Non-crossing partition = 2-cabled Temperley-Lieb planar algebra

V.S. Sunder
Institute of Mathematical Sciences
Chennai, India
sunder@imsc.res.in

March 24-28, 2008
Topics in von Neumann algebras
Banff

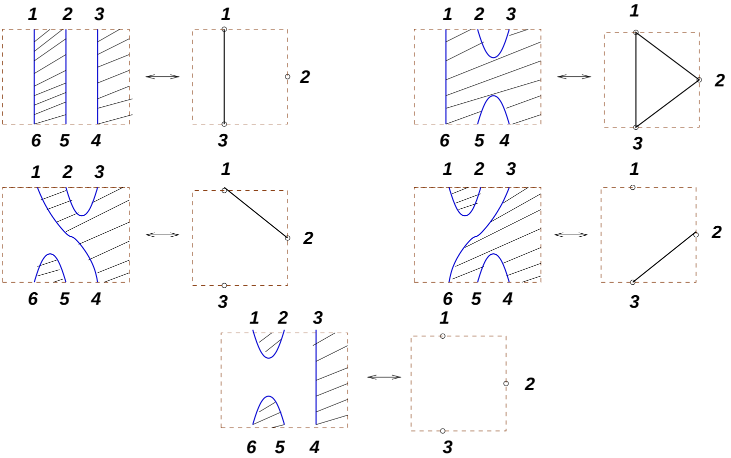
$TL(\delta)$

Recall that $TL_n(\delta)$ has $\mathbb{C}$-basis $\mathcal{K}_n$ (the set of **Kauffman diagrams** on $2n$ points). Our starting point is the observation that $\mathcal{K}_n$ is in bijection with the set $\mathcal{N}_n$ of non-crossing partitions on a set of n points.

$TL(\delta)$

Recall that $TL_n(\delta)$ has $\mathbb{C}$-basis $\mathcal{K}_n$ (the set of **Kauffman diagrams** on $2n$ points). Our starting point is the observation that $\mathcal{K}_n$ is in bijection with the set $\mathcal{N}_n$ of non-crossing partitions on a set of n points.

We illustrate with $n = 3$:



Thus, the i-th point of $\tilde{S} \in \mathcal{N}_n$ is defined as the point that is mid-way between the $(2i-1)$-th and $2i$-th points of $S \in \mathcal{K}_n$.

The correspondence $\mathcal{K}_n \ni S \leftrightarrow \tilde{S} \in \mathcal{N}_n$

Thus, the i-th point of $\tilde{S} \in \mathcal{N}_n$ is defined as the point that is mid-way between the $(2i-1)$-th and $2i$-th points of $S \in \mathcal{K}_n$.

And the $i-th$ and $j-th$ points of $\tilde{S}$ are declared to be in the same equivalence class of $\tilde{S}$ if, viewed in S, the two points lie in the same connected black region of S.

Thus, the i-th point of $\tilde{S} \in \mathcal{N}_n$ is defined as the point that is mid-way between the $(2i-1)$-th and $2i$-th points of $S \in \mathcal{K}_n$.

And the $i-th$ and $j-th$ points of $\tilde{S}$ are declared to be in the same equivalence class of $\tilde{S}$ if, viewed in S, the two points lie in the same connected black region of S.

This correspondence ($\mathcal{K}_n \ni S \leftrightarrow \tilde{S} \in \mathcal{N}_n$) will be at the heart of the isomorphism we shall establish

(We first recall definitions, when $\delta \geq 2$.)

Notice that an element of $\mathcal{K}_n$ may be viewed as an *n-tangle without any internal boxes* - in other words, tailor-made for inserting into an internal n-box of a planar tangle. Recall that $TL(\delta)$ is the planar algebra with

- the set of $TL_n(\delta)$ of n-boxes being the $\mathbb{C}$-vector space with basis $\mathcal{K}_n$, and
- the result of feeding inputs of Kauffman diagrams of appropriate sizes into each internal box of a tangle being defined as $\delta^c \times$ (resulting tangle after removing all loops obtained in this process), where c is the number of removed loops.

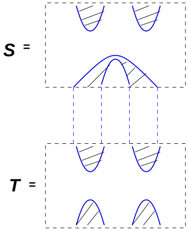
The planar algebra $TL(\delta)$

(We first recall definitions, when $\delta \geq 2$.)

Notice that an element of $\mathcal{K}_n$ may be viewed as an *n-tangle without any internal boxes* - in other words, tailor-made for inserting into an internal n-box of a planar tangle. Recall that $TL(\delta)$ is the planar algebra with

- the set of $TL_n(\delta)$ of n-boxes being the $\mathbb{C}$-vector space with basis $\mathcal{K}_n$, and
- the result of feeding inputs of Kauffman diagrams of appropriate sizes into each internal box of a tangle being defined as $\delta^c \times$ (resulting tangle after removing all loops obtained in this process), where c is the number of removed loops.

Thus, for example



$$\boldsymbol{S}\,\boldsymbol{T} = \delta\,\boldsymbol{T}$$

$$\boldsymbol{T}^{2} = \delta^{2}\,\boldsymbol{T}$$

The planar algebra $NC(\delta)$

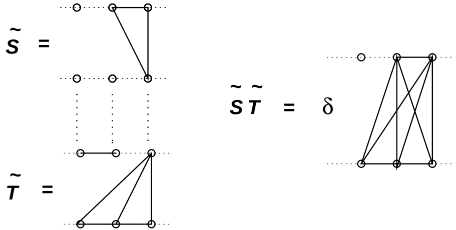
Elements of $\mathcal{N}_{2n}$ will be viewed at as having n points arrayed on the top and bottom of a square. Then $NC(\delta)$ is the planar algebra almost exacly as TL, with

- the set of $NC_n(\delta)$ of n-boxes being the $\mathbb{C}$-vector space with basis $\mathcal{N}_{2n}$, and
- the result of feeding inputs of non-crossing partitions of appropriate even sizes into each internal box of a tangle being defined as $\delta^c \times$ (resulting tangle after removing all 'internal components' obtained in this process), where c is the number of removed internal components.

Elements of $\mathcal{N}_{2n}$ will be viewed at as having n points arrayed on the top and bottom of a square. Then $NC(\delta)$ is the planar algebra almost exacly as TL, with

- the set of $NC_n(\delta)$ of n-boxes being the $\mathbb{C}$-vector space with basis $\mathcal{N}_{2n}$, and
- the result of feeding inputs of non-crossing partitions of appropriate even sizes into each internal box of a tangle being defined as $\delta^c \times$ (resulting tangle after removing all 'internal components' obtained in this process), where c is the number of removed internal components.

Thus, for example



Recall that the *2-cabling* of a planar algebra P is the planar algebra $P^{(2)}$ with the space of n-boxes given by $P_n^{(2)} = P_{2n}$ and with the action on $P^{(2)}$ of a tangle T given by

$$Z_T^{P^{(2)}}(x_1 \otimes \cdots \otimes x_b) = Z_{T^{(2)}}^P(x_1 \otimes \cdots \otimes x_b)$$

where $T^{(2)}$ is the tangle obatined by 2-cabling T.

Recall that the *2-cabling* of a planar algebra P is the planar algebra $P^{(2)}$ with the space of n-boxes given by $P_n^{(2)} = P_{2n}$ and with the action on $P^{(2)}$ of a tangle T given by

$$Z_T^{P^{(2)}}(x_1 \otimes \cdots \otimes x_b) = Z_{T^{(2)}}^{P}(x_1 \otimes \cdots \otimes x_b)$$

where $T^{(2)}$ is the tangle obatined by 2-cabling T.

Theorem The mapping

$$TL_{2n}(\delta \ni \frac{S}{\tau(S)} \mapsto \frac{\tilde{S}}{\widetilde{\tau}(\tilde{S})} \in NC_{2n}(\delta^2)$$

induces a planar algebra isomorphism $P = NC(\delta^2) \cong TL^{(2)}(\delta) = \tilde{P}$. (Here, τ and $\widetilde{\tau}$ denote the non-normalised *picture traces* on the two planar algebras which assign δ^{2n} to the identity element in the space of n-boxes.) $\square$

Reformulation of what needs to be proved

Suppose T is a k_0-tangle with b internal boxes of colours $k_1, \cdots, k_b$ and that $S_i \in \mathcal{K}_{2b_i}$ for $1 \leq i \leq b$. Then by definition of the tangle operations in P, there exist $S \in \mathcal{K}_{2k_0}$ and some integer $l \geq 0$ so that

$$Z_T^P(S_1 \otimes \cdots \otimes S_b) = \delta^l S.$$

Suppose T is a k_0-tangle with b internal boxes of colours $k_1, \cdots, k_b$ and that $S_i \in \mathcal{K}_{2b_i}$ for $1 \leq i \leq b$. Then by definition of the tangle operations in P, there exist $S \in \mathcal{K}_{2k_0}$ and some integer $l \geq 0$ so that

$$Z_T^P(S_1 \otimes \cdots \otimes S_b) = \delta^l S.$$

It is not hard to see then that there must exist some integer $c \geq 0$ so that

$$Z_T^{\widetilde{P}}(\widetilde{S}_1 \otimes \cdots \otimes \widetilde{S}_b) = \delta^{2c} \widetilde{S}.$$

Reformulation of what needs to be proved

Suppose T is a k_0-tangle with b internal boxes of colours $k_1, \cdots, k_b$ and that $S_i \in \mathcal{K}_{2b_i}$ for $1 \leq i \leq b$. Then by definition of the tangle operations in P, there exist $S \in \mathcal{K}_{2k_0}$ and some integer $l \geq 0$ so that

$$Z_T^P(S_1 \otimes \cdots \otimes S_b) = \delta^l S.$$

It is not hard to see then that there must exist some integer $c \geq 0$ so that

$$Z_T^{\widetilde{P}}(\widetilde{S}_1 \otimes \cdots \otimes \widetilde{S}_b) = \delta^{2c} \widetilde{S}.$$

Minor computation shows that what needs to be verified is that

$$\frac{\tau(Z_T^P(S))}{\prod_{i=1}^b \tau(S_i)} = \frac{\tilde{\tau}(Z_T^{\widetilde{P}}(\tilde{S}))}{\prod_{i=1}^b \tilde{\tau}(\widetilde{S}_i)} \tag{0.1}$$

In fact, it is true that every planar tangle may be expressed as an appropriate composition of tangles from the set

$$\mathcal{T}_0 = \{1_{0\pm}, R_k^k, E_k^{k+1}, I_k^{k+1}, M_k^k : k \geq 0\} ,$$

where

- $1_{0\pm}$ are as in the next slide
- R_k^k is *the roatation tangle of colour k*
- E_k^{k+1} is the *conditional expectation tangle*
- I_k^{k+1} is the *inclusion tangle*
- $M_{k,k}^k$ is the *multiplication tangle*

(Special cases are illustrated in the next slide.)

In fact, it is true that every planar tangle may be expressed as an appropriate composition of tangles from the set

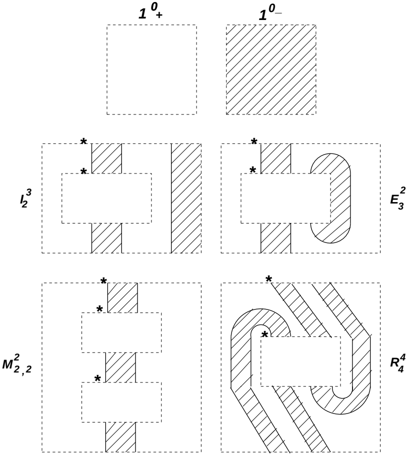
$$\mathcal{T}_0 = \{1_{0\pm}, R_k^k, E_k^{k+1}, I_k^{k+1}, M_k^k : k \geq 0\} ,$$

where

- $1_{0\pm}$ are as in the next slide
- R_k^k is *the roatation tangle of colour k*
- E_k^{k+1} is the *conditional expectation tangle*
- I_k^{k+1} is the *inclusion tangle*
- $M_{k,k}^k$ is the *multiplication tangle*

(Special cases are illustrated in the next slide.)

Hence it is not necessary to verify equation (0.1) for *every* tangle T; it is enough to verify this condition for tangles from $\mathcal{T}_0$.
This verification is fairly straightforward for all but the conditional expectation and rotation tangles is straightforward.



1^0_+
1^0_-
*
I^3_2
E^2_3
$M^2_{2,2}$
R^4_4

The requisite verification for the conditional expectation and rotation tangles depends on the following

Proposition: Consider a configuration consisting of:

- a system of C disjoint closed curves in the plane
- a checkerboard shading of the resulting regions
- a line intersecting each of the curves (with the number P of points of intersection being $2m$).

Then,

$$C - 2B = m - B_+ - B_- \ , \tag{0.2}$$

where B_+ (resp., B_-) denotes the number of black regions above (resp., below) the line, and $B = B_+ + B_-$. □

The proof of the proposition depends essentially on Euler characteristic considerations.