# *Hopf $C^*$-algebras and their quantum doubles - from the point of view of subfactors*

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Workshop on QG and NCG,
Max Planck Institute, Aug. '07

Much of this talk is based on the doctoral thesis of my student Jijo and inspired by the intuition of my colleague Vijay Kodiyalam.

Our motivation stems from:

- *(Ocneanu-Szymanski)* Finite-dimensional Kac algebras (=Hopf $C^*$-algebras) are in bijective correspondence with subfactors of depth two.

- *(Ocneanu)* The subfactor analogue of the quantum double construction is the *asymptotic inclusion*.

- *(Jones)* 'Good' subfactors are equivalent to *planar algebras*.

"Every finite-dimensional *Kac algebra* (=*Hopf $C^*$-algebra*) $H$ admits a canonical 'outer action' on the *$II_1$ factor $R$*, and the associated 'fixed subalgebra $R^H \subset R$ is the 'protoypical subfactor of depth 2."

In the first part of this talk, I shall try to explain the terms of the above paragraph, and give a model for this action.

*Ocneanu*: "The subfactor analogue of the quantum double is the asymptotic inclusion"

In the second part of the talk, I shall describe the asymptotic inclusion of the 'Kac-algebra subfactor'.

Finally, I shall describe the planar algebraic descriptions of these two subfactors.

Let $H = H(\mu, 1, \Delta, \epsilon, S, *)$ be a Kac algebra and A be a unital * algebra (both finite dimensional).

**Definition (action) :** An action of $H$ on a *-algebra A is a linear map $\alpha : H \to End_C(A)$ satisfying:
(i) $\alpha_1 = Id_A$
(ii) $\alpha_a(1_A) = \epsilon(a)1_A, \forall a \in H$
(iii) $\alpha_{ab} = \alpha_a \circ \alpha_b$
(iv) $\alpha_a(xy) = \Sigma\alpha_{a_1}(x)\alpha_{a_2}(y)$
(v) $\alpha_a(x)^* = \alpha_{Sa^*}(x^*)$

(We use (slightly modified) Sweedler-notation: $\Delta(a) = a_1 \otimes a_2$.)

**Example** The dual $H^*$ of a Kac algebra $H$ is also a Kac algebra, and $H^*$ acts on $H$ by the rule $\alpha_f(a) = f(a_2)a_1$

**The crossed product $A \rtimes H$** is the unital associative *-algebra, with underlying vector space $A \otimes H$, and multiplication and involution defined by

$$\begin{aligned}(x \rtimes a)(y \rtimes b) &= x\alpha_{a_1}(y) \rtimes a_2 b \\ (x \rtimes a)^* &= \alpha_{a_1^*}(x^*) \rtimes a_2^*.\end{aligned}$$

**The iterated crossed products:** With $H, A, \alpha$ as above, the action of $H^*$ on $H$ can be promoted to an action - call it $f \mapsto \beta_f$ - of $H^*$ on $A \rtimes H$ by 'ignoring the $A$-component' thus:

$$\beta_f(x \rtimes a) \;=\; x \rtimes \alpha_f(a) \;,$$

and we can define

$$A \rtimes H \rtimes H^* = (A \rtimes H) \rtimes H^* \;.$$

For integers $k < l$, we iteratively define

$$A_{[k,l]} = A_{[k,l-1]} \rtimes H_l = H_k \rtimes H_{k+1} \rtimes \ldots\ldots \rtimes H_l$$

where

$$H_i = \begin{cases} H \text{ if } i \text{ is odd} \\ H^* \text{ if } i \text{ is even} \end{cases}$$

We may, and do, regard $A_{[k,l]}$ as a *-subalgebra of $A_{[k_1,l_1]}$ whenever $k_1 \leq k \leq l \leq l_1$.

Let us write

$$\phi^{(k)} = \begin{cases} \phi & \text{if } k \text{ is even} \\ h & \text{if } k \text{ is odd} \end{cases}$$

where $h$ and $\phi$ respectively denote suitably normalised Haar integrals in $H$ and $H^*$. It is then true* that there is a unique consistent trace (= faithful normalised positive tracial functional) 'tr' defined on the grid $\{A_{[k,l]} : -\infty < k \leq l < \infty\}$ satisfying

$$tr(x^{(k)} \rtimes \cdots \rtimes x^{(l)}) = \prod_{j=k}^{l} \phi^{(j)}(x^{(j)}) \ .$$

*The only way we know to prove this seemingly elementary fact relies on the use of diagrammatic computations in the sense of Jones' planar algebras.

With the foregoing notation, write $A_{(-\infty,l]}$ for the weak closure of $\cup_{j=0}^{\infty} A_{[l-j,l]}$ in the GNS representation afforded by 'tr'. Specifically, let $N = A_{(-\infty,-1]}$ and $M = A_{(-\infty,0]}$ We summarise some facts about these objects below.

**Theorem:**
(a) $N$ and $M$ are both isomorphic to the hyperfinite $II_1$ factor $R$.
(b) There is a natural action - call it $\alpha$ - of $H$ on $M$ (by piecing together the consistently defined actions on the $A_{[-n,0])}$).
(c) $N' \cap M = \mathbb{C}$.
(d) $M^H := \{x \in M : \alpha_a(x) = \epsilon(a)x \ \forall a \in H\} = N$, so the action $\alpha$ is *outer*.
(e) The tower $\{A_{(-\infty,n]} : n \geq 1\}$ is isomorphic to the tower $\{M_n : n \geq 1\}$ of **Jones' basic construction**.

## The asymptotic inclusion:

For a general finite-index subfactor $N \subset M$ with associated 'Jones tower'

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

of $II_1$ factors, there is a consistent trace 'tr' on the tower $\{M_n\}$ (because a $II_1$ factor admits a unique trace). It follows that if we define $M_\infty$ to be the weak closure of $\cup_{n=1}^{\infty} M_n$ in the GNS representation afforderd by 'tr', then $\mathcal{M} = M_\infty$ is again a $II_1$ factor. In fact, it turns out that $\mathcal{N} = (M \cup (M' \cap M_\infty))''$ is also a $II_1$ factor and in fact a finite-index subfactor of $\mathcal{M}$.

The subfactor $\mathcal{N} \subset \mathcal{M}$ is the **asymptotic inclusion** of $N \subset M$.

We now consider our model

$$N = A_{(-\infty,-1]} \subset A_{(-\infty,0]} = M$$

and want to describe the Jones towers for the subfactors $N \subset M$ and $\mathcal{N} \subset \mathcal{M}$, which we denote by

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

and

$$\mathcal{N} = \mathcal{M}_{-1} \subset \mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$$

**Lemma (rel.comm):** If $k+2 \leq n$, then

$$A'_{(-\infty,k]} \cap A_{(-\infty,n)} = A_{[k+2,n]}$$

**Corollary:**

$$N' \cap M_n = A_{[1,n]}$$

$$\mathcal{M} = A_{(-\infty,\infty)}, \mathcal{N} = (A_{(-\infty,0]} \cup A_{[2,\infty)})''$$

## Planar algebras:

A planar algebra is a collection $\{P_n : n \geq 0\}$ of $\mathbb{C}$-vector spaces which admits an action by the *coloured operad of planar tangles*. Here is an example of a planar tangle:

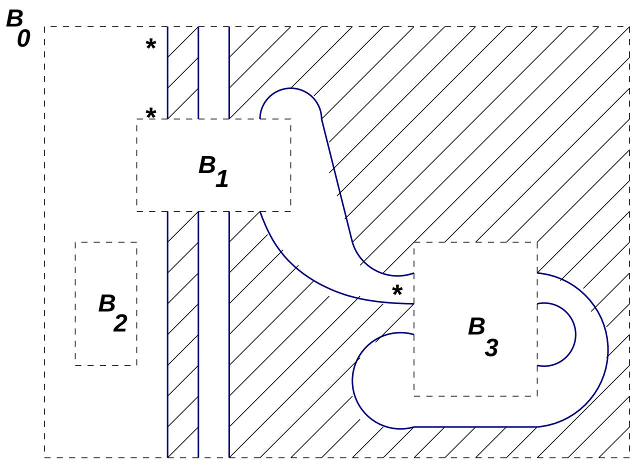



Figure 1: Tangle $T$

A planar tangle $T$ has the following features:

(a) its boundary consists of an external box (labelled $B_0$), and some number $b$ (which is 3 in this example, and can, in general, even be 0) of internal boxes (labelled $B_1, \cdots B_b$).

(b) each box $B_i$ has an even number $2k_i$ of marked points, and is said to be of *colour* $k_i$. In this example,

$$k_0 = 3, k_1 = 4, k_2 = 0, k_3 = 3.$$

(c) There are a number of non-crossing 'strings' which are either closed curves or have their two ends on a marked point of one of the boxes, in such a way that every marked point is the end-point of some string.

(d) The entire configuration comes equipped with a checkerboard shading.

(e) One special marked point on each box of non-zero colour is labelled with a '*' in such a way that as one travels outward (resp., inward) from the *-point of an internal (resp., the external) box, the black region is to the right.

The one thing one can do with tangles is *composition*, when that makes sense: thus, if $S$ and $T$ are tangles, such that the external box of $S$ has the same colour as the $i$-th internal box of $T$, then we may form a new tangle $T \circ_i S$ by 'glueing $S$ into the $i$-th internal box of $T$ in such a way that the *-points and the strings at the common boundary are aligned.

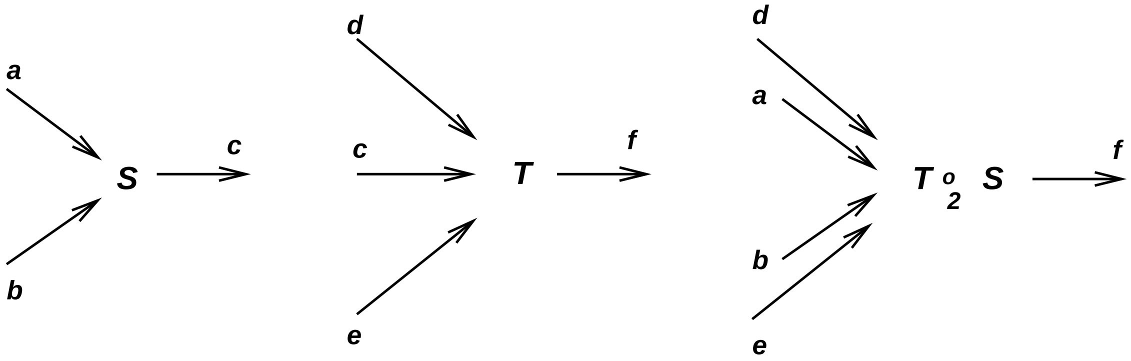
A tangle $T$ with boxes coloured $k_0, \cdots, k_b$ is required to induce a linear map

$$(Z_T^P =) Z_T : \otimes_{i=1}^b P_{k_i} \to P_{k_0}$$

and these maps are to satisfy some natural compatibility requirements, the most important being compatibility with composition of tangles.

*Compatibility with composition:*

If tangles $S$ and $T$ have colour attributes as below,




then

$$\begin{aligned} Z(S) : P_a \otimes P_b &\rightarrow P_c, \\ Z(T) : P_d \otimes P_c \otimes P_e &\rightarrow P_f, \\ Z(T \circ_2 S) : P_d \otimes P_a \otimes P_b \otimes P_e &\rightarrow P_f \ . \end{aligned}$$

and it is required that

$$Z(T \circ_2 S) = Z(T) \circ (id_{P_d} \otimes Z(S) \otimes id_{P_e})$$

## The planar algebra of a Kac algebra $H$:

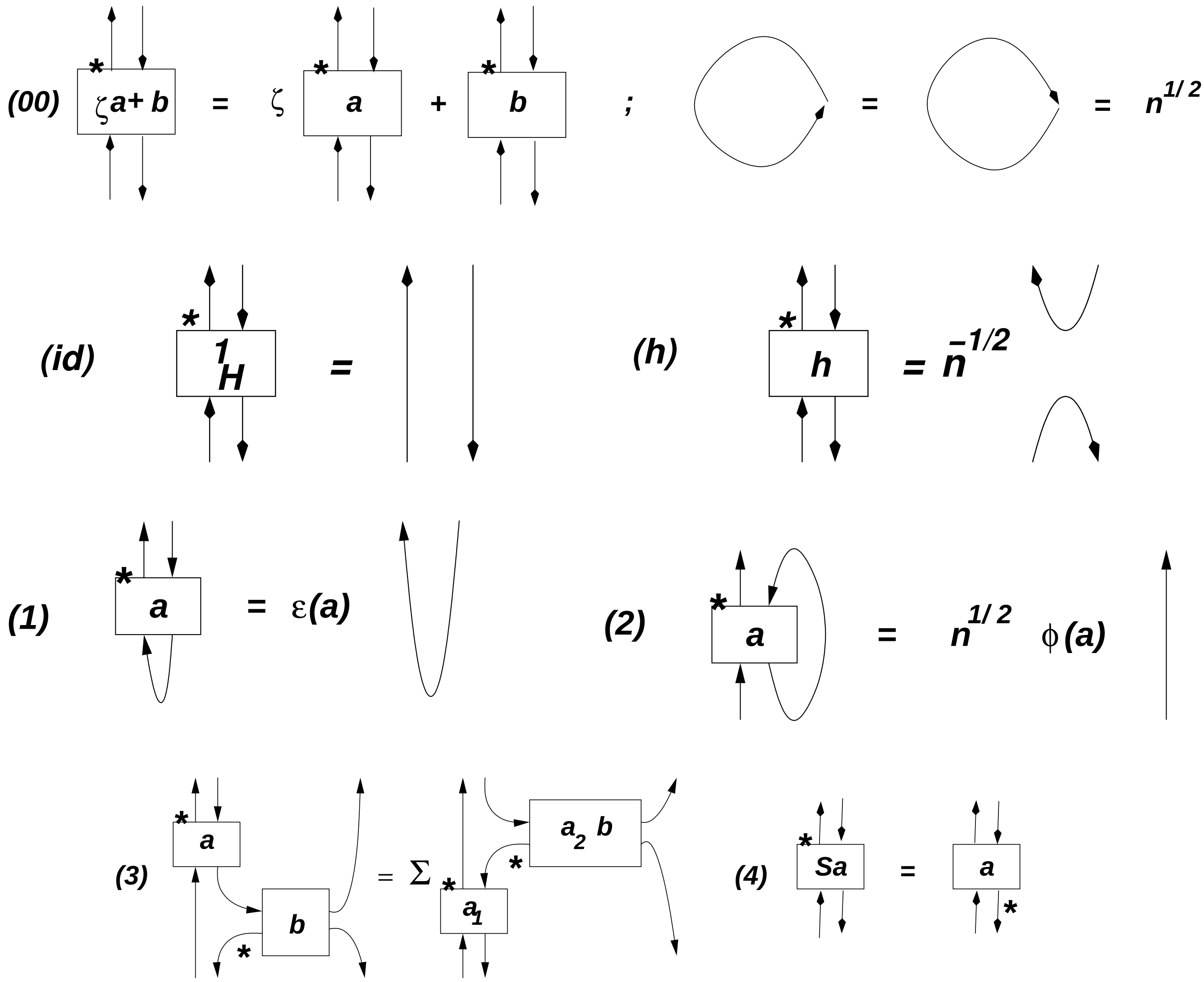
Define $\mathcal{P}_k(H)$ to be the vector space with basis consisting of '$H$-labelled k-tangles': so a basis vector is a $k$-tangle such that:

- every internal box has colour two and is labelled by an element of $H$

- there are no loops in the tangle

The collection $\mathcal{P}(H) = \{\mathcal{P}_k(H)\}$ admits a natural action by planar tangles. The planar algebra $P(H)$ is the quotient of this 'free planar algebra' $\mathcal{P}(H)$ by the following set of relations - where $n = dim(H)$, $h$ denotes the Haar integral, and we have used standard Hopf algebra notation:

## The relations in $P(H)$:

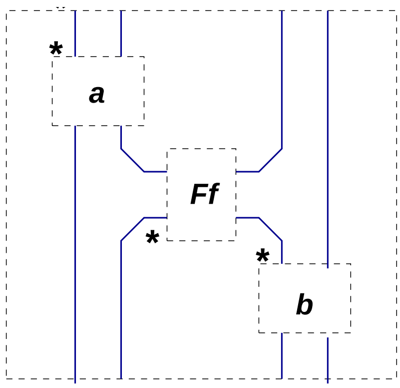


It must be mentioned that if $N \subset M$ is a 'good' subfactor, then the space $P_k$ of the associated planar algebra is nothing but $N' \cap M_{k-1}$, where

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$$

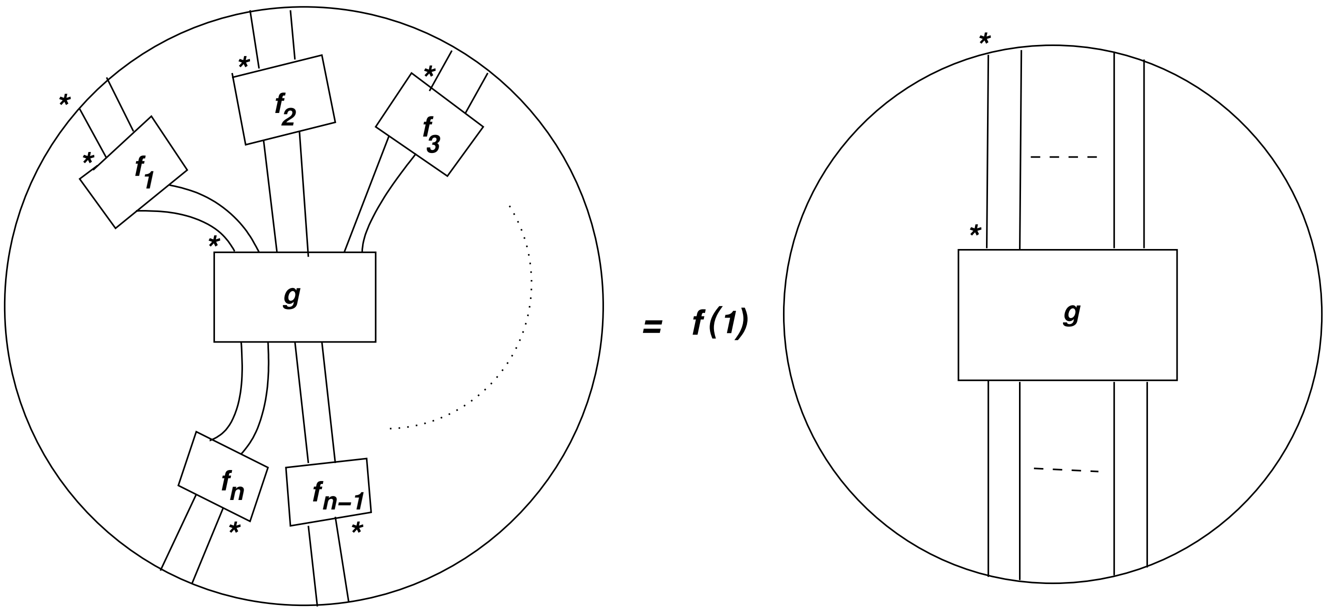
is the asociated Jones' basic construction tower.

It follows easily from our model for the subfactor $N = A_{(-\infty,-1]} = M^H \subset A_{(-\infty,-0]} = M$, by using Lemma (rel.comm), that $P_k = A_{[1,k-1]}$ for $k \geq 2$. (Also, $P_1 = \mathbb{C}$.) For instance, the isomorphism $\phi_4 : A_{[1,3]} \to P_4$ is the map which sends $a \rtimes f \rtimes b$ to the labelled tangle given below, where $F : H^* \to H$ is the 'Fourier transform.:



One of the crowning results of Jijo's thesis is:

**Theorem:** $\mathcal{P}(H)$ may be identified with the planar subalgebra of $P(H^{*op})$, with $\mathcal{P}_n(H)$ consisting of those elements $g \in \mathcal{P}_n(H^{*op})$ which satisfy




for all $f \in P_2(H^{*op}) = H^{*op}$. (Recall our 'Sweedler-like notation', whereby $\Delta_n(f) = f_1 \otimes \cdots \otimes f_n$, with $\Delta_n$ denoting iterated comultiplication.)

In particular,

$$\mathcal{P}_{2k}(H) = P_{2k}(H^{*op}) \cap \Delta_k(H^{*op})' \ .$$

**Corollary:** If $H^*$ is commutative, then $\mathcal{P}(H) = P(H^{*op})$ and so the subfactor $R^{H^*} \subset R$ is isomorphic to the asymptotic inclusion of $R^H \subset R$. (Thus, $R \subset R \rtimes G$ is isomorphic to the asymptotic inclusion of $R^G \subset R$.)