Hopf $C^*$-algebras and their quantum doubles - from the point of view of subfactors

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Much of this talk is based on the doctoral thesis of my student Jijo and inspired by the intuition of my colleague Vijay Kodiyalam.

Our motivation stems from:

- *(Ocneanu-Szymanski)* Finite-dimensional Kac algebras (≡Hopf C*-algebras) are in bijective correspondence with subfactors of depth two.

- *(Ocneanu)* The subfactor analogue of the quantum double construction is the *asymptotic inclusion*.

- *(Jones)* ‘Good’ subfactors are equivalent to *planar algebras*.
“Every finite-dimensional Kac algebra (=Hopf C*-algebra) $H$ admits a canonical ‘outer action’ on the $II_1$ factor $R$, and the associated ‘fixed subalgebra $R^H \subset R$ is the ‘protoypical subfactor of depth 2.’”

In the first part of this talk, I shall try to explain the terms of the above paragraph, and give a model for this action.

Ocneanu: “The subfactor analogue of the quantum double is the asymptotic inclusion”

In the second part of the talk, I shall describe the asymptotic inclusion of the ‘Kac-algebra subfactor’.

Finally, I shall describe the planar algebraic descriptions of these two subfactors.
Let $H = H(\mu, 1, \Delta, \epsilon, S, \ast)$ be a Kac algebra and $A$ be a unital * algebra (both finite dimensional).

**Definition (action)**: An action of $H$ on a *-algebra $A$ is a linear map $\alpha : H \to \text{End}_C(A)$ satisfying:

(i) $\alpha_1 = \text{Id}_A$

(ii) $\alpha_a(1_A) = \epsilon(a)1_A, \forall a \in H$

(iii) $\alpha_{ab} = \alpha_a \circ \alpha_b$

(iv) $\alpha_a(xy) = \sum \alpha_{a_1}(x)\alpha_{a_2}(y)$

(v) $\alpha_a(x)^* = \alpha_{s_{a^*}}(x^*)$

(We use (slightly modified) Sweedler-notations: $\Delta(a) = a_1 \otimes a_2$.)

**Example** The dual $H^*$ of a Kac algebra $H$ is also a Kac algebra, and $H^*$ acts on $H$ by the rule $\alpha_f(a) = f(a_2)a_1$
The crossed product $A \rtimes H$ is the unital associative $*$-algebra, with underlying vector space $A \otimes H$, and multiplication and involution defined by

$$(x \rtimes a)(y \rtimes b) = x\alpha_a(y) \rtimes a_2b$$

$$(x \rtimes a)^* = \alpha_{a_1^*}(x^*) \rtimes a_2^*.$$ 

The iterated crossed products: With $H, A, \alpha$ as above, the action of $H^*$ on $H$ can be promoted to an action - call it $f \mapsto \beta_f$ - of $H^*$ on $A \rtimes H$ by ‘ignoring the $A$-component’ thus:

$$\beta_f(x \rtimes a) = x \rtimes \alpha_f(a),$$

and we can define

$$A \rtimes H \rtimes H^* = (A \rtimes H) \rtimes H^*.$$ 

For integers $k < l$, we iteratively define

$$A[k,l] = A[k,l-1] \rtimes H_l = H_k \rtimes H_{k+1} \rtimes \ldots \rtimes H_l$$

where

$$H_i = \begin{cases} 
H & \text{if } i \text{ is odd} \\
H^* & \text{if } i \text{ is even}
\end{cases}.$$
We may, and do, regard $A_{[k,l]}$ as a $*$-subalgebra of $A_{[k_1,l_1]}$ whenever $k_1 \leq k \leq l \leq l_1$.

Let us write

$$\phi^{(k)} = \begin{cases} 
\phi & \text{if } k \text{ is even} \\
h & \text{if } k \text{ is odd}
\end{cases}$$

where $h$ and $\phi$ respectively denote suitably normalised Haar integrals in $H$ and $H^*$. It is then true* that there is a unique consistent trace (≡ faithful normalised positive tracial functional) ‘tr’ defined on the grid $\{A_{[k,l]} : -\infty < k \leq l < \infty\}$ satisfying

$$tr(x^{(k)} \times \cdots \times x^{(l)}) = \prod_{j=k}^{l} \phi^{(j)}(x^{(j)}).$$

*The only way we know to prove this seemingly elementary fact relies on the use of diagrammatic computations in the sense of Jones’ planar algebras.
With the foregoing notation, write $A_{(-\infty,l]}$ for the weak closure of $\bigcup_{j=0}^{\infty}A_{[l-j,l]}$ in the GNS representation afforded by ‘tr’. Specifically, let $N = A_{(-\infty,-1]}$ and $M = A_{(-\infty,0]}$. We summarise some facts about these objects below.

**Theorem:**

(a) $N$ and $M$ are both isomorphic to the hyperfinite $II_1$ factor $R$.

(b) There is a natural action - call it $\alpha$ - of $H$ on $M$ (by piecing together the consistently defined actions on the $A_{[-n,0]}$).

(c) $N' \cap M = \mathbb{C}$.

(d) $M^H := \{x \in M : \alpha_a(x) = \epsilon(a)x \ \forall a \in H\} = N$, so the action $\alpha$ is outer.

(e) The tower $\{A_{(-\infty,n]} : n \geq 1\}$ is isomorphic to the tower $\{M_n : n \geq 1\}$ of Jones’ basic construction.
The asymptotic inclusion:

For a general finite-index subfactor $N \subset M$ with associated ‘Jones tower’

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

of $II_1$ factors, there is a consistent trace ‘tr’ on the tower $\{M_n\}$ (because a $II_1$ factor admits a unique trace). It follows that if we define $M_\infty$ to be the weak closure of $\bigcup_{n=1}^{\infty} M_n$ in the GNS representation afforded by ‘tr’, then $M = M_\infty$ is again a $II_1$ factor. In fact, it turns out that $\mathcal{N} = (M \cup (M' \cap M_\infty))''$ is also a $II_1$ factor and in fact a finite-index subfactor of $M$.

The subfactor $\mathcal{N} \subset M$ is the asymptotic inclusion of $N \subset M$. 
We now consider our model

\[ N = A_{(-\infty,-1]} \subset A_{(-\infty,0]} = M \]

and want to describe the Jones towers for the subfactors \( N \subset M \) and \( \mathcal{N} \subset \mathcal{M} \), which we denote by

\[ N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots \]

and

\[ \mathcal{N} = \mathcal{M}_{-1} \subset \mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \]

**Lemma (rel.comm):** If \( k + 2 \leq n \), then

\[ A'_{(-\infty,k]} \cap A_{(-\infty,n)} = A_{[k+2,n]} \]

**Corollary:**

\[ N' \cap M_n = A_{[1,n]} \]

\[ \mathcal{M} = A_{(-\infty,\infty)}, \mathcal{N} = (A_{(-\infty,0]} \cup A_{[2,\infty)})'' \]
**Planar algebras:**

A planar algebra is a collection \( \{P_n : n \geq 0\} \) of \( \mathbb{C} \)-vector spaces which admits an action by the *coloured operad of planar tangles*. Here is an example of a planar tangle:

![Planar Tangle](image)

**Figure 1: Tangle T**

A planar tangle \( T \) has the following features:

(a) its boundary consists of an external box (labelled \( B_0 \)), and some number \( b \) (which is 3 in this example, and can, in general, even be 0) of internal boxes (labelled \( B_1, \ldots B_b \)).
(b) each box $B_i$ has an even number $2k_i$ of marked points, and is said to be of colour $k_i$. In this example,

$$k_0 = 3, k_1 = 4, k_2 = 0, k_3 = 3.$$  

(c) There are a number of non-crossing ‘strings’ which are either closed curves or have their two ends on a marked point of one of the boxes, in such a way that every marked point is the end-point of some string.

(d) The entire configuration comes equipped with a checkerboard shading.

(e) One special marked point on each box of non-zero colour is labelled with a ‘*’ in such a way that as one travels outward (resp., inward) from the *-point of an internal (resp., the external) box, the black region is to the right.
The one thing one can do with tangles is **composition**, when that makes sense: thus, if $S$ and $T$ are tangles, such that the external box of $S$ has the same colour as the $i$-th internal box of $T$, then we may form a new tangle $T \circ_i S$ by ‘glueing $S$ into the $i$-th internal box of $T$ in such a way that the $*$-points and the strings at the common boundary are aligned.

A tangle $T$ with boxes coloured $k_0, \cdots, k_b$ is required to induce a linear map

$$(Z_P^T) = Z_T : \bigotimes_{i=1}^b P_{k_i} \to P_{k_0}$$

and these maps are to satisfy some natural compatibility requirements, the most important being compatibility with composition of tangles.
Compatibility with composition:

If tangles $S$ and $T$ have colour attributes as below,

\[
\begin{align*}
\text{Z}(S) & : P_a \otimes P_b \rightarrow P_c, \\
\text{Z}(T) & : P_d \otimes P_c \otimes P_e \rightarrow P_f, \\
\text{Z}(T \circ_2 S) & : P_d \otimes P_a \otimes P_b \otimes P_e \rightarrow P_f.
\end{align*}
\]

and it is required that

\[
\text{Z}(T \circ_2 S) = \text{Z}(T) \circ (id_{P_d} \otimes \text{Z}(S) \otimes id_{P_e})
\]
The planar algebra of a Kac algebra $H$:

Define $\mathcal{P}_k(H)$ to be the vector space with basis consisting of ‘$H$-labelled $k$-tangles’: so a basis vector is a $k$-tangle such that:

- every internal box has colour two and is labelled by an element of $H$
- there are no loops in the tangle

The collection $\mathcal{P}(H) = \{\mathcal{P}_k(H)\}$ admits a natural action by planar tangles. The planar algebra $P(H)$ is the quotient of this ‘free planar algebra’ $\mathcal{P}(H)$ by the following set of relations - where $n = \dim(H)$, $h$ denotes the Haar integral, and we have used standard Hopf algebra notation:
The relations in $P(H)$:

\[(00) \quad \zeta a + b = \zeta a + \zeta b = n^{1/2}(00) ; \]

\[(id) \quad \frac{1}{H} = \quad \quad \quad (h) \quad h = \bar{n}^{1/2} \]

\[(1) \quad a = \varepsilon(a) \quad (2) \quad a = n^{1/2} \phi(a) \]

\[(3) \quad a + b = \Sigma a_{2} b \quad (4) \quad Sa = a \]
It must be mentioned that if $N \subset M$ is a ‘good’ subfactor, then the space $P_k$ of the associated planar algebra is nothing but $N' \cap M_{k-1}$, where

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

is the associated Jones’ basic construction tower.

It follows easily from our model for the subfactor $N = A_{(-\infty,-1]} = M^H \subset A_{(-\infty,0]} = M$, by using Lemma (rel.comm), that $P_k = A_{[1,k-1]}$ for $k \geq 2$. (Also, $P_1 = \mathbb{C}$.) For instance, the isomorphism $\phi_4 : A_{[1,3]} \to P_4$ is the map which sends $a \times f \times b$ to the labelled tangle given below, where $F : H^* \to H$ is the ‘Fourier transform.'
One of the crowning results of Jijo’s thesis is:

**Theorem:** $\mathcal{P}(H)$ may be identified with the planar subalgebra of $P(H^{*op})$, with $\mathcal{P}_n(H)$ consisting of those elements $g \in \mathcal{P}_n(H^{*op})$ which satisfy

$$g \cdot (f_1 \otimes \cdots \otimes f_n)$$

for all $f \in P_2(H^{*op}) = H^{*op}$. (Recall our ‘Sweedler-like notation’, whereby $\Delta_n(f) = f_1 \otimes \cdots \otimes f_n$, with $\Delta_n$ denoting iterated comultiplication.)
In particular,

\[ P_{2k}(H) = P_{2k}(H^{*\text{op}}) \cap \Delta_k(H^{*\text{op}})' . \]

**Corollary:** If \( H^* \) is commutative, then \( \mathcal{P}(H) = P(H^{*\text{op}}) \) and so the subfactor \( R^{H^*} \subset R \) is isomorphic to the asymptotic inclusion of \( R^H \subset R \). (Thus, \( R \subset R \rtimes G \) is isomorphic to the asymptotic inclusion of \( R^G \subset R \).)