0.5 setg
5 setgray1

Braids

V.S. Sunder

IMSc

Chennai

Braids

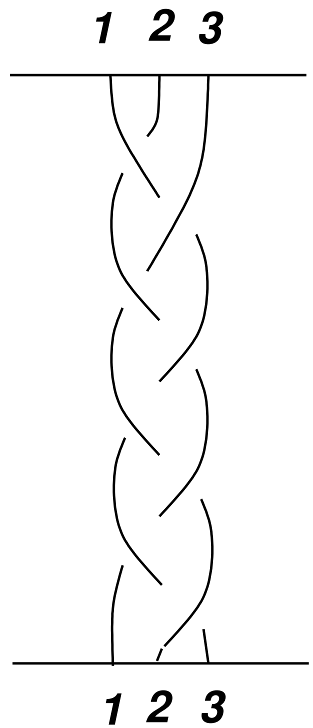
- What are braids?

Braids

- What are braids?
- Ask your mother!

Braids

- What are braids?
- Ask your mother!
-



3–strand braid

$$(= (b_1^{-1} b_2)^k)$$

n-strand braids

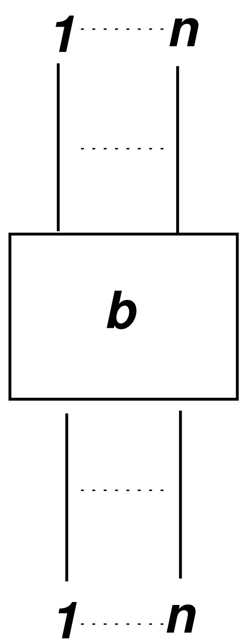
- The previous example was of a 3-strand braid.

n-strand braids

- The previous example was of a 3-strand braid.
- Your grandmother was probably familiar with 5-strand braids!

n-strand braids

- The previous example was of a 3-strand braid.
- Your grandmother was probably familiar with 5-strand braids!
-

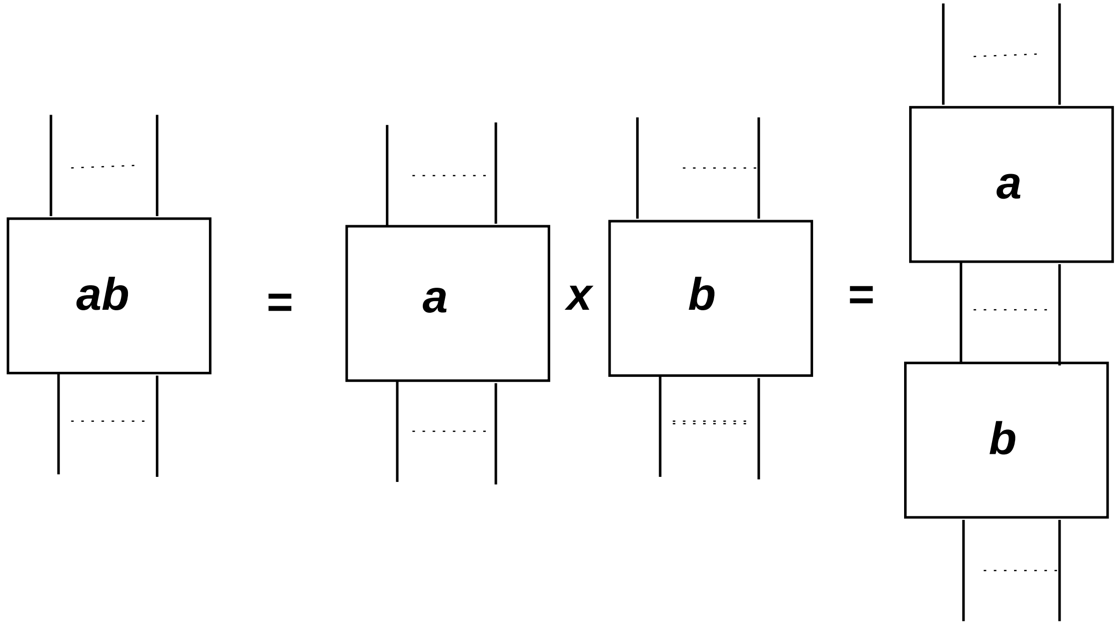


general

n–strand braid

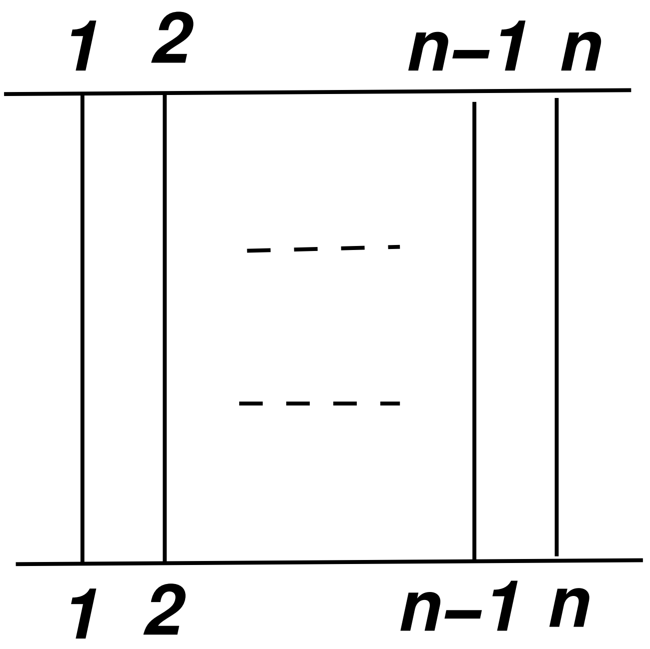
Multiplying braids

We equip the collection B_n of all n-strand braids with a product structure thus:



The Braid Group

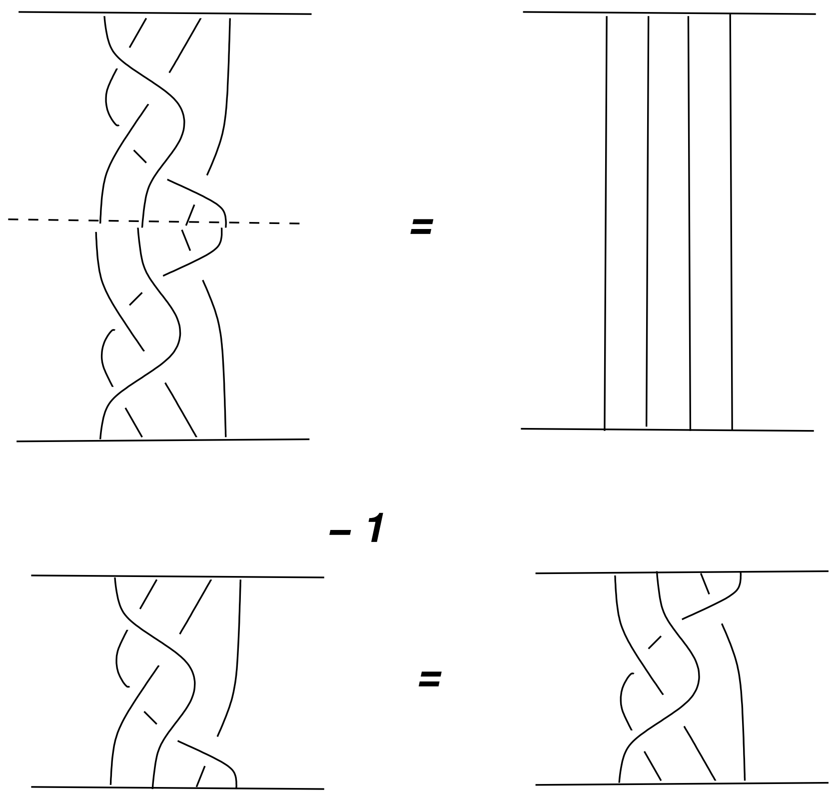
B_n turns out to be a group with this multiplication - provided we agree that two braids are the same if one may be continuously deformed into the other. (This is needed even for associativity.)



= *identity braid*

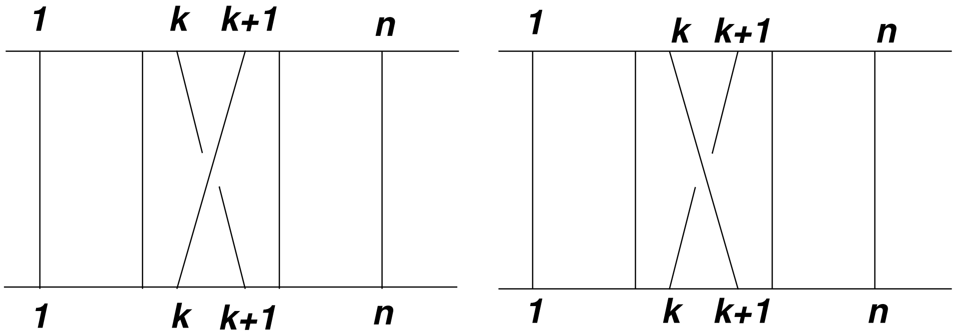
Braid inversion

- The inverse of a braid is obtained by reflecting in a horizontal mirror placed at the level of the lower frame of the braid: for example,



The generators

Since braids can be built up ‘one crossing at a time’ it is clear that B_n is generated , as a group, by the braids $b_1, b_2, \cdots, b_{n-1}$ shown below - together with their inverses:

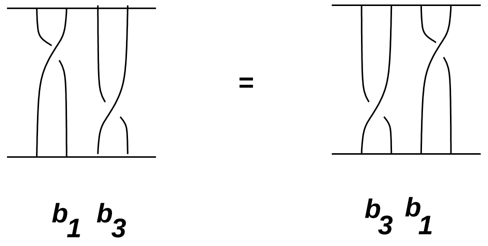


b_k $\quad$ b_k^{-1}

The braid relations

The b_j's satisfy the following relations:

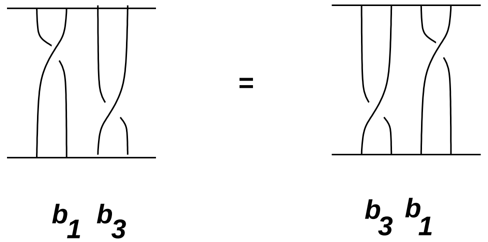
- $b_i b_j = b_j b_i$ if $|i - j| \geq 2$



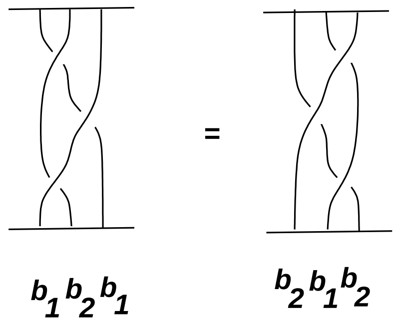
The braid relations

The b_j's satisfy the following relations:

- $b_i b_j = b_j b_i$ if $|i - j| \geq 2$



- $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for all $i < n - 1$



Free groups

- $G = \langle g_1, \cdots, g_n \rangle$ is said to be **the free group with generators** $\{g_1, \cdots, g_n\}$ if for any set $\{h_1, \cdots, h_n\}$ of elements in any group H, there exists a **unique** homomorphism $\phi : G \to H$ with the property that $\phi(g_k) = h_k$ for each $k = 1, \cdots, n$. Such a group is unique up to isomorphism.

Free groups

- $G = \langle g_1, \cdots, g_n \rangle$ is said to be **the free group with generators** $\{g_1, \cdots, g_n\}$ if for any set $\{h_1, \cdots, h_n\}$ of elements in any group H, there exists a **unique** homomorphism $\phi : G \to H$ with the property that $\phi(g_k) = h_k$ for each $k = 1, \cdots, n$. Such a group is unique up to isomorphism.
- For example, $\mathbb{Z} = \langle 1 \rangle$ is **the** free group on one generator.

Presentations of groups

A group G is said to have **presentation** $G = \langle g_1, \cdots, g_n | r_1, \cdots, r_m \rangle$ if:

- (i) it is generated by the set $\{g_1, \cdots, g_n\}$

Presentations of groups

A group G is said to have **presentation** $G = \langle g_1, \cdots, g_n | r_1, \cdots, r_m \rangle$ if:

- (i) it is generated by the set $\{g_1, \cdots, g_n\}$
- (ii) the g_i's satisfy each **relation** r_j for $j = 1, \cdots, m$; and

Presentations of groups

A group G is said to have **presentation** $G = \langle g_1, \cdots, g_n | r_1, \cdots, r_m \rangle$ if:

- (i) it is generated by the set $\{g_1, \cdots, g_n\}$
- (ii) the g_i's satisfy each **relation** r_j for $j = 1, \cdots, m$; and
- (iii) for any set $\{h_1, \cdots, h_n\}$ of elements in any group H, which 'satisfy each of the relations $r_1, \cdots, r_m$', there exists a **unique** homomorphism $\phi : G \to H$ with the property that $\phi(g_k) = h_k$ for each $k = 1, \cdots, n$.

Examples of presentations

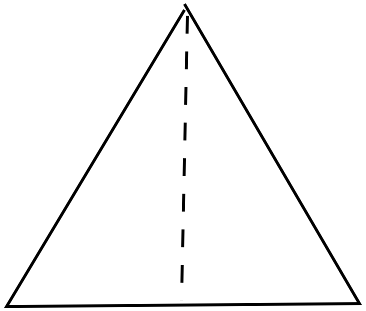
A group G with a given presentation is unique up to isomorphism.

- (i) $C_n = \langle g | g^n = 1 \rangle$ is **the** cyclic group of order n.

Examples of presentations

A group G with a given presentation is unique up to isomorphism.

- (i) $C_n = \langle g | g^n = 1 \rangle$ is **the** cyclic group of order n.
- (ii) $D_n = \langle g, t | g^n = 1, tgt^{-1} = t^{-1} \rangle$ is the **dihedral group** of symmetries of an n-gon. (D_n has $2n$ elements.)



g = rotation by 120°

t = reflection about an altitude

Artin’s theorem

The Braid group is often referred to as *Artin’s Braid Group*, partly because of the following theorem he proved:

- **Theorem:** (Artin) B_n has the presentation

$$B_n = \langle b_1, \cdots, b_{n-1} | r_1, r_2 \rangle \ ,$$

where

- $(r_1) b_i b_j = b_j b_i$ if $|i - j| \geq 2$
- $(r_2) b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for all $i < n - 1$

The symmetric group

In the symmetric group Σ_n, consider the transpositions defined by

$$t_i = (i, i+1)\ ,\ \ for\ i = 1, \cdots, n-1\ .$$

We have the following facts:

- Σ_n has the presentation

$$\Sigma_n = \langle t_1, \cdots, t_{n-1} | r_1, r_2, r_3 \rangle\ ,$$

where r_1, r_2 are the braid relations encountered earlier, and

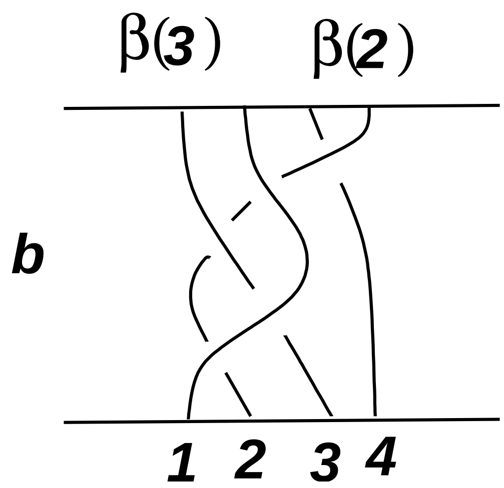
$$(r_3)\ t_i^2 = 1\ \ \text{for all } i < n$$

The quotient map $B_n \mapsto \Sigma_n$

- Hence there exists a unique homomorphism $\phi : B_n \to \Sigma_n$ such that $\phi(b_i) = t_i$ for each i. (Since the t_i's generate Σ_n, we see that ϕ is onto and hence Σ_n is a quotient of B_n.)

The quotient map $B_n \mapsto \Sigma_n$

- Hence there exists a unique homomorphism $\phi : B_n \to \Sigma_n$ such that $\phi(b_i) = t_i$ for each i. (Since the t_i's generate Σ_n, we see that ϕ is onto and hence Σ_n is a quotient of B_n.)
- If $\phi(b) = \beta$, it is not hard to see that



Remarks

- The generators b_i are all pairwise conjugate in B_n; in fact, if $b = b_1 b_2 \cdots b_n$, then $b b_i b^{-1} = b_{i+1} \; \forall i < n-1$. (For example:

$$b_1 b_2 b_3 \cdot b_1 = b_1 b_2 b_1 b_3 = b_2 \cdot b_1 b_2 b_3$$

and

$$b_1 b_2 b_3 \cdot b_2 = b_1 b_2 b_3 b_2 = b_1 b_3 b_2 b_3 = b_3 \cdot b_1 b_2 b_3$$

Remarks

- The generators b_i are all pairwise conjugate in B_n; in fact, if $b = b_1 b_2 \cdots b_n$, then $b b_i b^{-1} = b_{i+1}\ \forall i < n-1$. (For example:

$$b_1 b_2 b_3 \cdot b_1 = b_1 b_2 b_1 b_3 = b_2 \cdot b_1 b_2 b_3$$

and

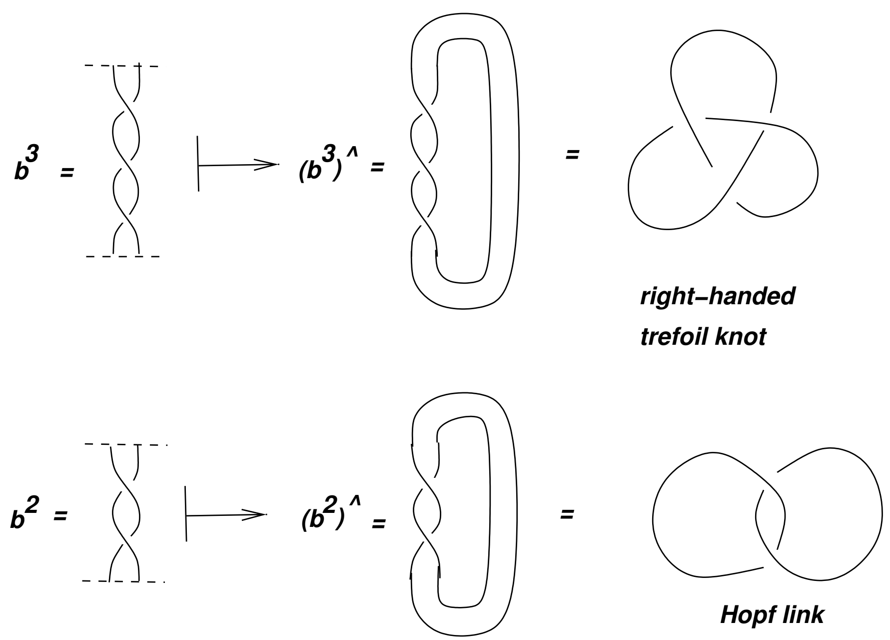
$$b_1 b_2 b_3 \cdot b_2 = b_1 b_2 b_3 b_2 = b_1 b_3 b_2 b_3 = b_3 \cdot b_1 b_2 b_3$$

- There exist 1-1 homomorphisms $B_n \hookrightarrow B_{n+1}$ given by $b_k^{(n)} \mapsto b_k^{(n+1)}$ for each $k < n$.

Braids to knots

- The **closure** of a braid $b \in B_n$ is obtained by sticking together the strings connected to the j-th pegs at the top and bottom. The result is a many component knot $\hat{b}$.

Braids to knots

- The **closure** of a braid $b \in B_n$ is obtained by sticking together the strings connected to the j-th pegs at the top and bottom. The result is a many component knot $\hat{b}$.
- 

Two theorems

What makes this ‘closure operation’ useful are:

- Theorem (**Alexander**):
  Every *tame* link is the closure of some braid (on some number of strands).

  and

Two theorems

What makes this 'closure operation' useful are:

- Theorem (**Alexander**):
  Every *tame* link is the closure of some braid (on some number of strands).

  and

- Theorem(**Markov)**:
  Two braids have equivalent closures iff you can pass from one to the other by a finite sequence of moves of one of two types.

The Markov move of type I

- **Type I Markov move:**

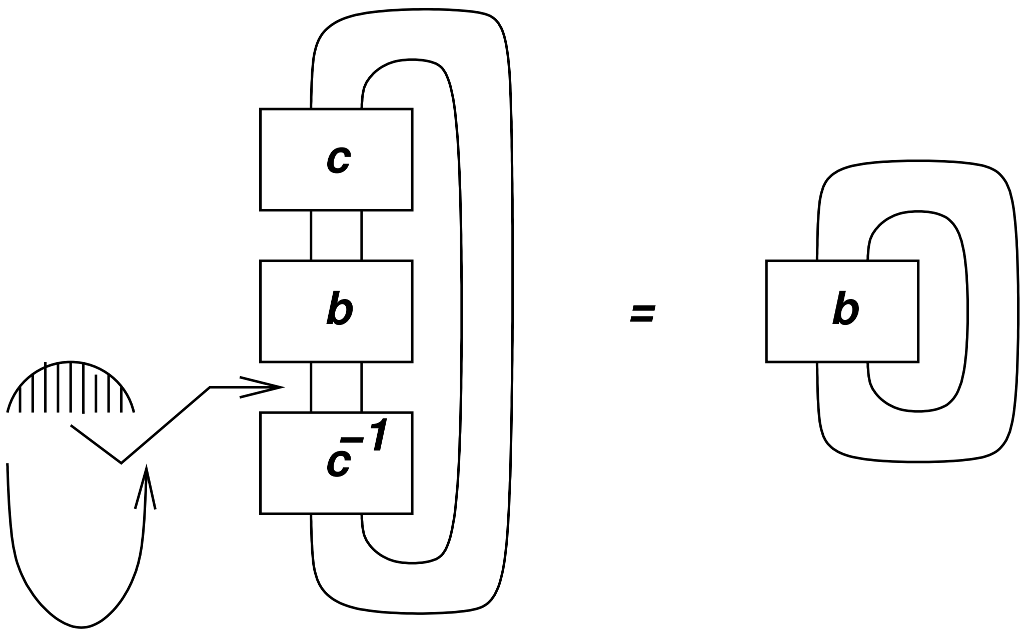
$$c^{(n)} b^{(n)} \left(c^{(n)}\right)^{-1} \sim b^{(n)}$$

The Markov move of type I

- **Type I Markov move:**

$$c^{(n)}b^{(n)}(c^{(n)})^{-1} \sim b^{(n)}$$

-



The Markov move of type II

- **Type II Markov move:**

$$b^{(n)} \sim b^{(n+1)}(b_n^{(n+1)})^{-1}$$

The Markov move of type II

- **Type II Markov move:**

$$b^{(n)} \sim b^{(n+1)}(b_n^{(n+1)})^{-1}$$

-