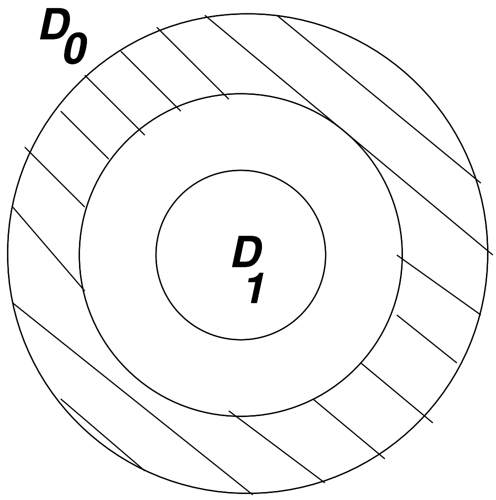
$$tr^{(r)}_k \circ Z_{E^k_{k+1}} = tr^{(r)}_{k+1}.$$

Two final requirements of a 'good' planar algebra:

Modulus: A (connected) planar algebra is said to have *modulus* δ if $Z_{T^{0_-}_{0_+}} = Z_{T^{0_-}_{0_+}} = id_{\mathbb{C}}$, where these 'toggling tangles' are given by



$T^{0_+}_{0_-}$



$T^{0_-}_{0_+}$

Finally in a connected planar algebra with non-zero modulus δ, the inclusion tangles induce injective maps, since

$$Z_{E^k_{k+1}} \circ Z_{I^{k+1}_k} = \delta \ id_{P_k} \ ,$$

Local finite-dimensionality: A planar algebra has this property if

$$dim\ P_k < \infty\ \forall k\ .$$

Theorem:(Jones) There is a bijective correspondence between

(i) extremal finite-index subfactors

and

(ii) locally finite-dimensional connected planar algebras with positive modulus, which are 'spherical' in the sense that

$$Z_{tr_k^{(l)}} = Z_{tr_k^{(r)}}\ \forall k\ ,$$

and further has an involutive structure (which Jones calls a C^*-*planar algebra*).

(Unitary) (1+1)-dimensional TQFTs

are functors on 'cobordism categories'*

$$\begin{array}{ccc} \text{Closed 1-manifolds} & \mapsto & \text{fin. dim. Hilbert spaces} \\ \sigma & \mapsto & V(\sigma) \end{array}$$

satisfying

$$\begin{array}{rcl} V(\bar{\sigma}) & = & V(\sigma)^* \\ V(\sigma_1 \coprod \sigma_2) & = & V(\sigma_1) \otimes V(\sigma_2) \\ (\text{Commutative} \coprod & \leftrightarrow & \text{unordered tensor products}) \end{array}$$

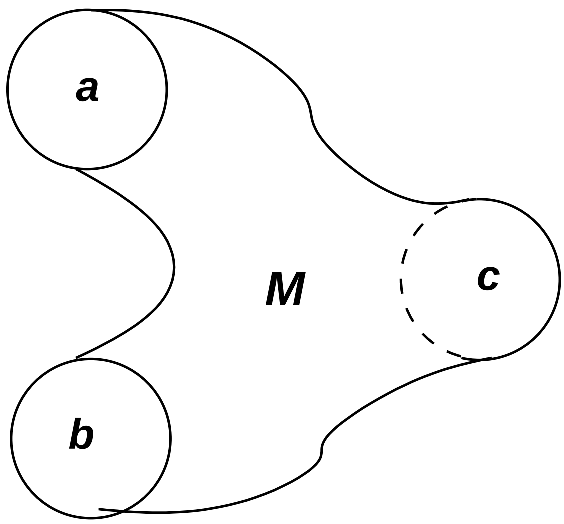
And

$$\begin{array}{rcl} \Sigma & \in & Cob(\sigma_1, \sigma_2) \\ \Rightarrow \; \partial\Sigma & = & \bar{\sigma}_1 \coprod \sigma_2 \\ \Rightarrow \; V(\partial\Sigma) & = & V(\bar{\sigma}_1) \otimes V(\sigma_2) \\ & = & V(\sigma_1)^* \otimes V(\sigma_2) \\ & = & Hom(V(\sigma_1), V(\sigma_2)) \end{array}$$

*objects are compact, oriented, often *decorated* manifolds, and morphisms are (decorated) cobordisms

Thus, to any cobordism $\Sigma : \sigma_1 \to \sigma_2$, a TQFT associates an operator $Z_\Sigma \in Hom(V(\sigma_1), V(\sigma_2))$ as also a vector $\zeta_\Sigma \in V(\partial\Sigma)$ which are related by the natural identifications.

If the boundary of a 2-manifold M has n components, then M may be regarded as a cobordism in 2^n ways.



For instance, if $\partial M = a \coprod b \coprod c$, (as above), then M yields cobordisms $\Sigma : \bar{a} \to b \coprod c$ as well as $\Sigma_1 : \bar{a} \coprod \bar{c} \to b$ and we would want ζ_Σ and ζ_{Σ_1} to correspond under the natural identification.

In particular, if $\Sigma : \sigma_1 \to \sigma_2$ is a cobordism, then $\partial\Sigma = \bar{\sigma}_1 \coprod \sigma_2$, so $\partial\bar{\Sigma} = \bar{\sigma}_2 \coprod \sigma_1$ and we may view $\bar{\Sigma}$ as a cobordism from σ_2 to σ_1; and the **unitarity requirement** on the TQFT is:

$$Z_{\bar{\Sigma}} = Z^*_{\Sigma} \ .$$

TQFTs from PAs - still first approximation

Imagine a category where the *simple objects* are in bijection with $Col = \{0, 1, 2, \cdots\}$, and general objects are 'disjoint unions' of finitely many simple objects; and suppose k is thought of as the equivalence class of a circle* with k marked points - where equivalence means existence of a diffeomorphism which preserves the structure (orientation, set of marked points, the $*$-points, etc.):

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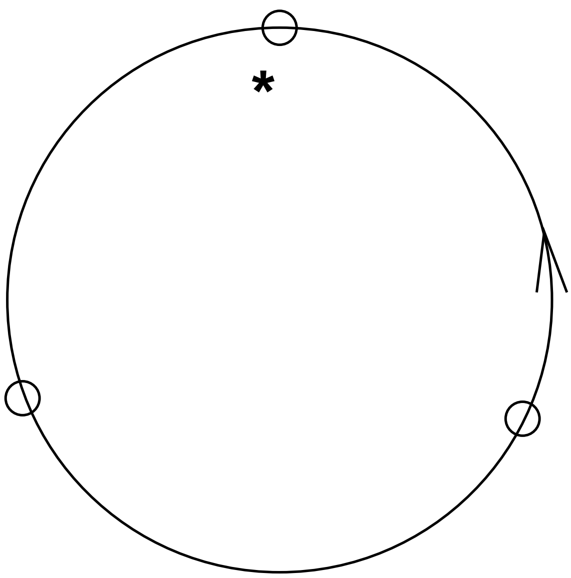


*all planar 1-manifolds are thought of as being oriented anti-clockwise.

Given a planar algebra P, define

$$V(\coprod_{i=1}^{b} k_i) = \otimes_{i=1}^{b} P_{k_i} \ ,$$

and note that a planar tangle $T \in \mathcal{T}$ yields an operator

$$Z(T) : V(\coprod_{i=1}^{b} k_i) = \otimes_{i=1}^{b} P_{k_i} \to P_{k_0} = V(k_0) \ . \tag{1}$$

One of the ‘reasonable requirements’ in the definition of a PA, which is related to the ‘operadic structure’ of $\mathcal{T}$, ensures a natural ‘permutation invariance’ in the association

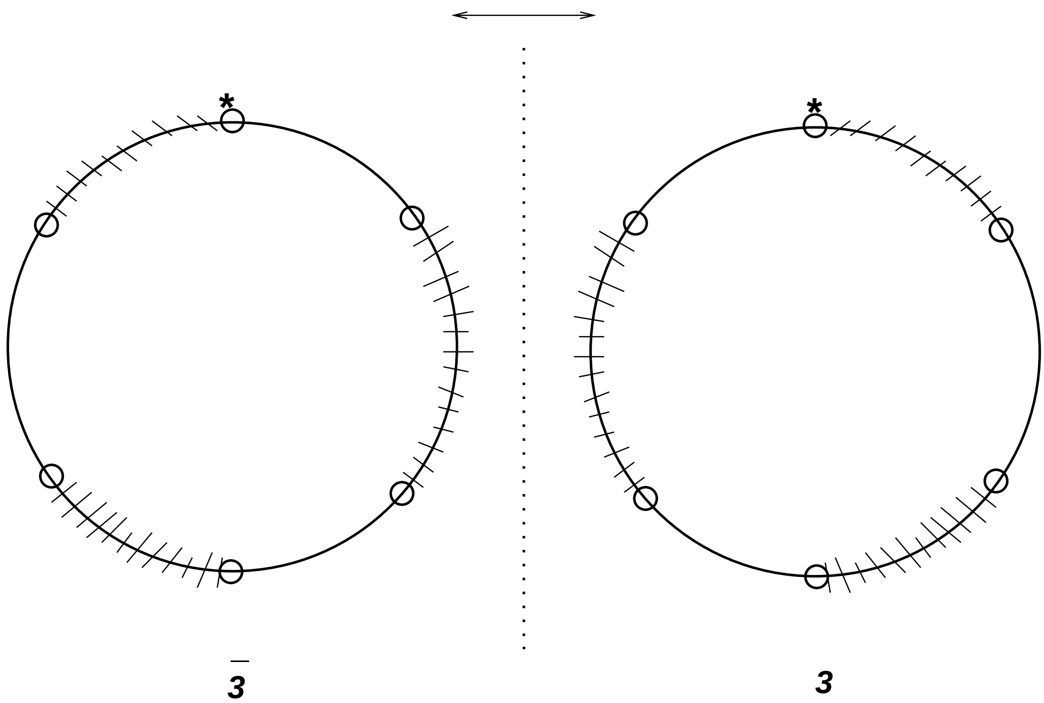
$$T \mapsto Z(T) \ ,$$

and hence eq. (1) would be consistent with the requirement of ‘unordered tensor products’ in a TQFT.

Note that we cannot expect k and $\bar{k}$ to be the same, since, in general, $P_k \neq P_k^*$. So we need to modify our simple objects.

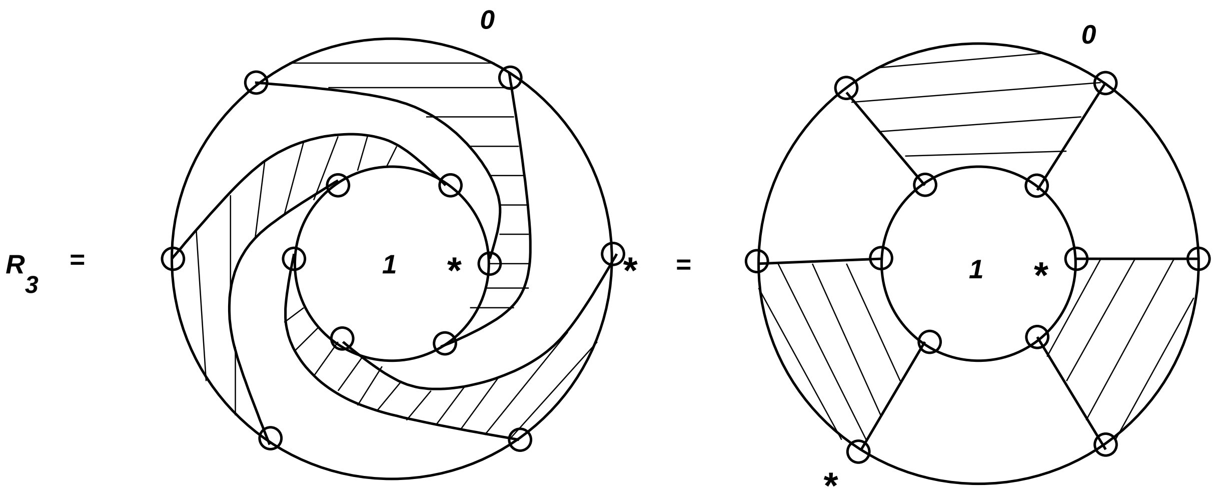
Relax parity requirement on $*$

This leads to the correct interpretation of $\bar{k}$:



(Recall: all planar 1-manifolds are oriented anti-clockwise.)

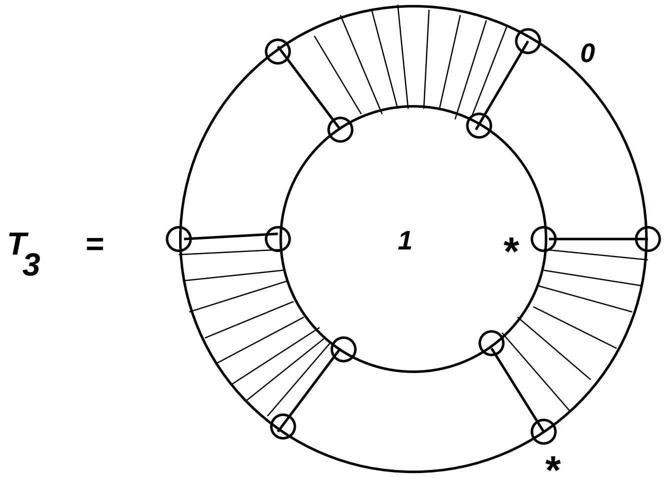
Among the most important of Jones' tangles are the **rotation tangles** (one for each k):



$$Z(R_3) : P_3 \to P_3\ , \quad \text{or} \quad \zeta_{R_3} \in P_3^* \otimes P_3$$

(Note that $*$ is 'shifted by two'.)

Now, 'rotation by 1' is possible unlike Jones' case:



This 'half-rotation' (or 'rotation by 1') is also very important (although not quite a 'Jones' tangle'). In fact it turns out that the operator

$$Z_{T_3} : P_3 \to P_{\bar{3}} \equiv P_3^*$$

corresponds to the element

$$\zeta_{T_3} \in P_3^* \otimes P_3^*$$

given by

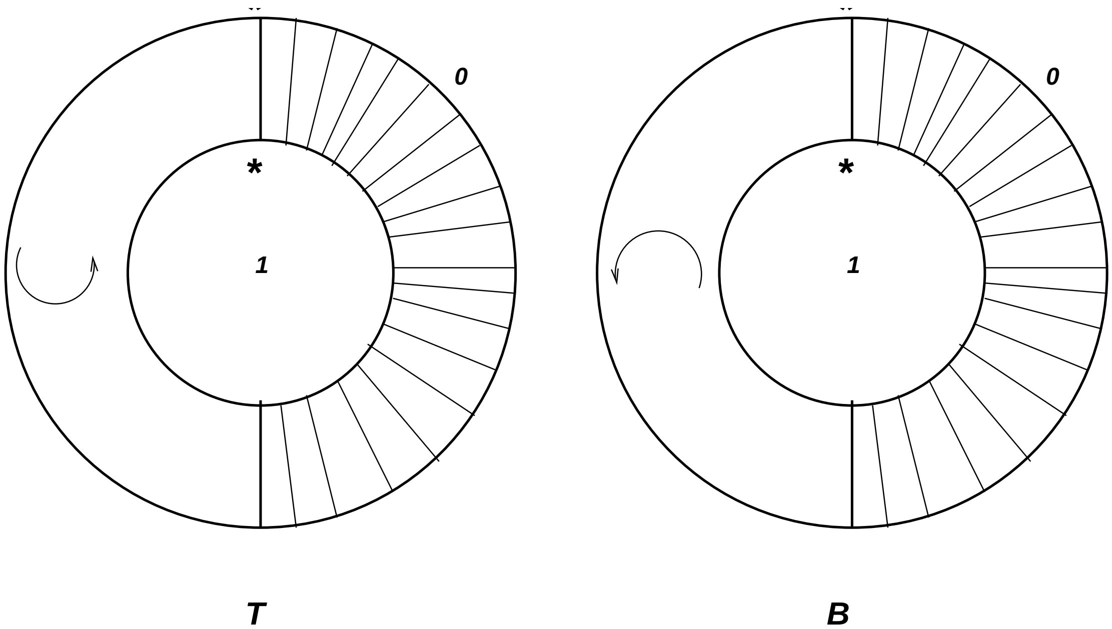
$$\langle x \otimes y, \zeta_{T_3} \rangle = tr(xy)$$

dvipdf

The idea is to define a 'cobordism category' $\mathcal{D}$ - on which our TQFT will be defined - where our 'quasi-tangles' (where the constraint on * has been relaxed) are 'building blocks' and general morphisms are obtained by 'glueing finitely many of our quasi-tangles - and can be quite general 2-manifolds carrying a 'decoration' featuring 'strings, $*$'s and 'checkerboard shadings'.

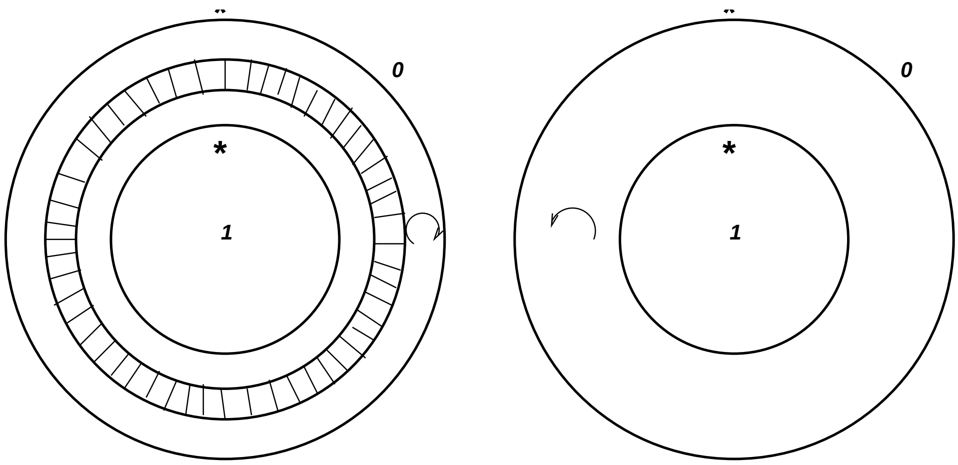
But there is a problem - owing to the *non-triviality of the mapping class group* of surfaces of higher genera.

For instance, let T and B be the 'quasi-tangles'



denoting the top and bottom halves of a torus 'painted half-black'.

On the other hand, if A_t and A_b are the 'decorated annuli' indicated below,



A_t A_b

then the result $A_b \circ A_t$ of glueing them together is diffeomorphic to the result $B \circ T$, but the number 'obtained' for the former turns out to be different from that of the latter.

The reason for the discrepancy is that we have to keep track of how the surface is sliced into its 'planar building blocks'.

We now define our category $\mathcal{D}$

Objects: simple objects are of the form k or $\bar{k}$, as in the earlier illustration for $k = 3$, where $k \in Col \ = \{0_B, 0_W, 1, 2, \cdots\}$.*

General objects are appropriate equivalence classes of triples $(\sigma, P, *, sh)$ where

- σ is a closed (possibly empty) 1-manifold
- P is a set of 'marked points' on it
- $*$ is a choice of distinguished marked point from every component meeting P, and
- sh is a 'checkerboard shading' on σ

*Due to the shading introduced, it bcomes necessary to allow two variations 0_B and 0_W of the basic 0 in Col; counting their 'bars' there are 4 variants altogether!

Morphisms have to be defined as 'equivalence classes of premorphisms' - a *premorphism* being defined to be a tuple $(\Sigma, \ell, *, sh, \Pi)$ where

- Σ is a surface

- ℓ is a 1-submanifold of Σ - the strings

- $*$ is a choice of distinguished marked point on the boundary of each component of $\partial\Sigma$ which meets ℓ

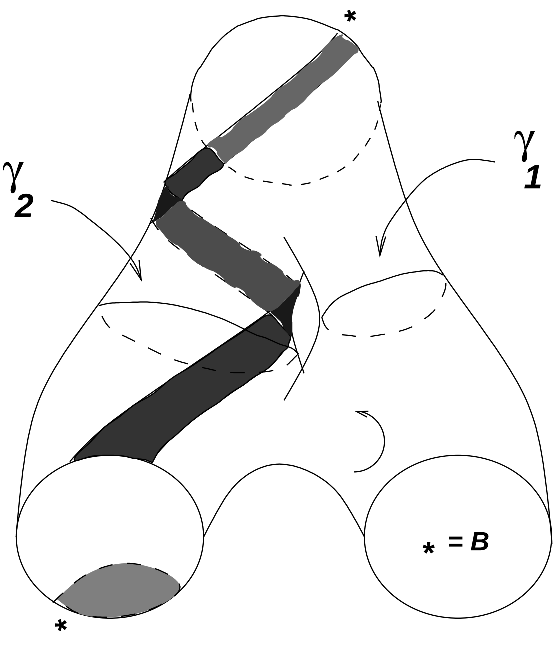
- sh is a checkerboard shading; and

- Π is a *planar decomposition* of Σ - i.e., a system of closed curves which (i) meet ℓ transversally, if at all, and (ii) 'carve' Σ up into finitely many 'planar pieces'

Two pre-morphisms are *equivalent* if it is possible to pass from one to the other by a finite sequence of moves of one of 3 types:

- structure preserving diffeomorphism
- replacing $(\Sigma, \ell, *, sh, \Pi)$ by $(\Sigma, \ell, *, sh, \Pi_1)$ if Π is carried to Π_1 by an ambient isotopy of Σ
- replacing $(\Sigma, \ell, *, sh, \Pi)$ by $(\Sigma, \ell, *, sh, \Pi_1)$ if either $\Pi \subset \Pi_1$ or $\Pi_1 \subset \Pi$

Here is an example of a premorphism, with 'planar decomposition' afforded by the family $\{\gamma_1, \gamma_2\}$ of curves:



This is viewed most naturally as a morphism from $\bar{\mathbf{2}} \coprod \mathbf{0_W}$ to $\mathbf{1}$.

The following result should be compared with the well-known equivalence between 1+1-dim. TQFTs (on $2Cob$) and Frobenius algebras.

Theorem:

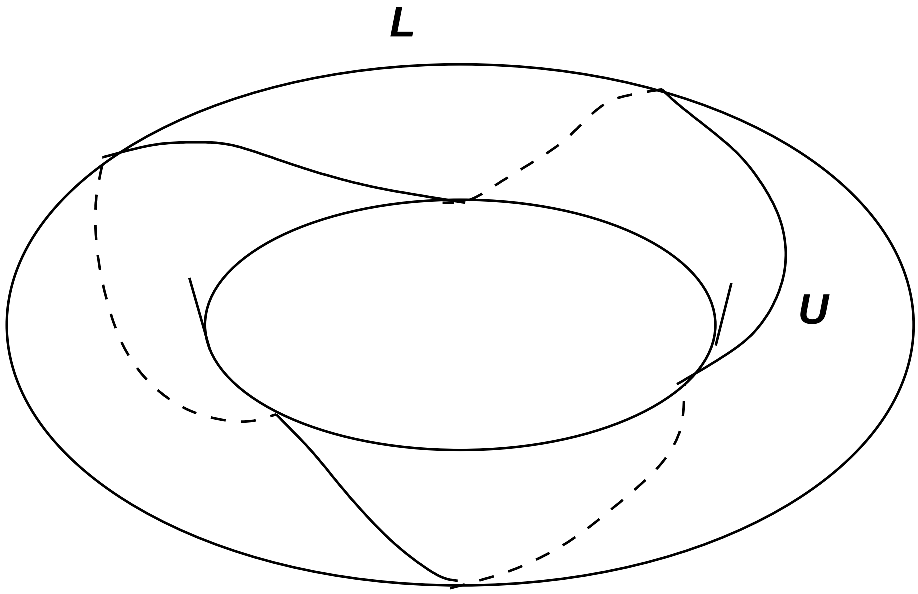
- There is a unique (1+1)-dim.) 'cobordism category' $\mathcal{D}$ where objects and morphisms are defined as in the last few slides - with naturally defined composition, disjoint unions, conjugates (or adjoints).

- Each 'subfactor planar algebra' gives rise uniquely to a unitary TQFT defined on $\mathcal{D}$.

- Every unitary TQFT defined on $\mathcal{D}$ arises as above from a subfactor planar algebra.

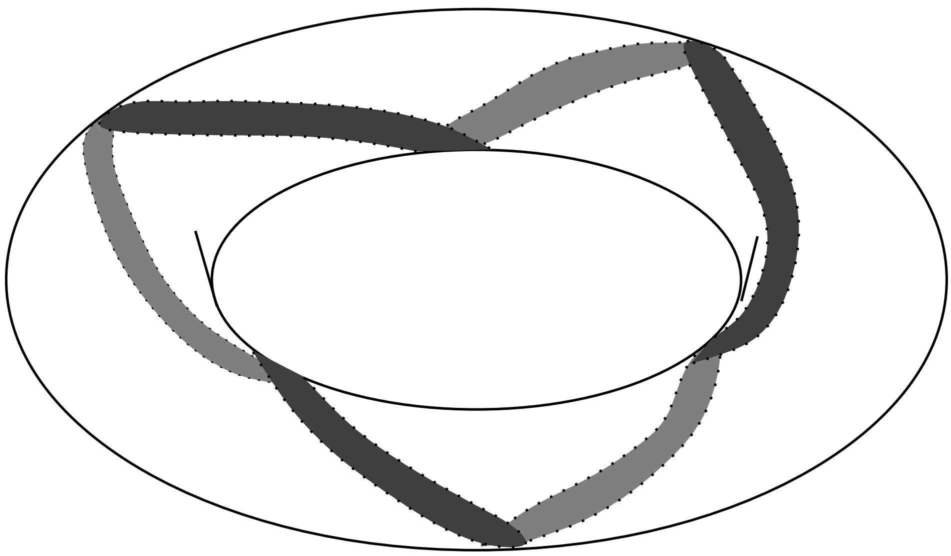
Our result leads to an alternative description of Kuperberg's 'quantum invariant' of 3-manifolds associated to semisimple, cosemisimple Hopf algebras.

Recall that a **Heegaard diagram** is a tuple consisting of an oriented smooth surface Σ, say, of genus g, and two systems of smoothly embedded circles on Σ, which we will denote by $U^1, ..., U^g$ and $L_1, ..., L_g$ (referred to by Kuperberg as upper and lower circles), such that each is a non-intersecting system of curves that does not disconnect Σ. An example, with $g = 1$ is given by



A well-known prescription leads from Heegaard diagrams to closed oriented 3-manifolds. The previous example leads to the lens space $L(3,1)$.

A Heegaard diagram also leads to a (pre-morphism and hence a) 'morphism' in our category. To see this, thicken the U-curves to a black band, and consider the surface Σ with this 'decoration', and with planar decomposition given by the L-curves; thus the Lens space example yields:



As in the above example, the morphism has both domain and range given by the 'empty object' $\emptyset$ and thus, each planar algebra leads from a Heegaard diagram via our prescription to a scalar.

The fact is that if we start with the planar algebra associated to a (semisimple and cosemisimple) Hopf algebra, the result of the prescription above coincides with Kuperberg's invariant, and is hence seen to depend only on the 3-manifold.

For example, we find that the invariant thus associated to $L(3,1)$ is given by $Tr(Z_{R_3})$. (More generally, the invariant associated to $L(p,q)$ is the trace of the the q-th power of the rotation R_3 on P_3.)