von Neumann algebras

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The six lectures in this course will be devoted to covering the topics listed below:

1. **Two density theorems**

   After defining the strong and weak operator topologies on the space $\mathcal{B}(\mathcal{H})$, we shall discuss the density theorems of von Neumann and Kaplanski, as well as the important corollary of the former, *viz.,* the *Double Commutant Theorem* of von Neumann.

2. **Some operator theory and a first exposure to von Neumann algebras**

   After introducing concrete and abstract von Neumann algebras, we give a crash course on the two pillars of operator theory in Hilbert space: the spectral theorem and the polar decomposition theorem. And we finish with some basic von Neumann algebra theory.

3. **Normality**

   We discuss some of the equivalent descriptions of what it takes to belong to the pre-dual of a von Neumann algebra, as well as demonstrate the equivalence of the concrete and abstract descriptions of a von Neumann algebra.

4. **The standard form of von Neumann algebras**

   We discuss what may be considered the analogue in von Neumann algebra theory of the existence and uniquenes of Haar measure in the theory of locally compact groups.

5. **The Tomita-Takesaki Theorem**

   We discuss: states and weights, the GNS construction for weights, the Tomita-Takesaki theorem, modular automorphism groups, Connes' unitary cocycle theorem, ...

6. **Factors** We conclude with a brief discussion of factors, the Murray-von Neumann classification of them, the crossed-product construction, and examples of factors.

1 Two density theorems

All the action in this course of lectures, will take place in the ambiance of the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a **separable** Hilbert space. This space is rich in structures, e.g.:

- It is a Banach space with respect to the norm defined by

$$\begin{aligned} \|x\| &= \sup\{\{x\xi\|\xi \in \mathcal{H}, \|\xi\| = 1\} \\ &= \sup\{|\langle x\xi, \eta\rangle| : \xi, \eta \in \mathcal{H}, \|\xi\| = \|\eta\| = 1\} \end{aligned}$$

- It is an involutive algebra with product given by composition of operators and involution given by the adjoint operation (which satisfies and is determined by $\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle \ \forall \xi, \eta \in \mathcal{H}$).

- It is a C^*-algebra with respect to the preceding two structures: i.e., this involutive Banach algebra satisfies (for all $x, y \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$):

  1. $(\alpha x + y)^* = \bar{\alpha} x^* + y^*$
  2. $(xy)^* = y^* x^*$
  3. $(x^*)^* = x$, and
  4. $\|x^* x\| = \|x\|^2$.

What is crucial for us, however, are three other topologies, the so-called **strong operator topology** (SOT), **weak operator topology** (WOT), and the **σ-weak topology**, each of which is a locally convex topology, induced by a family of seminorms as follows. Consider the seminorms defined on $\mathcal{B}(\mathcal{H})$ by:

$$\begin{aligned} p_\xi(x) &= \|x\xi\| \ , \xi \in \mathcal{H} \\ p_{\xi,\eta}(x) &= \langle x\xi, \eta\rangle \ , \xi, \eta \in \mathcal{H} \\ p_{\{\xi_n,\eta_n\}}(x) &= \sum_n |\langle x\xi_n, \eta_n\rangle|^2, \xi_n, \eta_n \in \mathcal{H}, \sum_n (\|\xi_n\|^2 + \|\eta_n\|^2) < \infty \end{aligned}$$

These topologies typically **do not** satisfy the first axiom of countability so it is not sufficient to work with sequences in order to deal with them. We should deal with *nets*. Thus, the above topologies have the following descriptions:

a net $\{x_\alpha : \alpha \in \Lambda\}$ converges to x in the:

- SOT if and only if $\|(x_\alpha - x)\xi\| \to 0 \ \forall \xi \in \mathcal{H}$;

- WOT if and only if $\langle (x_\alpha - x)\xi, \eta\rangle \to 0 \ \forall \xi, \eta \in \mathcal{H}$

- σ-weak topology if and only if $\sum_n \langle (x_\alpha - x)\xi_n, \eta_n\rangle \to 0$ whenever $\xi_n, \eta_n \in \mathcal{H}$ satisfy $\sum_n (\|\xi_n\|^2 + \|\eta_n\|^2) < \infty$.

An equivalent way of describing these topologies is by describing a system of basic open neighbourhoods of a point. Thus, for instance, a set $\Omega \subset \mathcal{B}(\mathcal{H})$ is an SOT-open neighbourhood of x if and only if there exist finitely many vectors $\xi_1, \cdots, \xi_n$ and an $\epsilon > 0$ such that $\Omega \supset \{y \in \mathcal{B}(\mathcal{H}) : \|(y - x)\xi_i\| < \epsilon \ \forall 1 \leq i \leq n\}$.

EXERCISE 1.1. *Suppose $\{x_i : i \in I\}$ is a net in $\mathcal{B}(\mathcal{H}, \mathcal{K})$ which is uniformly bounded - i.e., $\sup\{||x_i|| : i \in I\} = K < \infty$. Let $\mathcal{S}_1$ (resp., $\mathcal{S}_2$) be a* **total** *set in $\mathcal{H}$ (resp., $\mathcal{K}$) - i.e., the set of finite linear combinations of elements in $\mathcal{S}_1$ (resp., $\mathcal{S}_2$) is dense in $\mathcal{H}$ (resp., $\mathcal{K}$).*

(a) $\{x_i\}$ converges strongly to 0 if and only if $lim_i||x_i\xi|| = 0$ for all $\xi \in \mathcal{S}_1$; and

(b) $\{x_i\}$ converges weakly to 0 if and only if $lim_i\langle x_i\xi, \eta\rangle = 0$ for all $\xi \in \mathcal{S}_1, \eta \in \mathcal{S}_2$.

EXAMPLE 1.2. (1) Let $u \in \mathcal{B}(\ell^2)$ be the unilateral (or right-) shift. Then the sequence $\{(u^*)^n : n = 1, 2, \cdots\}$ converges strongly, and hence also weakly, to 0. (Reason: the standard basis $\{e_m : m \in \mathbb{N}\}$ is total in ℓ^2, the sequence $\{(u^*)^n : n = 1, 2, \cdots\}$ is uniformly bounded, and $(u^*)^n e_m = 0 \ \forall n > m$. On the other hand, $\{u^n = (u^{*n})^* : n = 1, 2, \cdots\}$ is a sequence of isometries, and hence certainly does not converge strongly to 0. Thus, the adjoint operation is not 'strongly continuous'.

(2) On the other hand, it is obvious from the definition that if $\{x_i : i \in I\}$ is a net which converges weakly to x in $\mathcal{B}(\mathcal{H}, \mathcal{K})$, then the net $\{x_i^* : i \in I\}$ converges weakly to x^* in $\mathcal{B}(\mathcal{K}, \mathcal{H})$. In particular, conclude from (1) above that both the sequences $\{(u^*)^n : n = 1, 2, \cdots\}$ and $\{u^n : n = 1, 2, \cdots\}$ converge weakly to 0, but the sequence $\{(u^*)^n u^n : n = 1, 2, \cdots\}$ (which is the constant sequence given by the identity operator) does not converge to 0; thus, multiplication is not even 'sequentially jointly weakly continuous'.

(3) Multiplication is 'separately strongly (resp., weakly) continuous' - i.e., if $\{x_i\}$ is a net which converges strongly (resp., weakly) to x in $\mathcal{B}(H, \mathcal{K})$, and if $a \in \mathcal{B}(\mathcal{K}, \mathcal{K}_1), b \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$, then the net $\{as_ib\}$ converges strongly (resp., weakly) to asb in $\mathcal{B}(\mathcal{H}_1, \mathcal{K}_1)$. (For instance, the 'weak' assertion follows from the equation $\langle as_ib\xi, \eta\rangle = \langle s_i(b\xi), a^*\eta\rangle$.)

(4) Multiplication is 'jointly strongly continuous' if we restrict ourselves to uniformly bounded nets - i.e., if $\{x_i : i \in I\}$ (resp., $\{y_j :$

$j \in J\}$) is a uniformly bounded net in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ (resp., $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$) which converges strongly to x (resp., y), and if $K = I \times J$ is the directed set obtained by coordinate-wise ordering (thus $(i_1, j_1) \leq (i_2, j_2) \Leftrightarrow i_1 \leq i_2$ and $j_1 \leq j_2$), then the net $\{y_j \circ x_i : (i, j) \in K\}$ converges strongly to $y \circ x$. (Reason: assume, by translating if necessary, that $x = 0$ and $y = 0$; (the assertion in (3) above is needed to justify this reduction); if $\xi \in \mathcal{H}_1$ and if $\epsilon > 0$, first pick $i_0 \in I$ such that $||x_i\xi|| < \frac{\epsilon}{M}$, $\forall i \geq i_0$, where $M = 1 + sup_j||y_j||$; then pick an arbitrary $j_0 \in J$, and note that if $(i, j) \geq (i_0, j_0)$ in K, then $||y_j x_i \xi|| \leq M||S_i x|| < \epsilon$.)

(5) The purpose of the following example is to show that the asserted joint strong continuity of multiplication is false if we drop the restriction of uniform boundedness. Let $\mathcal{N} = \{t \in \mathcal{B}(\mathcal{H}) : t^2 = 0\}$, where $\mathcal{H}$ is infinite-dimensional.

We assert that $\mathcal{N}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. To see this, note that sets of the form $\{x \in \mathcal{B}(\mathcal{H}) : ||(x - x_0)\xi_i|| < \epsilon, \ \forall 1 \leq i \leq n\}$, where $x_0 \in \mathcal{B}(\mathcal{H})$, $\{\xi_1, \cdots, \xi_n\}$ is a linearly independent set in $\mathcal{H}$, $n \in \mathbb{N}$ and $\epsilon > 0$, constitute a base for the strong operator topology on $\mathcal{B}(\mathcal{H})$. Hence, in order to prove the assertion, we need to check that every set of the above form contains elements of $\mathcal{N}$. For this, pick vectors $\eta_1, \cdots, \eta_n$ such that $\{\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n\}$ is a linearly independent set in $\mathcal{H}$, and such that $||x_0\xi_i - \eta_i|| < \epsilon \ \forall i$; now consider the operator t defined thus: $t\xi_i = \eta_i, t\eta_i = 0 \ \forall i$ and $t\zeta = 0$ whenever $\zeta \perp \{\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n\}$; it is seen that $t \in \mathcal{N}$ and that t belongs to the open set exhibited above.

Since $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, the assertion (of the last paragraph) shows three things: (a) multiplication is not jointly strongly continuous; (b) if we choose a net in $\mathcal{N}$ which converges strongly to an operator which is not in $\mathcal{N}$, then such a net is not uniformly bounded (because of (4) above); thus, strongly convergent nets need not be uniformly bounded; and (c) since a strongly convergent sequence is necessarily uniformly bounded (by the uniform boundedness principle), the strong operator topology cannot be described with 'just sequences'.

(6) This point of this item is that when dealing with separable Hilbert spaces, 'just sequences will do' if we restrict ourselves to norm-bounded subsets of $\mathcal{B}(\mathcal{H})$. Indeed, it follows from Exercise 1.1 that if $\S \subset \mathcal{B}(\mathcal{H})$ and if $sup\{\|x\| : x \in S\} < \infty$, and if $\{\xi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$, then a net $\{x_\alpha : \alpha \in \Lambda\}$ in S converges in the SOT (resp. WOT) to an x if and only if $x_\alpha \xi_n \to x\xi_n \ \forall n \in \mathbb{N}$. Thus, the countable family $\{p_{\xi_n}(\cdot) = \|(\cdot)\xi_n\|, \ n \in \mathbb{N}\}$ (resp.,

$\{p_{\xi_m,\xi_n}(\cdot) = |\langle(\cdot)\xi_m, \xi_n\rangle|, \ m, n \in \mathbb{N}\}$) of seminorms determines the SOT (resp., WOT) on the set S, and consequently these topologies on S are metrisable. (For instance, SOT restricted to S is topologised by the metric given by $d(x, y) = \sum_{n\in\mathbb{N}} 2^{-n} p_{\xi_n}.$) $\square$

We introduce some notation which we shall employ throughout these notes.

DEFINITION 1.3. *If $S \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{S} \subset \mathcal{H}$, we shall write:*

1. *sp S and sp $\mathcal{S}$ to denote the linear subspaces spanned by S and $\mathcal{S}$.*
2. *$[S]$ and $[\mathcal{S}]$ for the SOT-closure and norm-closure of sp S and sp $\mathcal{S}$ respectively.*
3. *$S' = \{x' \in \mathcal{B}(\mathcal{H}) : xx' = x'x \ \forall x \in S\}$*

We list some useful facts whose proofs may be found in the first chapter of {SZ}. (The reader should be warned that what we call the σ-weak topology is called the 'w-topology' in this book.)

PROPOSITION 1.4.

1. *A linear functional on $\mathcal{B}(\mathcal{H})$ is σ-weakly continuous if and only if its restriction to the unit ball of $\mathcal{B}(\mathcal{H})$ is WOT-continuous.*
2. *The weak operator topology and the σ-weak topology coincide when restricted to norm bounded subsets of $\mathcal{B}(\mathcal{H})$.*
3. *A linear functional on $\mathcal{B}(\mathcal{H})$ is SOT-continuous if and only if it is WOT-continuous; and consequently, (by the Hahn-Banach separation theorem), the closed convex sets (in particular, the closed subspaces) in these two topologies are the same.*

Part 3 of the previous proposition is the reason why $[S]$ is also equal to the WOT-closure of *sp* S. We now come to a crucial notion in the theory of von Neumann algebras.

DEFINITION 1.5. *For $S \subset \mathcal{B}(\mathcal{H})$, define its* **commutant** *to be the set*

$$S' = \{x' \in \mathcal{B}(\mathcal{H}) : xx' = x'x \ \forall x \in S\}.$$

EXERCISE 1.6. *Let $S, T \subset \mathcal{B}(\mathcal{H})$; show that*

1. *$S \subset T \Rightarrow T' \subset S'$;*

2. *If we write $S^{(n+1)} = S^{(n)\prime}$ deduce from the first part of this problem that $S' = S^{(3)}$ $\forall S$ and hence that*

$$S \subset S^{(2)} = \cdots S^{(2n)} \; \forall n$$

and

$$S' = S^{(2n+1)} \; \forall n \; ;$$

3. *S' is a WOT-closed unital subalgebra of $\mathcal{B}(\mathcal{H})$ for any subset $S \subset \mathcal{B}(\mathcal{H})$.;*

4. *If S is *-closed (meaning $x \in S \Rightarrow x^* \in S$), so is S'.*

Now all the pieces are in place for our first density theorem and its celebrated consequence - barring (a) the remark that it is more customary to write S'' in place of what was called $S^{(2)}$ in the previous exercise, and () the following trivial lemma:

LEMMA 1.7. *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$, let p' denote the orthogonal projection onto $\mathcal{M}$, and let $a \in \mathcal{B}(\mathcal{H})$. Then,*

1. *$a\mathcal{M} \subset \mathcal{M} \Rightarrow p'ap' = ap'$;*

2. *$a\mathcal{M} \subset \mathcal{M}$ for all $a \in A$ where A is a self-adjoint (i.e., *-closed subset of $\mathcal{B}(\mathcal{H})$ if and only if $p' \in A'$*

Proof. (1) is a triviality.

As for (2), we see that if $a \in A$, then (since also $a^* \in A$)

$$p'a = (a^*p')^* = (p'a^*p')^* = p'ap' = ap'.$$

$\square$

THEOREM 1.8. *[**von Neumann density Theorem**] If A is a *-closed unital subalgebra of $\mathcal{B}(\mathcal{H})$, then A is SOT dense in A''.*

Proof. We need to show that if $x \in A''$, and if Ω is a basic SOT-open neighbourhood of x, then $A \cap \Omega \neq \emptyset$. We may assume that $\Omega = \{y \in \mathcal{B}(\mathcal{H}) : \|(x-y)\xi_i\| < \epsilon \; \forall 1 \leq i \leq n\}$.

We first consider the case when $n = 1$, i.e., we assume that $\Omega = \{y \in \mathcal{B}(\mathcal{H}) : \|(x-y)\xi\| < \epsilon\}$. Let $\mathcal{S} = A\xi = \{a\xi : a \in A\}$, let p' be the orthogonal projection onto the subspace $\mathcal{M} = [\mathcal{S}]$. Since $\mathcal{S}$ and consequently $\mathcal{M}$ is invariant under A, it follows from Lemma 1.7 that $p' \in A'$ and hence x commutes with p' and hence leaves $\mathcal{M}$ invariant (as does its adjoint). Since $\xi \in \mathcal{S}$, it follows that $x\xi \in \mathcal{M} = \underline{\mathcal{S}}$; in particular there exists $a \in A$ such that $\|(a-x)\xi\| < \epsilon$ as desired.

Now for the case of general $n > 1$. The (Hilbert) direct sum $\mathcal{H}^{(n)}$ of n copies of $\mathcal{H}$ will be identified with the space $M_{n\times 1}(\mathcal{H})$ of column vectors $\eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}$ of n elements from $\mathcal{H}$. It is known that there is a natural identification $M_n(\mathcal{B}(\mathcal{H})) \ni ((a_{ij}))\leftrightarrow a \in \mathcal{B}(\mathcal{H}^{(n)})$ whereby $a\eta = \zeta$ where $\zeta_i = \sum_{j=1}^n a_{ij}\xi_j$.

The proof is completed by reducing the case of general n to the case $n = 1$ by the useful trick contained in the following exercise. □

EXERCISE 1.9. *Consider the mapping $\mathcal{B}(\mathcal{H}) \ni a \stackrel{\pi}{\to} a^{(n)} \in M_n(\mathcal{B}(\mathcal{H}))$ given by $a_{ij}^{(n)} = \delta_{ij}a$.*

1. *Show that π is a *-algebra homomorphism, which is a homeomorphism if domain and range are equipped with the SOT.*
2. *Show that $\pi(A)' = M_n(A')$ and $\pi(A)'' = \pi(A'')$.*
3. *Complete the proof of the case of general n in the von Neumann density theorem by applying the case $n = 1$ to $\pi(A)$ and $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$.*

COROLLARY 1.10. *[**von Neumann's Double Commutant theorem**] The following conditions on a unital self-adjoint subalgebra M of $\mathcal{B}(\mathcal{H})$ are equivalent:*

1. *$M = M''$.*
2. *M is WOT-closed.*
3. *M is SOT-closed.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial, while (3) ⇒ (1) is an immediate consequence of von Neumann's density theorem. □

REMARK 1.11. *The remarkable fact about the Double Commutant Theorem (referred to as DCT in the sequel) is that asserts the equivalence of the purely algebraic condition (1) to the purely topological conditions (2) and (3).*

It is a fact that the three conditions are also equivalent to a fourth condition that M is σ-weakly closed. This is proved by an adaptation of the trick in Exercise 1.9 with n replaced by $\aleph_0$. The details are spelt out in the next exercise.

EXERCISE 1.12.

1. *Identify the (Hilbert) direct sum* $\mathcal{H}^{(\infty)}$ *of countably infinitely many copies of* $\mathcal{H}$ *with the space* $M_{\infty\times 1}(\mathcal{H})$ *of column vectors* $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix}$ *of countably infinitely many elements from* $\mathcal{H}$ *which satisfy*

$$\|\eta\|^2 = \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty \ .$$

2. *Show that there is a natural identification between* $\mathcal{B}(\mathcal{H}^{(\infty)})$ *and that subspace of matrices* $((a_{ij})) \in M_\infty(\mathcal{B}(\mathcal{H}))$ *for which the transform* $a\eta = \zeta$ *where* $\zeta_i = \sum_{j=1}^{\infty} a_{ij}\xi_j$ *(the series being convergent in* $\mathcal{H}$ *for each* i*) defines the correspondence* $((a_{ij}))\leftrightarrow a$.

3. *Consider the mapping* $\mathcal{B}(\mathcal{H}) \ni a \stackrel{\pi}{\to} a^{(\infty)} \in M_\infty(\mathcal{B}(\mathcal{H}))$ *given by* $a^{(n)}_{ij} = \delta_{ij}a$ *and show that:*
   (a) *Show that* π *is a *-algebra homomorphism, which is a homeomorphism if domain and range are equipped with the WOT and the* σ*-weak (subspace-) topology.*
   (b) *Show that* $\pi(A)' = M_n(A')$ *and* $\pi(A)'' = \pi(A'')$.
   (c) *Complete the proof of the case of general* n *in the von Neumann density theorem by applying the case* $n = 1$ *to* $\pi(A)$ *and* $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$.

EXERCISE 1.13. *Let* $\{x_i : i \in I\}$ *be a uniformly bounded monotone net of positive elements in a von Neumann algebra* $M \subset \mathcal{B}(\mathcal{H})$*, ie, it satisfies*

1. $i \leq j \Rightarrow 0 \leq x_i \leq x_j$*; and*
2. $\sup_i \|x_i\| < \infty$. .

Then show that there exists a positive element $x := \sup_{i\in I} x_i$ *in* M *satisfying (i)* $x := SOT - \lim_{i\in I} x_i = WOT - lim_{i\in I} x_i = \sigma - weak - lim_{i\in I} x_i$*, and (ii)* $y \in M$ *and* $x_i \leq y \ \forall i \in I$ *imply* $x \leq y$.

(Thus uniformly bounded monotone nets converge in a von Neumann algebra - in the natural topology of a von Neumann algebra, and the next exercise shows that the collection $\mathcal{P}(M)$ of projections in M is a complete lattice.)

EXERCISE 1.14. *In the notation of the last paragraph, show that:*

1. *$\mathcal{P}(M)$ is a lattice w.r.t. to the natural order (whereby $p \leq q \Rightarrow pq = p \Rightarrow qp = p \Rightarrow p(\mathcal{H}) \subset q(\mathcal{H})$)*

2. *$\mathcal{P}(M)$ is a complete lattice in the sense that every sub-collection Λ of $\mathcal{P}(M)$ has a supremum and an infimum. (*Hint: *Apply Exercise 1.13 to the net $\{e_I = \sup\{p : p \in I\} : I$ a finite subset of Λ of suprema of finite subsets of Λ.)*

We come now to the second basic density theorem.

THEOREM 1.15. *[**Kaplansky's density theorem**] If A is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, then the unit ball A_1 of A is SOT dense in the unit ball $(A'')_1$ of A''.*

COROLLARY 1.16. *If A is as in the above theorem, and if $x \in A''$, then there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset A$ which SOT-converges to x such that also $\|x_n\| \leq \|x\| \; \forall x$.*

Proof. This follows from Kaplansky's density theorem and Proposition 1.2 (6). □

We now use this theorem to give an alternative proof of the following fact (already proved in the Exercise preceding Kaplansky's density theorem):

COROLLARY 1.17. *A unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is weakly closed if and only of it is σ-weakly closed.*

Proof. Since σ-weak convergence clearly implies WOT convergence, it follows that WOT-closed sets are automatically σ-weakly closed. Conversely, if A is σ-weakly closed, then so is $A_1 = A \cap (\mathcal{B}(\mathcal{H}))_1$. Deduce from parts (2) and (3) of Proposition 1.4 that A_1 is WOT-closed and hence also SOT-closed. Deduce now from Kaplansky's density theorem that $A_1 = (A'')_1$; hence $A = A''$. An appeal to Exercise 1.6 (3) now shows that indeed A is weakly closed. □

2 Basic von Neumann algebra theory

DEFINITION 2.1. *1. A* concrete von Neumann algebra *is a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which satisfies the equivalent conditions of DCT.*

2. *An* abstract von Neumann algebra *is a C^*-algebra M which admits a 'predual' M_* in the sense that M is isometrically isomorphic to the Banach dual space $(M_*)^*$.*

We will eventually see that the two notions are equivalent. The following remarks are meant to indicate why concrete von Neumann algebras are abstract von Neumann algebras:

REMARK 2.2. 1. *What should be clear is that $L^\infty(X, \mathcal{B}, \mu)$ is a commutative (abstract) von Neumann algebra (with pre-dual $L^1(X, \mathcal{B}, \mu)$). It is a fact that any abstract commutative von Neumann algebra with a separable predual is isomorphic to an $L^\infty(X, \mathcal{B}, \mu)$.*

2. *$\mathcal{B}(\mathcal{H})$ is known to be an abstract von Neumann algebra, with its predual $\mathcal{B}_*(\mathcal{H})$ identifiable with the closure in $\mathcal{B}(\mathcal{H})^*$ of the WOT-continuous linear functions, which can be shown to be the same as the space of linear functionals in $\mathcal{B}(\mathcal{H})$ which are σ-weakly continuous. Thus the σ-weak topology on $\mathcal{B}(\mathcal{H})$ is identifiable with the weak*-topology that M inherits by virtue of being the dual space of $\mathcal{B}_*(\mathcal{H})$.*

3. *Let M be any abstract von Neumann algebra, with pre-dual M_*; Suppose N is a C^*-subalgebra of M which is closed in the weak* topology that M inherits by virtue of being the dual space of M_*. Then it is a fact that N is again an abstract von Neumann algebra with pre-dual identifiable with $M_*/N_\perp$, where $N_\perp = \{\phi \in M_* : x(\phi) = 0 \ \forall x \in N\}$.*

4. *It is a consequence of the two previous remarks that any σ-weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ is automatically an abstract von Neumann algebra.*

5. *We may now deduce that concrete von Neumann algebras are indeed abstract von Neumann algebras. Indeed, if N is a concrete von Neumann algebra, then N is a weakly closed unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, and hence also σ-weakly closed by Corollary 1.17. So by parts (2) and (3) of this rmark, it follows that N is an abstract von Neumann algebra.*

6. *It is a fact (proved by Sakai) that the predual of an abstract von Neumann algebra is unique up to isomorphism, and consequently M inherits a* canonical *weak*-topology from its predual, which we shall henceforth simply refer to as 'the σ-weak*

topology on M'. (This terminology is in fact completely unambiguous.)

Remark 2.2 (3) gives us a way to define the von Neumann subalgebra generated by any subset S of an abstract von Neumann algebra: simply take the 'weak*-closure of the C^*-subalgebra generated by S. (In fact, concrete von Neumann algebras are nothing but von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$.) We write $W^*(S)$ for the on Neumann algebra generated by S.

EXERCISE 2.3. *1. For any $S \subset \mathcal{B}(\mathcal{H})$, let S_1 denote the multiplicative sub-semigroup of $\mathcal{B}(\mathcal{H})$ generated by $S \cup S^* \cup \{1\}$, where $S^* = \{x^* : x \in S\}$. Show that $W^*(S) = [S_1]$ (in the notation of Definition 1.3(2).*

2. Prove that the following conditions on an $x \in \mathcal{B}(\mathcal{H})$ are equivalent:

(a) $W^*(\{x\})$ is commutative.

(b) $xx^* = x^*x$

Any such x is said to be **normal**.

We shall now give one formulation of the celebrated *spectral theorem for a normal operator.*

THEOREM 2.4. *[**The spectral Theorem**] Let x be a normal operator on a separable Hilbert space $\mathcal{H}$. Then,*

1. There exists a probability measure μ defined on the Borel subsets of $sp\ x$ $(= \{\lambda \in \mathbb{C} : (x-\lambda)$ does not have an inverse in $\mathcal{B}(\mathcal{H}\})$ and a von Neumann algebra isomorphism

$$L^\infty(sp\ x, \mu) \ni f \to f(x) \in W^*(\{x\}) .$$

*(This means an isomorphism of *-algebras, which is a homeomorphism when domain and range are equipped with their respective σ-weak topologies.) This isomorphism will be called the (bounded measurable) functional calculus for x and satisfies $id(x) = x$, where id denotes the identity function $sp\ x \ni z \mapsto z \in \mathbb{C}$.*

2. The functional calculus also satisfies

$$f \in C(sp\ x) \Leftrightarrow f(x) \in C^*(\{1, x\}) ,$$

where we write $C^*(S)$ for the C^*-subalgebra generated by a set S contained in a C^*-algebra.

3. *The measure μ is uniquely determined up to mutual absolutely continuous.*

4. *the mapping $\mathcal{B}_{sp\ x} \ni E \mapsto 1_E(x)$ is often called* **the spectral measure of** *x, which is a countably additive projection-valued measure on $\mathcal{B}_{sp\ x}$.*

A consequence of the spectral theorem is that one can deal with normal operators as comfortably as with functions. The next Proposition is an illustration of this phenomenon.

PROPOSITION 2.5. *Let $x \in \mathcal{B}(\mathcal{H})$. Then:*

1. *$x = x^* \Leftrightarrow x$ is normal andsp!$x \subset \mathbb{R}$.*

2. *x is unitary $\Leftrightarrow x$ is normal and sp $x \subset \mathbb{T}$.*

3. *The following conditions are equivalent:*

   (a) *x is normal and sp $x \subset [0, \infty)$.*

   (b) *There exists a self-adjoint operator y such that $y^2 = x$ Any such x is said to be* **positive** *and we simply write $x \geq 0$*

   (c) *If $x \geq 0$ there exists a unique $y \geq 0$ such that $x = y^2$ and in fact $y = x^{\frac{1}{2}}$ in the notation of the functional calculus.*

Proof. (1), (2), (3)(a) and (3)(b) are immediate consequences of the spectral theorem. As for uniqueness in 3(c), one needs to observe that if y is a positive square root of x, then $x^{\frac{1}{2}}$ and y are both positive elements of $W^*(\{y\})$ whose squares are positive and hence have to be equal. □

LEMMA 2.6. *There exists a decomposition $x = x_+ - x_-$, where $x_\pm$ are positive and $x_+ x_- = 0$.*

Proof. Set $f_\pm(t) = \max\{\pm t, 0\}$ and define $x_\pm = f_\pm(x)$. □

Here are two other facts which are very useful and important.

PROPOSITION 2.7. *Let A be a unital C^*-algebra.*

1. *Every element $z \in A$ has a unique decomposition $z = x + iy$ with self-adjoint $x, y \in A$.*

2. *Every element of A is a linear combination of four unitary elements of A*

Proof. 1. This is the usual Cartesian decomposition necessarily given by $x = \frac{1}{2}(z + z^*), y = \frac{1}{2i}(z - z^*)$.

2. In view of the Cartesian decomposition, it will suffice to prove that if x is a self-adjoint contraction (i.e., $\|x\| \leq 1$), then x may be written as an average of two unitary elements; and this is because for such an x, also $sp\ x^2 \subset [0, 1]$ with the result that $1 - x^2 \geq 0$. Now write $u_\pm = x \pm i(1 - x^2)^{\frac{1}{2}}$ and check that $x = \frac{1}{2}(u_+ + u_-)$ is a decomposition of the desired form.

□

EXERCISE 2.8. *1. If $u \in \mathcal{U}(\mathcal{H}, \mathcal{K})$ is an arbitrary unitary operator, verify that the mapping $x \mapsto uxu^*$ defines an isomorphism - which is usually denoted by ad u - of the von Neumann algebra $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$.*

2. Deduce from the uniqueness assertion in the spectral theorem that if x is a normal operator on $\mathcal{H}$ and if $u \in \mathcal{U}(\mathcal{H}, \mathcal{K})$ is unitary, then $1_E(uxu^*) = u1_E(x)u^*$.

3. Verify that the assignment $u \mapsto ad\ u$ defines a group isomorphism of $\mathcal{U}(\mathcal{H})$ into $Aut(\mathcal{B}(\mathcal{H}))$.

We now establish the very useful **polar decomposition** for bounded operators on Hilbert space. We begin with a few simple observations and then introduce the crucial notion of a **partial isometry**.

LEMMA 2.9. *Let $x \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then,*

$$ker\ x \ = \ ker\ (x^*x) \ = \ ker\ (x^*x)^{\frac{1}{2}} \ = \ ran^{\perp}x^* \ . \tag{2.1}$$

In particular, also

$$ker^{\perp}x \ = \ \overline{ran\ x^*} \ .$$

(In the equations above, we have used the notation $ran^{\perp}x^*$ and $ker^{\perp}x$, for $(ran\ x^*)^{\perp}$ and $(ker\ x)^{\perp}$, respectively.)

Proof : First observe that, for arbitrary $\xi \in \mathcal{H}$, we have

$$||x\xi||^2 \ = \ \langle x^*x\xi, \xi\rangle \ = \ \langle (x^*x)^{\frac{1}{2}}\xi, (x^*x)^{\frac{1}{2}}\xi\rangle \ = \ ||(x^*x)^{\frac{1}{2}}\xi||^2 \ , \tag{2.2}$$

whence it follows that $ker\ x \ = \ ker(x^*x)^{\frac{1}{2}}$.

Notice next that

$$\begin{aligned} \xi \in ran^{\perp} x^* &\Leftrightarrow \langle \xi, x^*\eta \rangle = 0 \ \forall \ \eta \in \mathcal{K} \\ &\Leftrightarrow \langle x\xi, \eta \rangle = 0 \ \forall \ \eta \in \mathcal{K} \\ &\Leftrightarrow x\xi = 0 \end{aligned}$$

and hence $ran^{\perp} x^* = ker \ x$. 'Taking perps' once again, we find - because of the fact that $V^{\perp\perp} = \overline{V}$ for any linear subspace $V \subset \mathcal{K}$ - that the last statement of the Lemma is indeed valid.

Finally, if $\{p_n\}_n$ is any sequence of polynomials with the property that $p_n(0) = 0 \ \forall \ n$ and such that $\{p_n(t)\}$ converges uniformly to $\sqrt{t}$ on $\sigma(x^*x)$, it follows that $||p_n(x^*x) - (x^*x)^{\frac{1}{2}}|| \to 0$, and hence,

$$\xi \in \ ker(x^*x) \ \Rightarrow \ p_n(x^*x)\xi = 0 \ \forall n \ \Rightarrow (x^*x)^{\frac{1}{2}}\xi = 0$$

and hence we see that also $ker(x^*x) \subset ker(x^*x)^{\frac{1}{2}}$; since the reverse inclusion is clear, the proof of the lemma is complete. □

PROPOSITION 2.10. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces; then the following conditions on an operator $u \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ are equivalent:*

(i) $u = uu^*u$;

(ii) $p = u^*u$ is a projection;

(iii) $u|_{ker^{\perp}u}$ is an isometry.

An operator which satisfies the equivalent conditions (i)-(iii) is called a **partial isometry**.

Proof : $(i) \Rightarrow (ii)$: The assumption (i) clearly implies that $p = p^*$, and that $p^2 = u^*uu^*u = u^*u = p$.

$(ii) \Rightarrow (iii)$: Let $\mathcal{M} = ran \ p$. Then notice that, for arbitrary $\xi \in \mathcal{H}$, we have: $||p\xi||^2 = \langle p\xi, \xi \rangle = \langle u^*u\xi, \xi \rangle = ||u\xi||^2$; this clearly implies that $ker \ u = ker \ p = \mathcal{M}^{\perp}$, and that u is isometric on $\mathcal{M}$ (since p is identity on $\mathcal{M}$).

$(iii) \Rightarrow (ii)$: Let $\mathcal{M} = ker^{\perp}u$. For $i = 1, 2$, suppose $\zeta_i \in \mathcal{H}$, and $\xi_i \in \mathcal{M}, \eta_i \in \mathcal{M}^{\perp}$ are such that $\zeta_i = \xi_i + \eta_i$; then note that

$$\begin{aligned} \langle u^*u\zeta_1, \zeta_2 \rangle &= \langle u\zeta_1, u\zeta_2 \rangle \\ &= \langle u\xi_1, u\xi_2 \rangle \\ &= \langle \xi_1, \xi_2 \rangle \ \ (\textit{since } u|_{\mathcal{M}} \textit{ is isometric}) \\ &= \langle \xi_1, \zeta_2 \rangle \ , \end{aligned}$$

and hence u^*u is the projection onto $\mathcal{M}$.

$(ii) \Rightarrow (i)$: Let $\mathcal{M} = ran \ u^*u$; then (by Lemma 2.9) $\mathcal{M}^{\perp} = ker \ u^*u = ker \ u$, and so, if $\xi \in \mathcal{M}, \eta \in \mathcal{M}^{\perp}$, are arbitrary, and if $\zeta = \xi + \eta$, then observe that $u\zeta = u\xi + u\eta = u\xi = u(u^*u\zeta)$. □

REMARK 2.11. Suppose $u \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a partial isometry. Setting $\mathcal{M} = ker^{\perp} u$ and $\mathcal{N} = ran\ u (= \overline{ran\ u})$, we find that u is identically 0 on $\mathcal{M}^{\perp}$, and u maps $\mathcal{M}$ isometrically onto $\mathcal{N}$. It is customary to refer to $\mathcal{M}$ as the **initial space**, and to $\mathcal{N}$ as the **final space**, of the partial isometry u.

On the other hand, upon taking adjoints in condition (i) of the previous Proposition, it is seen that $u^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is also a partial isometry. In view of the preceding lemma, we find that $ker\ u^* = \mathcal{N}^{\perp}$ and that $ran\ u^* = \mathcal{M}$; thus $\mathcal{N}$ is the inital space of u^* and $\mathcal{M}$ is the final space of u^*.

Finally, it follows from condition (ii) of the previous Proposition (and the proof of that proposition) that u^*u is the projection (of $\mathcal{H}$) onto $\mathcal{M}$ while uu^* is the projection (of $\mathcal{K}$) onto $\mathcal{N}$. $\square$

EXERCISE 2.12. *If $u \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a partial isometry with initial space $\mathcal{M}$ and final space $\mathcal{N}$, show that if $\eta \in \mathcal{N}$, then $u^*\eta$ is the unique element $\xi \in \mathcal{M}$ such that $u\xi = \eta$.*

Before stating the polar decomposition theorem, we introduce a convenient bit of notation: if $x \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a bounded operator between Hilbert spaces, we shall always use the symbol $|x|$ to denote the unique positive square root of the positive operator $|x|^2 = x^*x \in \mathcal{L}(\mathcal{H})$; thus, $|x| = (x^*x)^{\frac{1}{2}}$.

THEOREM 2.13. *(***Polar Decomposition***)*

(a) Any operator $x \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ admits a decomposition $x = ua$ such that

(i) $u \in \mathcal{L}(H, \mathcal{K})$ is a partial isomertry;

(ii) $a \in \mathcal{L}(\mathcal{H})$ is a positive operator; and

(iii) $ker\ x = ker\ u = ker\ a$.

(b) Further, if $x = vb$ is another decomposition of x as a product of a partial isometry v and a positive operator b such that $kerv = kerb$, then necessarily $u = v$ and $b = a = |x|$. This unique decomposition is called the polar decomposition of x.

(c) If $x = u|x|$ is the polar decomposition of x, then $|x| = u^*x$.

Proof : (a) If $\xi, \eta \in \mathcal{H}$ are arbitrary, then,

$$\langle x\xi, x\eta \rangle = \langle x^*x\xi, \eta \rangle = \langle |x|^2\xi, \eta \rangle = \langle |x|\xi, |x|\eta \rangle ,$$

whence it follows that there exists a unique unitary operator $u_0 : \overline{ran\ |x|} \to \overline{ran\ x}$ such that $u_0(|x|\xi) = x\xi\ \forall\ \xi \in \mathcal{H}$. Let $\mathcal{M} = \overline{ran\ |x|}$

and let $p = p_{\mathcal{M}}$ denote the orthogonal projection onto $\mathcal{M}$. Then the operator $u = u_0 p$ clearly defines a partial isometry with initial space $\mathcal{M}$ and final space $\mathcal{N} = \overline{ran\ x}$ which further satisfies $x = u|x|$ (by definition). It follows from Lemma 2.9 that $ker\ u = ker\ |x| = ker\ x$.

(b) Suppose $x = vb$ as in (b). Then v^*v is the projection onto $ker^{\perp} v = ker^{\perp} b = \overline{ran\ b}$, which clearly implies that $b = v^*vb$; hence, we see that $x^*x = bv^*vb = b^2$; thus b is a, and hence the, positive square root of $|x|^2$, i.e., $b = |x|$. It then follows that $v(|x|\xi) = x\xi = u(|x|\xi)\ \forall \xi$; by continuity, we see that v agrees with u on $\overline{ran\ |x|}$, but since this is precisely the initial space of both partial isometries u and v, we see that we must have $u = v$.

(c) This is an immediate consequence of the definition of u and Exercise 2.12. □

EXERCISE 2.14. *(1) Prove the 'dual' polar decomposition theorem; i.e., each $x \in \mathcal{L}(\mathcal{H},\mathcal{K})$ can be uniquely expressed in the form $x = bv$ where $v \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is a partial isometry, $b \in \mathcal{L}(\mathcal{K})$ is a positive operator and $ker b = ker v^* = ker x^*$. (Hint: Consider the usual polar decomposition of x^*, and take adjoints.)*

(2) Show that if $x = u|x|$ is the (usual) polar decomposition of x, then $u|_{ker^{\perp} x}$ implements a unitary equivalence between $|x|\ |_{ker^{\perp}|x|}$ and $|x^*|\ |_{ker^{\perp}|x^*|}$. (Hint: Write $\mathcal{M} = ker^{\perp} x$, $\mathcal{N} = ker^{\perp} x^*$, $w = u|_{\mathcal{M}}$; then $w \in \mathcal{L}(\mathcal{M},\mathcal{N})$ is unitary; further $|x^*|^2 = xx^* = u|x|^2u^*$; deduce that if a (resp., b) denotes the restriction of $|x|$ (resp., $|x^*|$) to $\mathcal{M}$ (resp., $\mathcal{N}$), then $b^2 = wa^2w^*$; now deduce, from the uniqueness of the positive square root, that $b = waw^*$.)

(3) Apply (2) above to the case when $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional, and prove that if $x \in L(V,W)$ is a linear map of vector spaces (over $\mathbb{C}$), then dim $V = rank(x) + nullity(x)$, where $rank(x)$ and $nullity(x)$ denote the dimensions of the range and null-space, respectively, of the map x.

(4) Show that an operator $x \in \mathcal{L}(\mathcal{H},\mathcal{K})$ can be expressed in the form $x = wa$, where $a \in \mathcal{L}(\mathcal{H})$ is a positive operator and $w \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is unitary if and only if $dim(ker\ x) = dim(ker\ x^*)$. (Hint: In order for such a decomposition to exist, show that it must be the case that $a = |x|$ and that the w should agree, on $ker^{\perp} x$, with the u of the polar decomposition, so that w must map $ker\ x$ isometrically onto $ker\ x^*$.)

(5) In particular, deduce from (4) that in case $\mathcal{H}$ is a finite-dimensional inner product space, then any operator $x \in \mathcal{L}(\mathcal{H})$ admits a decomposition as the product of a unitary operator and a positive operator. (Note that when $\mathcal{H} = \mathbb{C}$, this boils down to the usual polar

decomposition of a complex number.)

Several problems concerning a general bounded operator between Hilbert spaces can be solved in two stages: in the first step, the problem is 'reduced', using the polar decomposition theorem, to a problem concerning positive operators on a Hilbert space; and in the next step, the positive case is settled using the spectral theorem. This is illustrated, for instance, in exercise 2.15(2).

EXERCISE 2.15. *(1) Recall that a subset* Δ *of a (real or complex) vector space* V *is said to be* **convex** *if it contains the 'line segment joining any two of its points'; i.e.,* Δ *is convex if* $\xi, \eta \in \Delta, 0 \leq t \leq 1 \Rightarrow t\xi + (1-t)\eta \in \Delta$.

(a) If V *is a normed (or simply a topological) vector space, and if* Δ *is a closed subset of* V*, show that* Δ *is convex if and only if it contains the mid-point of any two of its points - i.e.,* Δ *is convex if and only if* $\xi, \eta \in \Delta \Rightarrow \frac{1}{2}(\xi + \eta) \in \Delta$. *(Hint: The set of dyadic rationals, i.e., numbers of the form* $\frac{k}{2^n}$ *is dense in* $\mathbb{R}$.)

(b) If $\mathcal{S} \subset V$ *is a subset of a vector space, show that there exists a smallest convex subset of* V *which contains* $\mathcal{S}$*; this set is called the* **convex hull** *of the set* $\mathcal{S}$ *and we shall denote it by the symbol* $co(\mathcal{S})$. *Show that* $co(\mathcal{S}) = \{\sum_{i=1}^n \theta_i \xi_i : n \in \mathbb{N}, \theta_i \geq 0, \sum_{i=1}^n \theta_i = 1\}$.

(c) Let Δ *be a convex subset of a vector space; show that the following conditions on a point* $\xi \in \Delta$ *are equivalent:*

(i) $\xi = \frac{1}{2}(\eta + \zeta),\ \eta, \zeta \in \Delta \Rightarrow \xi = \eta = \zeta$;

(ii) $\xi = t\eta + (1-t)\zeta, 0 < t < 1,\ \eta, \zeta \in \Delta \Rightarrow \xi = \eta = \zeta$.

The point ξ *is called an* **extreme point** *of a convex set* Δ *if* $\xi \in \Delta$ *and if* ξ *satisfies the equivalent conditions (i) and (ii) above.*

(d) It is a fact, called the Krein-Milman theorem - see [Yos], for instance - that if K *is a compact convex subset of a Banach space (or more generally, of a locally convex topological vector space which satisfies appropriate 'completeness conditions'), then* $K = \overline{co(\partial_e K)}$, *where* $\partial_e K$ *denotes the set of extreme points of* K. *Verify the above fact in case* $K = ball(\mathcal{H}) = \{\xi \in \mathcal{H} : ||\xi|| \leq 1\}$, *where* $\mathcal{H}$ *is a Hilbert space, by showing that* $\partial_e(ball\ \mathcal{H}) = \{\xi \in \mathcal{H} : ||\xi|| = 1\}$. *(Hint: Use the parallelogram law.)*

(e) Show that $\partial_e(ball\ X) \neq \{\xi \in X : ||\xi|| = 1\}$, *when* $X = \ell^1_n,\ n > 1$. *(Thus, not every point on the unit sphere of a normed space need be an extreme point of the unit ball.)*

(2) Let $\mathcal{H}$ *and* $\mathcal{K}$ *denote (separable) Hilbert spaces, and let* $\mathbb{B} = \{A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : ||A|| \leq 1\}$ *denote the unit ball of* $\mathcal{L}(\mathcal{H}, \mathcal{K})$. *The aim of the following exercise is to show that an operator* $x \in \mathbb{B}$ *is an extreme*

point of $\mathbb{B}$ *if and only if either* x *or* x^* *is an isometry. (See (1)(c) above, for the definition of an extreme point.)*

(a) Let $\mathbb{B}_+ = \{x \in \mathcal{L}(\mathcal{H}) : x \geq 0, \ ||x|| \leq 1\}$. *Show that* $x \in \partial_e \mathbb{B}_+ \Leftrightarrow x$ *is a projection. (Hint: suppose* p *is a projection and* $p = \frac{1}{2}(a+b), a, b \in \mathbb{B}_+$*; then for arbitrary* $\xi \in ball(\mathcal{H})$*, note that* $0 \leq \frac{1}{2}(\langle a\xi, \xi\rangle + \langle b\xi, \xi\rangle) \leq 1$*; since* $\partial_e[0,1] = \{0,1\}$*, deduce that* $\langle A\xi, \xi\rangle = \langle B\xi, \xi\rangle = \langle p\xi, \xi\rangle \ \forall \ \xi \in (ker \ p \cup \ ran \ p)$*; but* $a \geq 0$ *and* $ker \ p \subset \ ker \ a$ *imply that* $a(ran \ p) \subset \ ran \ p$*; similarly also* $b(ran \ p) \subset \ ran \ p$*; conclude that* $a = b = p$*. Conversely, if* $x \in \mathbb{B}_+$ *and* x *is not a projection, then it must be the case that there exists* $\lambda \in \sigma(x)$ *such that* $0 < \lambda < 1$*; fix* $\epsilon > 0$ *such that* $(\lambda - 2\epsilon, \lambda + 2\epsilon) \subset (0,1)$*; since* $\lambda \in \sigma(x)$*, deduce that* $p \neq 0$ *where* $p = 1_{(\lambda-\epsilon,\lambda+\epsilon)}(x)$*; notice now that if we set* $a = x - \epsilon p, b = x + \epsilon p$*, then the choices ensure that* $a, b \in \mathbb{B}_+, x = \frac{1}{2}(a+b)$*, but* $a \neq x \neq b$*, whence* $x \notin \partial_e \mathbb{B}_+$*.)*

(b) Show that the only non-zero extreme point of ball $\mathcal{L}(\mathcal{H}) = \{x \in \mathcal{L}(\mathcal{H}) : ||x|| \leq 1\}$ *which is a positive operator is* 1*, the identity operator on* $\mathcal{H}$*. (Hint: Prove that 1 is an extreme point of ball* $\mathcal{L}(\mathcal{H})$ *by using the fact that 1 is an extreme point of the unit disc in the complex plane; for the other implication, by (a) above, it is enough to show that if* p *is a projection which is not equal to 1, then* p *is not an extreme point in ball* $\mathcal{L}(\mathcal{H})$*; if* $p \neq 0, 1$*, note that* $p = \frac{1}{2}(u_+ + u_-)$*, where* $u_\pm = p \pm (1-p) \neq p$*.)*

(c) Suppose $x \in \partial_e \mathbb{B}$*; if* $x = u|x|$ *is the polar decomposition of* x*, show that* $|x| \ |_{\mathcal{M}}$ *is an extreme point of the set* $\{A \in \mathcal{L}(\mathcal{M}) : ||A|| \leq 1\}$*, where* $\mathcal{M} = ker^\perp |x|$*, and hence deduce, from (b) above, that* $x = u$*. (Hint: if* $|x| = \frac{1}{2}(c+d)$*, with* $c, d \in ball \ \mathcal{L}(\mathcal{M})$ *and* $c \neq |x| \neq d$*, note that* $x = \frac{1}{2}(a+b)$*, where* $a = uc, b = ud$*, and* $a \neq x \neq b$*.)*

(d) Show that $x \in \partial_e \mathbb{B}$ *if and only if* x *or* x^* *is an isometry. (Hint: suppose* x *is an isometry; suppose* $x = \frac{1}{2}(a+b)$*, with* $a, b \in \mathbb{B}$*; deduce from (1)(d) that* $x\xi = a\xi = b\xi \ \forall \xi \in \mathcal{H}$*; thus* $x \in \partial_e \mathbb{B}$*; similarly, if* x^* *is an isometry, then* $x^* \in \partial_e \mathbb{B}$*. Conversely, if* $x \in \partial_e \mathbb{B}$*, deduce from (c) that* x *is a partial isometry; suppose it is possible to find unit vectors* $\xi \in kerx, \quad \eta \in kerx^*$*; define* $u_\pm \zeta = x\zeta \pm \langle \zeta, \xi\rangle \eta$*, and note that* $u_\pm$ *are partial isometries which are distinct from* x *and that* $x = \frac{1}{2}(u_+ + u_-)$*.)*

(e) Imitate the reaasoning given in the above exercises to show that if M *is a von Neumann algebra, and if* $\mathcal{P}(M), M_+, (M)_1$ *denote the sets of elements of* M *which are projections, positive, and of norm at most one, respectively, then*

(i) $\partial(M_+ \cap (M)_1) = \mathcal{P}(M)$

(ii) $M_+ \cap \partial((M)_1) = \{0, 1\}$

(iii) $\partial((M)_1)$ consists only of partial isometries.

3 Normality

This section is devoted to a discussion of matters pertaining to the σ-weak topology on a von Neumann algebra. Through much of this section, many statements will not be proved at all. Instead, we will merely indicate the number of the corresponding result in {SZ} where the proof may be found.

We shall call linear functionals as *forms.* Given a form ϕ on a *-algebra, its adjoint is the form ϕ^* defined by $\phi^*(x) = \overline{\phi(x^*)}$; a form is said to be self-adjoint if $\phi^* = \phi$.

Exercise 3.1. *Show that:*

1. *A form (on a *-algebra) is self-adjoint if and only if it takes real values on self-adjoint elements.*

2. *any form ϕ admits a unique Cartesian decomposition $\phi = \phi_1 + i\phi_2$, with the ϕ_i being self-adjoint.*

3. *A form ϕ on a C^*-algebra is bounded if and only if its real and imaginary parts (ϕ_i as above) are bounded.*

A form ϕ on a C^*-algebra is said to be *positive* if it assumes non-negative values on positive self-adjoint elements. (Clearly, positive forms are self-adjoint.)

Proposition 3.2. *A positive form, say ϕ, on a C^*-algebra, say A is bounded.*

Proof. Observe first that the equation

$$\langle x, y\rangle = \phi(y^*x) \tag{3.3}$$

defines a semi-inner-product on A. Since the Cauchy-Schwarz inequality is valid for any semi-inner-product, we may conclude that

$$\phi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y)\ \forall x, y \in A \tag{3.4}$$

Put $y = 1$ to find that $|\phi(x)|^2 \leq \phi(x^*x)\phi(1)$. since $x^*x \leq \|x\|^2.1$, deduce finally from the assumed positivity of ϕ that $|\phi(x)| \leq \phi(1)\|x\|$. $\square$

Observe that the proof shows that $\|\phi\| = \phi(1)$. It is interesting to note - and not too hard to prove - that a linear functional on a unital C^*-algebra is positive if and only if it is positive and attains its norm at the identity element.

DEFINITION 3.3. *A* **state** *on a C^*-algebra is a positive form ϕ normalised so that $\|\phi\| = 1$.*

Thus for a unital C^*-algebras are just positive forms satisfying $\phi(1) = 1$. For non-unital C^*-algebras, there are analogous statements involving approximate identities, but we shall not have much to do with non-unital algebras, as out main interest is in von Neumann algebras.

One of the things we shall do in this lecture is to complete the proof of the fact that the equivalence of the notions of concrete and abstract von Neumann algebras. The easier half, *viz.* that concrete von Neumann algebras are abstract von Neumann algebras. For the other implication, we first remark that abstract von Neumann algebras are the objects of a category whose morphisms are *-homomorphisms which are continuous when domain and range are equipped with their σ-weak topologies (see Remark 2.2 (6).

DEFINITION 3.4. *A positive (=positivity-preserving) linear map between (abstract) von Neumann algebras will be called* **normal** *if it is continuous when domain and range are equipped with their σ-weak topologies.*

In order to prove that abstract von Neumann algebras 'are' concrete ones, we shall show that any abstract von Neumann algebra M is isomorphic to a concrete one. Thus what we shall need to prove is that there exists a normal representation (i.e, a morphism π from M into a $\mathcal{B}(\mathcal{H})$ which induces an isomorphism of M onto the von Neumann subalgebra $\pi(M)$. All this will take some work.

We begin by outlining, in the following Exercises, a proof of the corresponding fact - usually calle the (non-commutative) Gelfand-Naimark theorem - for abstract C^* algebras. We will only consider unital C^*-algebras, since our interest is in von Neumann algebras (which always have identity).

EXERCISE 3.5.
1. *Complete the proof of the fact that if ψ is a positive functional on a unital C^*-algebra, then $\|\psi\| = \psi(1)$.*
2. *Show that $z \mapsto xzx^*$ defines a positivity preserving map of A for each $x \in A$.*

3. *If ϕ is a positive functional on A, if $x \in A$, and if ψ is the form defined by*
$$\psi(z) = \phi(x^*zx) \ \forall z \in A \ ,$$
show that $\|\psi\| = \phi(x^*x)$.

EXERCISE 3.6. *Let ϕ be a positive functional on a C^*-algebra A. Consider the so-called radical $Rad(\phi) = \{x \in A : \phi(x^*x) = 0\}$. Deduce from the Cauchy-Schwarz inequality (3.4) that*
$$x \in Rad(\phi) \Leftrightarrow \phi(yx) = 0 \ \forall y \in A \ ,$$
and hence deduce that $Rad(\phi)$ is a left-ideal in A.

EXERCISE 3.7. *Let ϕ be a positive functional on a C^*-algebra A. For any $a \in A$, write $\hat{a}$ for $a + Rad(\phi)$.*

1. *Show that the equation*
$$\langle \hat{x}, \hat{y} \rangle = \phi(y^*x)$$
defines a genuine inner product on the quotient vector space $A/Rad(\phi)$.

2. *Show that if $x, y \in A$, then $\|\widehat{xy}\|_\phi \leq \|x\| \|\hat{y}\|_\phi$, where we write $\|\hat{a}\|_\phi = \phi(a * a)^{\frac{1}{2}}$ for the associate norm of $\hat{a}$.*

3. *Deduce that the equation $\pi_0(x)(\hat{y}) = \widehat{xy}$ defines a bounded (w.r.t. $\| \cdot \|_\phi$) operator $\pi_0(x)$.*

4. *If $\mathcal{H}_\phi$ denotes the Hilbert space completion of $A/Rad(\phi)$, and if $\pi_\phi(x)$ denotes the unique continuous extension of $\pi_0(x)$, show that π_ϕ is a representation of A on $\mathcal{H}_\phi$. The vector $\xi_\phi = \hat{1}$ is a cyclic vector (in the sense that $\pi_\phi(A)\xi_\phi$ is dense in $\mathcal{H}_\phi$) such that*
$$\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi \rangle \ \forall x \in A.$$

5. *If $(\pi, \mathcal{H}, \phi)$ is another triple of a representation π of A on $\mathcal{H}$ which admits a cyclic vector ξ satisfying $\langle \pi(x)\xi, \xi \rangle = \phi(x) \ \forall x \in A$, show that there exists a unique unitary operator $u : \mathcal{H}_\phi \to \mathcal{H}$ such that $u\pi_\phi(x)\xi_\phi = \pi(x)\xi \ \forall x \in A$. (For this reason, one refers to* **the** *GNS-triple associated to a positive functional - with the acronym standing for Gelfand-Naimark-Segal; in fact this entire construction of a cyclic representation from a positive functional is referred to as the GNS construction.)*

EXERCISE 3.8. *Suppose* $\pi : A \to B$ *is a unital morphism of unital* C^**-algebras. Then, show that:*

1. $sp(\pi(x)) \subset sp(x)\ \forall x \in A$

2. *If* $x \in A$ *is self-adjoint, show that* $\|\pi(x)\| = max\{|\lambda| : \lambda \in sp(\pi(x)\} \leq max\{|\lambda| : \lambda \in sp(x)\} = \|x\|$.

3. *Deduce from the* C^**-identity (since* x^*x *is self-adjoint) that* $\|pi(x)\| \leq \|x\|\ \forall x \in A$*, thus establishing that every* $*$*-homomorphism of* C^**-algebras is contractive.*

EXERCISE 3.9. *If* x *is an element of a unital* C^**-algebra* A*, and if* $A_x = C^*(\{1, x\})$*, deduce from*

1. *the Gelfand-Naimark theorem for commutative* C^**-algebras that there exists a state* ϕ_0 *on* A_x *such that* $|\phi_0(x)| = \|x\|$*;*

2. *the Hahn-Banach theorem and the fact that linear functionals which attain their norm at the identity are positive, that there exists a state* ϕ_x *on* A *which extends* ϕ_0*.*

3. *Apply the GNS construction to* ϕ_x*(and the previous exercise)e to obtain a representation* $\pi_x : A \to \mathcal{B}(\mathcal{H}_x)$ *such that* $\|\pi_x(x)\| = \|x\|$

4. *Deduce from the conttractive nature of represdntations of* C^**-algebras that* $\pi = \oplus_{x \in A} \pi_x$ *defines an isometric representation of* A*.*

   (The Hilbert space underlying the above representation is hugely non-separable. The next exercise addresses this problem)

5. *If* A *is separable, and if* $\{x_n\}_n$ *is a dense sequence in* A*, then show that in the notation of the previous exercise, the representation* $\oplus_{n=1}^{\infty} \pi_{x_n}$ *represents* A *faithfully on a separable Hilbert space.*

Our interest lies in the von Neumann algebraic version of this fact; the rest of this lecture is devoted to showing that the GNS representation corresponding to a normal state ϕ on a von Neumann algebra M yields a normal representation of M onto a von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_\phi)$; and that if M_* is separable, then M is isomorphic to a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. For this, we shall need one of several conditions equivalent to normality for a form on a von Neumann algebra. We shall say nothing about the (technical) proof.

THEOREM 3.10. *The following conditions on a positive form ϕ on a von Neumann algebra M are equivalent:*

1. *ϕ is normal*

2. *ϕ respects monotone convergence, i.e., if $\{x_i : i \in I\}$ is a uniformly bounded monotone (increasing) net in M, then*

$$\phi(\sup_{i \in I} x_i) = \phi(\lim_{i \in I} x_i) = \lim_{i \in I} \phi(x_i) = \sup_{i \in I} \phi(x_i)$$

3. *ϕ is completely additive, meaning that whenever $\{e_i : i \in I\}$ is a family of pairwise orthogonal projections in M, and $e = \sum_{i \in I} e_i$, we have*

$$\phi(e) = \sum_{i \in I} \phi(e_i)$$

4. *The restriction of ϕ to any commutative von Neumann subalgebra is normal*

We shall now make a minor digression involving supports, etc., which will eventually end with the conclusion that there are sufficiently many normal forms on von Neumann algebras.

For an $x \in \mathcal{B}(\mathcal{H})$, we write

$$\begin{aligned} \mathbf{n}(x) &= \text{projection onto } ker(x) \\ \mathbf{l}(x) &= \text{projection onto } \overline{ran(x)} \\ \mathbf{r}(x) &= 1 - \mathbf{n}(x) , \end{aligned}$$

observe that $\mathbf{r}(x) = \mathbf{l}(x^*)$ and these projections can be characterised (and defined for elements of abstract von Neumann algebras) as

$$\begin{aligned} \mathbf{l}(x) &= \inf\{p : p \text{ a projection}, px = x\} \\ \mathbf{r}(x) &= \inf\{p : p \text{ a projection}, xp = x\} ; \end{aligned}$$

for this reason they are called the 'left support projection' and 'right support projection' of x. When $x = x^*$, we write

$$\mathbf{s}(x) = \mathbf{l}(x) = \mathbf{r}(x)$$

and simply call it the 'support of x'.

LEMMA 3.11. 1. *If $\mathcal{I}$ is a left ideal in a von Neumann algebra M, then*

$$\begin{aligned} x \in \mathcal{I} &\Rightarrow 1_{[\epsilon,\infty)}(|x| \in \mathcal{I} \ \forall \epsilon > 0 \\ &\Rightarrow \mathbf{r}(x) \in \overline{\mathcal{I}} , \end{aligned}$$

where $\overline{\mathcal{I}}$ denotes the WOT-closure of $\mathcal{I}$.

2. *If $\mathcal{I}$ is a WOT-closed left ideal in M, then $\mathcal{P}(\mathcal{I})$ is a directed set with respect to the natural order, $z = \sup \mathcal{P}(\mathcal{I})$ is the largest projection in M, and $\mathcal{I} = Mz$.*

Proof. 1. Define

$$g(t) = \begin{cases} \frac{1}{t^2} & \text{if } t \geq \epsilon \\ 0 & \text{if } t < \epsilon \end{cases} ,$$

note that $g(a)a^2 = 1_{[\epsilon,\infty)}(a)$ for all positive $a \in M$, and deduce that if $x = u|x|$ is the polar decomposition of $x \in \mathcal{I}$, then $1_{[\epsilon,\infty)}(|x|) = g(|x|)|x|^2 = g(|x|)x^*x \in \mathcal{I}$. Since $\lim_{\epsilon \downarrow 0} 1_{[\epsilon,\infty)}(|x|) = 1_{(0,\infty)}(|x|) = \mathbf{r}(x)$, we see that indeed $\mathbf{r}(x) \in \overline{\mathcal{I}}$.

2. If $e, f \in \mathcal{P}(\mathcal{I})$, and $p = \mathbf{s}(e+f)$, it follows from part 1 of this lemma that $p \in \mathcal{P}(\mathcal{I})$, and clearly $e, f \leq p$. Thus $\mathcal{P}(\mathcal{I})$ is indeed a monotone (increasing) net of projections in $\mathcal{I}(\subset M)$ which converges, to z, say. It follows from Example 1.2 that z inherits idempotence from the members of $\mathcal{P}(\mathcal{I})$ and hence z is indeed idempotent and hence the largest projection in $\mathcal{I}$. Since $z \in \mathcal{I} \subset M$, it is clear that $Mz \subset \mathcal{I}$.

   Notice now that also $z = \sup \mathcal{P}(\mathcal{I}) = \sup \mathcal{P}(\mathcal{I}^*)$. As $\mathbf{r}(x) \in \mathcal{P}(\mathcal{I}^*)$, we find now that that if $x \in \mathcal{I}$, then $\mathbf{r}(x) = \mathbf{r}(x)z$ and hence

$$x = x\mathbf{r}(x) = x\mathbf{r}(x)z \in Mz ,$$

   thus showing that also $\mathcal{I} \subset Mz$.

□

COROLLARY 3.12. *The radical $Rad(\phi)$ of a normal form ϕ on a von Neumann algebra M admits a largest projection $1 - \mathbf{s}(\phi)$, and $\mathbf{s}(\phi)$ is called the support of the form ϕ. It is the smallest projection satisfying*

$$\phi(x) = \phi(x\mathbf{s}(\phi) \ \forall x \in M$$

REMARK 3.13. *It follows trivially from the definitions that the following conditions on a normal form ϕ are equivalent:*

1. *ϕ is faithful, i.e., $x \geq 0$ and $\phi(x) = 0$ imply $x = 0$*
2. *$\mathbf{s}(\phi) = 1$*

We next make the important observation that M^* and M_* have the natural $M - M$ bimodule structure defined by:

$$(a.\phi.b)(x) = \phi(bxa) .$$

This plays a vital role in the important **polar decomposition for forms**.

THEOREM 3.14. *Every σ-weakly continuous form ϕ on a von Neumann algebra M admits a decomposition $\phi = v \cdot |\phi|$, which is uniquely determined by the conditions*

1. *v is a partial isometry in M;*
2. *$|\phi|$ is a normal (positive) form on M; and*
3. *$v^* v = \mathbf{s}(\phi)$.*

Proof. See {SZ} 5.16. □

EXERCISE 3.15. 1. *Verify that $(a.\phi.b)^* = b^*.\phi^*.a^*$.*

2. *Verify the dual polar decomposition $\phi = |\phi^*|.u$ where u is a partial isometry with $uu^* = \mathbf{s}(|\phi^*|)$.*

The next step in showing that there exist 'sufficiently many' normal forms on a von Neumann algebra is the **Jordan decomposition**.

THEOREM 3.16. *If ϕ is a σ-weakly continuous self-adjoint form on a von Neumann algebra M, there exist unique normal forms $\phi_\pm$ on M satisfying (i) $\phi = \phi_+ - \phi_-$, and (ii) $\mathbf{s}(\phi_+)\mathbf{s}(\phi_-) = 0$.*

Proof. See {SZ} 5.17. □

THEOREM 3.17. *Let M be a von Neumann algebra with separabe predual. Then*

1. *there exists a faithful normal state ϕ on M; and*
2. *If ϕ is as above, then the associated GNS triple $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ satisfies:*
   (a) *$\mathcal{H}_\phi$ is separable;*
   (b) *π_ϕ is a normal isomorphism onto the (concrete) von Neumann subalgebra $\pi_\phi(M)$ of $\mathcal{B}(\mathcal{H}_\phi)$; and*
   (c) *ξ_ϕ is a* **separating** *(as well as* **cyclic***) vector for $\pi_\phi(M)$.*

Proof. 1. If $\{\psi_n : n \in \mathbb{N}\}$ is a dense sequence in M_*, it follows upon applying the Jordan decomposition to the real and imaginary parts (w.r.t. the Cartesian decomposition) of all the ψ_n's, and listing the resulting collection that there exists a sequence

$\{\phi_n : n \in \mathbb{N}\}$ of positive forms on M such that eacg ψ_m is a linear combination of four ϕ_n's. Now, let

$$\phi = \sum_{n=1}^{\infty} \frac{1}{2^n \|\phi_n\|} \phi_n \ .$$

Observe that ϕ is a normal form on M and that if $x \in M$, then

$$\begin{aligned} \phi(x^*x) = 0 \quad &\Rightarrow \quad \phi_n(x * x) = 0 \ \forall n \\ &\Rightarrow \quad \psi_n(x * x) = 0 \ \forall n \\ &\Rightarrow \quad \psi(x^*x) = 0 \ \forall \psi \in M_* \\ &\Rightarrow \quad x^*x = 0 \\ &\Rightarrow \quad x = 0 \ , \end{aligned}$$

there by establishing that $\frac{1}{\|\phi\|}\phi$ is indeed a faithful normal state on M.

2. (a) Since M_* is a separable Banach space, its unit ball $(M_*)_1$ is also separable. Pick a dense sequence $\{\psi_n : n \in \mathbb{N}\}$ in $(M_*)_1$. It is then not hard to see that the σ-weak topology on $(M)_1$ is induced by the metric defined by

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |\psi_n(x-y)| \ .$$

Then $(M)_1$ is a compact metric space and consequently separable in the σ-weak topology. But, on $(M)_1$, the weak and σ-weak topologies coincide, and hence the weak (equivalently the norm)- closure in $\mathcal{H}_\phi$ of $\{\pi_\phi(x_n)\xi_\phi : n \in \mathbb{N}\}$ contains $\pi((M)_1)\xi_\phi$, and it follows from the cyclicity of ξ_ϕ that $\mathcal{H}_\phi$ is indeed separable.

For the proof of the other parts of the theorem, see {SZ} 5.18.

□

4 The standard form of a von Neumann algebra

We saw that the GNS-representation associated to any faithful normal state on a von Neumann algebra M yields a faithful normal representation π which identifies M with the concrete von Neumann algebra$\pi(M)$; and the latter von Neumann algebra admits a cyclic and separating vector. It is a remarkable fact that any two such representations are unitarily equivalent - see {SZ} 5.25. The underlying Hilbert space is the **standard module for** M.

Let us begin by considering the group example. Consider a countable group Γ and lts left regular representation $\lambda : G \to \mathcal{B}(\ell^2(\Gamma))$ (given by $\lambda(s)\xi_t = \xi_{st}$, where $\{\xi_s : s \in \Gamma\}$ denotes the standard orthonormal basis of $\ell^2(\Gamma)$. It is true[1] then that the equation

$$\tau(x) = \langle x\xi_1, \xi_1\rangle \tag{4.5}$$

defines a faithful normal tracial state on the von Neumann algebra $M = \lambda(G)''$. In case Γ is abelian, its Pontrjagin dual $G = \hat{\Gamma}$ is a compact abelian group and the Fourier transform defines a unitary operator $\mathcal{F} : \ell^2(\Gamma) \to L^2(G, \mu)$ where μ denotes Haar measure on G so normalised as to be a probability measure; further $\mathcal{F}M\mathcal{F}^* = L^\infty(G, \mu)$, with the right side viewed as acting as multipliction operators, and $\mathcal{F}$ transports τ to integration against μ. Thus, the Haar measure on G may be 'identified with' the faithful normal tracial state τ. There are two bonuses with the τ-picture of Haar measure. Only when G is compact can Haar measure μ be a finite measure and thus define a linear functional on $L^\infty(G, \mu)$. For non-compact (still abelian) G, integration against μ will only yield a faithful normal semifinite tracial weight on $L^\infty(G, \mu)$. When Γ is not even abelian (possibly also not discrete), the von Neumann algebra $M = \lambda(G)''$ still makes sense and the role of the 'measure class of Haar measure on G' is taken over by any faithful normal semifinite weight on M. Akin to the fact that the measure class of a σ-finite measure always contains a probability measure. It is in this sense that the uniqueness of the measure-class of Haar measure (as a 'quasi-invariant' measure) has as counterpart the uniqueness, up to unitary equivalence' of the GNS-representation with respect to a faithful normal state.

The proof of the unitary equivalence of the GNS representation associated to all faithful normal states is achieved in [SZ] through a sequence of intermediate results some of which we shall merely state without proof.

LEMMA 4.1. *If a positive form ϕ on a C^*-algebra A satisfies $\phi \cdot a \geq 0$ for some $a \in A$, then $\phi \cdot a \leq \|a\|\phi$.*

Proof. If $x \in A$ and $x \geq 0$, then,

$$(\phi \cdot a)(x) = \phi(ax) = \phi((x^{\frac{1}{2}}a^*)^* x^{\frac{1}{2}}) \leq \phi(axa^*)^{\frac{1}{2}}\phi(x)^{\frac{1}{2}} \ , \tag{4.6}$$

[1]This will be seen in the sixth lecture.

by Cauchy-Schwarz inequality. Since $\phi \cdot a \geq 0$, we deduce that

$$\phi(a^2x) = (\phi \cdot a)(ax) = \overline{(\phi \cdot a)(xa^*)} = \overline{\phi(axa^*)} = \phi(axa^*)$$

and hence see that also $\phi \cdot a^2 \geq 0$.

Thus, we deduce from $\phi \cdot a \geq 0$ that also $\phi \cdot a^2 \geq 0$ and that

$$\begin{array}{rcl} (\phi \cdot a)(x) & \leq & \phi(axa^*)^{\frac{1}{2}}\phi(x)^{\frac{1}{2}} \\ & = & (\phi \cdot a^2)(x)^{\frac{1}{2}}\phi(x)^{\frac{1}{2}} \\ & \leq & (\phi \cdot a^4)(x)^{\frac{1}{4}}\phi(x)^{\frac{1}{2}+\frac{1}{4}} \\ & \leq & \cdots \\ & \leq & (\phi \cdot a^{2^n})(x)^{\frac{1}{2^n}}\phi(x)^{\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2^n}} \\ & \leq & \|\phi\|^{\frac{1}{2^n}}\|a\|\|x\|^{\frac{1}{2^n}}\phi(x)^{\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2^n}} \quad \text{for each } n \end{array}$$

.

Lettting $n \to \infty$, we find that

$$(\phi \cdot a)(x) \leq \|a\|\phi(x) \ ,$$

as desired. □

If $M \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, and if $\xi \in \mathcal{H}$, we shall write ω_ξ^M for the (obviously normal) from defined on M by

$$\omega_\xi^M(x) = \langle x\xi, \xi \rangle \ , \forall \ x \in M \ ,$$

and p_ξ^M for the projection onto $[M'\xi]$ - so that

$$p_\xi^M = \inf\{e \in \mathcal{P}(M) : e\xi = \xi\} \ .$$

EXERCISE 4.2. *With the foregoing notation, verify that* $\mathbf{s}(\omega_\xi^M) = p_\xi^M$*, and hence deduce that the following conditions on a vector* $\xi \in \mathcal{H}$ *are equivalent:*

1. ξ *is cyclic for* M'*;*
2. ξ *is separating for* M*.*
3. $p_\xi^M = 1$*;*
4. ω_ξ^M *is a faithful normal form on* M*.*

We now head towards the beautiful non-commutative Radon-Nikodym theorem due to Sakai. (For commutative M, this is equivalent to the usual Radon-Nikodym theorem, but restricted to the special case when the Radon-Nikodym derivative is bounded - by 1). But first, we establish an easy 'baby version'.

LEMMA 4.3. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, suppose $\xi \in \mathcal{H}$ is a cyclic and separating vector for M, and suppose ϕ is a normal form on M such that $\phi \leq \omega_\xi^M$. Then there exists $a' \in M'$ such that $\phi = \omega_{a'\xi}^M$ and $0 \leq a' \leq 1$.*

Proof. The equation

$$(x\xi, y\xi) = \phi(y^*x) \tag{4.7}$$

is seen to yield a well-defined (since ξ is separating for M) sesquilinear form on a dense subspace (since ξ is a cyclic vector for M) of $\mathcal{H}$, which satisfies

$$\begin{aligned} (x\xi, y\xi) &= \phi(y^*x) \\ &\leq \omega_\xi^M(y^*x) \\ &= \langle x\xi, y\xi\rangle \\ &\leq \|x\xi\|\|y\xi\| \ ; \end{aligned}$$

hence equation 4.7 extends to a bounded (in fact contractive) sesquilinear form on $\mathcal{H}$, and there exists a unique operator b' of norm at most one on $\mathcal{H}$ such that

$$\phi(y^*x) = \langle b'x\xi, y\xi\rangle \ \forall x, y \in M.$$

If $x, y, z \in M$, note that

$$\begin{aligned} \langle (zb')x\xi, y\xi\langle &= \langle b'x\xi, (z^*y)\xi\rangle \\ &= (x\xi, z^*y\xi) \\ &= \phi((z^*y)^*x) \\ &= \phi(y^*zx) \\ &= \langle b'zx\xi, y\xi\rangle \ , \end{aligned}$$

thereby establishing that indeed $b' \in M'$. Clearly b' is positive (and contractive). The proof is completed by setting $a' = (b')^{\frac{1}{2}}$ and noting that for $x \in M$,

$$\langle xa'\xi, a'\xi\rangle = \langle b'x\xi, 1\xi\rangle = \phi(x)$$

and so $\phi = \omega_{a'\xi}^M$. $\square$

THEOREM 4.4. **(Sakai's Radon-Nikodym Theorem)** *Suppose normal forms ϕ, ψ on a von Neumann algebra M satisfy $\phi \leq \psi$. Then there exists $a \in M$ which satisfies and is uniquely determined by the following properties:*

1. $0 \leq a \leq 1$
2. $\mathbf{s}(a) \leq \mathbf{s}(\psi)$*; and*
3. $\phi = a \cdot \psi \cdot a$

In fact, it is true that $\mathbf{s}(a) = \mathbf{s}(\phi)$.

Proof. Since $\phi \leq \psi \Rightarrow \mathbf{s}(\phi) \leq \mathbf{s}(\psi)$, wwe may - by replacing M by $\mathbf{s}(\psi)M\mathbf{s}(\psi)$, if necessary - assume, without loss of generality, that $\mathbf{s}(\psi) = 1$ or, equivalently, that ψ is faithful. Then, by Theorem 3.17(2), we may - by replacing $(M, \mathcal{H})$ by $(\pi_\psi(M), \mathcal{H}_\psi)$, if necessary - that $\psi = \omega_\xi^M$ where ξ is a cyclic and separating vector for M in $\mathcal{H}$. We may, by Lemma 4.3, then deduce that there exists $a' \in M'$ such that $\phi = \omega_{a'\xi}^M$ and $0 \leq a' \leq 1$.

Now consider the forms on M' defined by

$$\psi' = \omega_\xi^{M'} \ , \ \phi' = \psi' \cdot a' \ .$$

Let

$$\phi' = |\phi'^*| \cdot v' \tag{4.8}$$

be the dual polar decomposition (see Exercise 3.15) of ϕ'. Then $v'v'^* = \mathbf{s}(|\phi'^*|)$, and we have

$$|\phi'^*| = \phi' \cdot v'^* = \psi' \cdot (a'v'^*) \ , \tag{4.9}$$

and

$$\phi' = |\phi'^*| \cdot v' = \psi' \cdot (a'v'^*v') \ . \tag{4.10}$$

Deduce now from Lemma 4.1 that $|\phi'^*| \leq \psi'$, and then from Lemma 4.3 that there exists $b \in M$ such that $0 \leq b \leq 1$ and $|\phi'^*| = \omega_{b\xi}^{M'}$. Now observe that

$$\begin{aligned}
\langle x'\xi, b^2\xi\rangle &= |\phi'^*|(x') \\
&= (\psi' \cdot (a'v'^*))(x') \\
&= \langle a'v'^*x'\xi, \xi\rangle \\
&= \langle x'\xi, v'a'\xi\rangle \ ,
\end{aligned}$$

while

$$\begin{aligned}\langle x'\xi, a'\xi\rangle &= \phi'(x') \\ &= (\psi' \cdot (a'v'^*v'))(x') \\ &= \langle a'v'^*v'x'\xi, \xi\rangle \\ &= \langle x'\xi, v'^*v'a'\xi\rangle\end{aligned}$$

and deduce from the fact that ξ is a cyclic vector for M' that

$$b^2\xi = v'a'\xi \ , \ a'\xi = v'^*v'a'\xi \ .$$

Now, let $a = b^2$, and note that

$$\begin{aligned}\phi(x) &= \langle xa'\xi, a'\xi\rangle \\ &= \langle xv'^*v'a'\xi, a'\xi\rangle \\ &= \langle xv'a'\xi, v'a'\xi\rangle \\ &= \langle xa\xi, a\xi\rangle\end{aligned}$$

and finally $\phi = a \cdot \psi \cdot a$.

We shall omit the proof of the uniqueness asserion (as it is neither very illuminating nor difficult and can be found in [SZ]).

As for the last assertion concerning supports, we may assume that ξ is a separating and cyclic vector for M as above, and that $\phi = \omega^M_{a\xi}$ Then $\mathbf{s}(\phi) = p^M_{a\xi}$ (by Exercise 4.2, while

$$ran(\mathbf{s}(a)) = [ran(a)] = [aM'\xi] = [M'a\xi] = ran(p^M_{a\xi})] \ .$$

□

LEMMA 4.5. *Let ϕ be a normal form on a von Neumann subalgebra M of $\mathcal{B}(\mathcal{H})$ If*

$$\phi \geq \omega^M_\xi \quad \textit{and} \quad \mathbf{s}(\phi) = p^M_\xi \ ,$$

then there exists an $\eta \in \mathcal{H}$ such that

$$\phi = \omega^M_\eta \quad \textit{and} \quad p^{M'}_\eta = p^{M'}_\xi \ .$$

Proof. It is an immediate consequence of Theorem 4.4 that there exists $a \in M$ such that $0 \leq a \leq 1$ and $\omega^M_\xi = a \cdot \phi \cdot a$, and that, further,

$$\mathbf{s}(a) = \mathbf{s}(\omega^M_\xi) = p^M_\xi = \mathbf{s}(\phi)$$

Define

$$f_n(t) = \frac{1}{t} 1_{[\frac{1}{n},\infty)}(t)$$

and $x_n = f_n(a)$, so it is clear that $ax_n = 1_{[\frac{1}{n},\infty)}(a) \nearrow \mathbf{s}(a)$.

Set $\eta_n = x_n\xi$ and observe that if $m < n$, then

$$\begin{aligned} \|\eta_n - \eta_m\|^2 &= \|(x_n - x_m)\xi\|^2 \\ &= a \cdot \phi \cdot a(x_n - x_m) \\ &= \phi(a(x_n - x_m)a) \\ &= \phi(1_{[\frac{1}{m},\frac{1}{n})}) \ ; \end{aligned}$$

and the fact that $1_{[\frac{1}{n},\infty)}(a) \nearrow \mathbf{s}(a)$ implies, in view of normality of ϕ, that $\{\eta_n : n \in \mathbb{N}\}$ is a Cauchy sequence and hence convergent to some vector, call it η. Then, for any $x \in M$, observe that

$$\begin{aligned} \phi(x) &= \phi(\mathbf{s}(a)x\mathbf{s}(a)) \\ &= \lim_n \phi(1_{[\frac{1}{n},\infty)}(a)x1_{[\frac{1}{n},\infty)}(a)) \\ &= \lim_n \phi(ax_nxx_na) \\ &= \lim_n \langle x\eta_n, \eta_n\rangle \\ &= \langle x\eta, \eta\rangle \end{aligned}$$

and so $\phi = \omega_\eta^M$.

Now,

$$\begin{aligned} \eta \in [M\xi] &\Rightarrow \eta \in [M\xi] \\ &\Rightarrow [M\eta] \subset [M\xi] \ ; \end{aligned}$$

and conversely,

$$\xi = p_\xi^M \xi = \mathbf{s}(a)\xi = \lim_n 1_{[\frac{1}{n},\infty)}(a)\xi = \lim_n ax_n\xi = a\eta \ ,$$

and so $\xi \in [M\eta]$ and $[M\xi] \subset [M\eta]$.

Therefore $[M\eta] = [M\xi]$ and hence $p_\eta^{M'} = p_\xi^{M'}$. $\square$

We identify a simple fact about the support of normal forms, which will be used in the proof of the next theorem, in the next exercise.

EXERCISE 4.6. *Let ϕ be a normal form on a von Neumann algebra M. Then,*

1. $\mathbf{s}(\phi) = \inf\{e \in \mathcal{P}(M) : \phi = \phi \cdot e\}$.
2. *If $\phi_i, i = 1, 2$ are normal forms on M such that $\phi = \phi_1 + \phi_2$, and if $\mathbf{s}(\phi_1) = \mathbf{s}(\phi_2) = p$(say), show that also $\mathbf{s}(\phi) = p$.*

(*Hint:* By positivity, observe that $\phi(1-e) = 0 \Rightarrow \phi(1-e_1) = \phi(1-e_2) = 0$, so that $\mathbf{s}(\phi)$ and $\mathbf{s}(\phi_i), i = 1, 2$ are the minima over the same collections of projections.)

THEOREM 4.7. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\xi \in \mathcal{H}$. If $\mathbf{s}(\psi) \leq p_\xi^M$, then there exists $\eta \in [M\xi] \cap [M'\xi]$ such that $\psi = \omega_\eta^M$. Further, if $\mathbf{s}(\psi) = p_\xi^M$, then then there exists $\eta \in \mathcal{H}$ such that $\psi = \omega_\eta^M$ and $p_\eta^{M'} = p_\xi^{M'}$.*

Proof. Define $\phi = \psi + \omega_\xi^M$. It follows from $\mathbf{s}(\psi) \leq p_\xi^M = \mathbf{s}(\omega_\xi^M)$ that $\mathbf{s}(\phi) = p_\xi^M$. It follows from Lemma 4.5 that there exists $\eta_0 \in \mathcal{H}$ such that $\phi = \omega_{\eta_0}^M$ and $p_{\eta_0}^{M'} = p_\xi^{M'}$. Since $\psi \leq \phi$, deduce from Theorem 4.4 that there exists $a \in M$ with $0 \leq a \leq 1$ such that $\psi = a \cdot \phi \cdot a = \omega_{a\eta_0}^M$ and $\mathbf{s}(a) = \mathbf{s}(\phi) = p_\xi^M$. Setting $\eta = a\eta_0$, we find that $\psi = \omega_\eta^M$. Then

$$\eta \in ran(\mathbf{s}(\psi)) = ran(p_\xi)) \Rightarrow \eta \in [M'\xi]$$

and hence $\eta \in [M\xi] \cap [M'\xi]$.

Now, if $\mathbf{s}(\psi) = p_\xi^M = \mathbf{s}(\omega_\xi^M)$, deduce from Exercise 4.6(2) that also $\mathbf{s}(a) = \mathbf{s}(\psi) = \mathbf{s}(\phi) = p_{\eta_0}^M$. Now, with $x_n = f_n(a)$ as in Lemma 4.5, and $e_n = 1_{[\frac{1}{n},\infty)}(a)$, notice that

$$f_n(a)\eta = e_n\eta_0 \to \mathbf{s}(a)\eta_0$$

and deduce that

$$\eta_0 = p_{\eta_0}^M\eta_0 = \mathbf{s}(a)\eta_0 \in [M\eta] \ .$$

Finally, deduce that

$$p_\xi^{M'} = p_{\eta_0}^{M'} \leq p_\eta^{M'} .$$

Clearly, conversely, $\eta \in M\xi \Rightarrow p_\eta^{M'} \leq p_\xi^{M'}$, and the proof is complete. ☐

COROLLARY 4.8. *For $i = 1, 2$, let $M_i \subset \mathcal{B}(\mathcal{H}_i)$ be a von Neumann algebra and let $\xi_i \in \mathcal{H}_i$ be a cyclic and separating vector for M_i.*

If $\pi : M_1 \to M_2$ is any normal isomorphism, then there exists a unique unitary operator $u : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi(x_1) = ux_1u^* \ \forall x_1 \in M_1.$$

Proof. Define a normal form ϕ on M_1 by $\phi = \omega_{\xi_2}^{M_2} \circ \pi$. Deduce from the fact that ξ is cyclic and separating for M_1 that $p_{\xi_1}^{M_1} = p_{\xi_1}^{M_1'} = id_{\mathcal{H}_1}$.

Deduce from Theorem 4.7 that there exists a vector $\eta_1 \in \mathcal{H}_1$ such that $\phi = \omega_{\eta_1}^{M_1}$ and $p_{\eta_1}^{M_1} = p_{\eta_1}^{M_1'} = id_{\mathcal{H}_1}$.

The desired u is easily seen to be the unique unitary operator satisfying

$$u(x_1\eta_1) = \pi(x_1)\xi_2 \ \forall x_1 \in M_1.$$

□

We shall later try to formulate (at the end of the next lecture) the general version of this fact - with state replaced by weight - in such a way that this 'standard module' is very like the regular representation of a locally compact group!

5 The Tomita-Takesaki theorem

A measure μ can be considered a normal form on $L^\infty(\mu)$ only f it is finite. Thus limiting ourselves to normal states will result in our denying ourselves the liberty of using such naturl objects as Lebesgue measure on $\mathbb{R}$. Also, integration with respect to an infinite measure of a non-negative function leads to no problems of the kind invlved with such functions as $1_e - 1_F$ where both E and F have infinite measure.

Definition 5.1. *A* **weight** *on a von Neumann algebra M is a map $\phi : M^+ \to [0, \infty]$ satisfying: $\phi(\lambda x + y) = \lambda\phi(x) + \phi(y) \ \forall \lambda \geq 0, x, y \in M^+$ (with the understanding that $0 \times \infty = 0$). A weight ϕ is said to be*

1. *faithful if $\phi(x^*x) = 0 \Rightarrow x = 0$*
2. *normal if $\phi(\sup_i x_i) = \sup_i \phi(x_i)$ for every monotonically increasing net $\{x_i : i \in I\}$ in M^+.*
3. *a trace if $\phi(x^*x) = \phi(xx^*) \ \forall x \in M$.*

Fndamental to the study of a weight ϕ on M is three associated subsets of M defined as follows:

$$\mathcal{D}_\phi = \{x \in M^+ : \phi(x) < \infty\} \tag{5.11}$$

$$\mathcal{N}_\phi = \{x \in M : x^*x \in \mathcal{D}_\phi\} \tag{5.12}$$

$$\mathcal{M}_\phi = \mathcal{N}_\phi^*\mathcal{N}_\phi = \{\sum_{i=1}^n x_i^* y_i : x_i, y_i \in \mathcal{N}_\phi, n \in \mathbb{N}\} \tag{5.13}$$

Example 5.2. *1. Set $M = L^\infty(X,\mu)$ where μ is an infinite measure; here we see that $\mathcal{D}_\phi = L^\infty(x,\mu)^+ \cap L^1(X,\mu), \mathcal{N}_\phi = L^\infty(x,\mu) \cap L^2(X,\mu), \mathcal{M}_\phi = L^\infty(x,\mu) \cap L^1(X,\mu)$.*

2. *For a non-commutative example, we shall merely make some unproved statements, which are discussed in detail in Exercise 2.4.4 of [S]. Let $\mathcal{H}$ be a sepaable infinte-dimensional Hilbert space, and $M = \mathcal{B}(\mathcal{H})$.*

   (a) *If $x \in M$ then the quantity $\sum_{n=1}^\infty \|x\xi_n\|^2$ (as a number in $[0,\infty]$ is independent of the orthonormal basis $\{\xi_n\}$.*

   (b) *x is said to be a Hilbert-Schmidt operator if the value of the series in 1 above is finite; in that case, we write $\|x\|_2^2$ for the value of this series.*

   (c) *If $x \in M^+$, and if $\{\xi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$, then the series $\sum_{n=1}^\infty \langle x\xi_n, \xi_n\rangle$ converges if and only if $x^{\frac{1}{2}}$ is a Hilbert-Schmidt operator; and in particular, the quantity $\sum_{n=1}^\infty \langle x\xi_n, \xi_n\rangle$ (as a number in $[0,\infty]$ is independent of the choice of basis $\{\xi_n\}$*

   (d) *The equation $Tr\ x = \sum_{n=1}^\infty \langle x\xi_n, \xi_n\rangle$ dafines a faithful, normal, tracial weight on M.*

   (e) *$\mathcal{N}_{Tr}$ is the set of Hilbert-Schmidt opertors.*

   (f) *$\mathcal{M}_{Tr}$ coincides with the collection M_* of trace-class operators.*

The general weight satisfies many properties suggested by (and possessed by the 'trivial' commutative example 5.2(1). The not very difficult proof of this result may be found in [S] Proposition 2.4.5.

Proposition 5.3. *Let ϕ be a weight on M. With the notation, we have:*

1. *$\mathcal{D}_\phi$ is a hereditary positive cone, i.e., $x, y \in \mathcal{D}_\phi, z \in M^+, \lambda \geq 0$ and $z \leq x \Rightarrow z, \lambda x + y \in \mathcal{D}_\phi$;*

2. *$\mathcal{N}_\phi$ is a left ideal in M (not necessarily unital or closed in any topology); $\mathcal{M}_\phi$ is a self-adjoint subalgebra of M (again not necessarily unital or closed in any topology);*

3. *$\mathcal{D}_\phi = \mathcal{M}_\phi \cap M^+$, and in fact, any element of $\mathcal{M}_\phi$ is a complex linear combination of four elements of $\mathcal{D}_\phi$.*

4. *$x, z \in \mathcal{N}_\phi, y \in M \Rightarrow x^*yz \in \mathcal{M}_\phi$;*

5. *there exists a unique linear functional (which we shall also denote by ϕ) on $\mathcal{M}_\phi$ such that $\phi|_{\mathcal{D}_\phi} = \phi$.*

Exercises 2.4.7 and 2.4.8 in [S] outline a proof of the following result:

PROPOSITION 5.4. *The following conditions on a weight are equivalent:*

1. *$\mathcal{M}_\phi$ is σ-weakly dense in M.*

2. $1 = \sup\{e : e \in \mathcal{D}_\phi \cap \mathcal{P}(M)\}$

3. *there exists an increasing net $\{x_i : i \in I\} \subset \mathcal{D}_\phi$ such that $\|x_i\| \leq 1 \ \forall i$ and $1 = \lim_i x_i$.*

The GNS construction goes through smoothly for weights, as below:

PROPOSITION 5.5. *If ϕ is a faithful normal semifinite weight on M, there exists a triple $(\mathcal{H}_\phi, \pi_\phi, \eta_\phi)$ consisting of a Hilbert space $\mathcal{H}_\phi$, a normal representation $\pi_\phi : M \to \mathcal{B}(\mathcal{H}_\phi)$ and a linear map $\eta_\phi : \mathcal{N}_\phi \to \mathcal{H}_\phi$ such that whenever $x, y \in$
CN_ϕ and $z \in M$, we have*

1. $\langle \eta_\phi(x), \eta_\phi(y) \rangle = \phi(y^* x)$;

2. $\pi_\phi(z)\eta_\phi(x) = \eta_\phi(zx)$; *and*

3. *$\eta(\mathcal{N}_\phi)$ is dense in $\mathcal{H}_\phi$*

and the triple $(\mathcal{H}_\phi, \pi_\phi, \eta_\phi)$ is uniquely determined up to unitary equivalence by these three properties.

Further, π_ϕ is a normal isomorphism onto the von Neumann subalgebra $\pi_\phi(M)$ of $\mathcal{B}(\mathcal{H}_\phi)$.

As is natural, we shall write $L^2(M, \phi)$ for what was called $\mathcal{H}_\phi$ above and identify $\pi_\phi(x)$ with x, thereby viewing M as a von Neumann subalgebra of $\mathcal{B}(L^2(M, \phi))$; as can be expected, the Tomita-Takesaki theorem extends almost verbatim to this set-up thus:

THEOREM 5.6. **(The Tomita-Takesaki theorem for weights)** *Let ϕ be a faithful, normal semifinite weight on M. Consider the (obviously conjugate-linear) map S_0 mapping $\eta_\phi(\mathcal{N}_\phi \cap \mathcal{N}_\phi^*)$ onto itself defined by*

$$S_0(\eta_\phi(x)) = \eta_\phi(x^*) \ \forall x \in dom(S_0) \ .$$

1. *Then S_0 is a densely-defined conjugate-linear closable operator.*

2. *Let S denote its closure, and let $S = J\Delta^{\frac{1}{2}}$ be the polar decomposition of the closed and densely-defined conjugate-linear operator S - thus, J is conjugate-linear and $\Delta = (\Delta^{\frac{1}{2}})^2$ is the possibly unbounded positive self-adjoint operator $\Delta = S^*S$. Then*

   (a) *J is a anti-unitary involution ($J = J^* = J^{-1}$) satisfying $JMJ = M'$.*

   (b) *Δ is an injective self-adjoint operator with dense range, satisfying $J\Delta J = \Delta^{-1}$.*

   (c) *$\Delta^{it} M \Delta^{-it} = M \ \forall t$*

REMARK 5.7. 1. *The operator J (which should really be denoted by J_ϕ to denote its dependence on ϕ) is usually referred to as Tomita's modular conjugation operator, while Δ (actually Δ_ϕ) is called emerely the modular operator. The adjectie 'the' is not really appropriate since the dependence on ϕ of the operatrors in question is not non-eistent (see (2) below.*

2. *The following conditions on the weight ϕ are equivalent:*

   (a) *S_0 is norm-preserving*

   (b) *$S = J$*

   (c) *$\Delta = 1$*

   (d) *ϕ is a trace*

   Thus there could be two weights (even states in fact) with $\Delta_\phi = 1 \neq \Delta_\psi$!

3. *Each faithful normal weight ϕ on M is seen to induce a one-parameter group $\sigma^\phi = \{\sigma_t^\phi : t \in \mathbb{R}\}$ of automorphisms of M defined by*
$$\sigma_t^\phi(x) = \pi_\phi^{-1}(\Delta_\phi^{it}\pi_\phi(x)\Delta_\phi^{-it})$$
   and called the group of modular automorphisms.

4. *It is a consequence of Connes' 'unitary cocycle theorem' that these modular groups are 'outer equivalent' (in the sense that if ϕ and ψ are two faithful normal semifinite weights on M, then there exists a one-parameter family $\{u_t : t \in \mathbb{R}\} \subset \mathcal{U}(M)$ such that $\sigma_t^\psi(x) = u_t \sigma_t^\phi(x) u_t^* \ \forall t, x$.*

5. *It may not always be possible to explicitly define a weight on all positive elements of a von Neumann algebra, as in 'the group example' which we shall shortly discuss. One way out is to define it on a 'sufficiently large class' and prove that there exists a weight on the generated von Neumann algebra which agrees on the subclass with the earlier definition. A way of fomralising all this uses the language of 'left (or generalised) Hilbert algebras'.*

We now come to the all important group example. In view of the importance of this class of examples, we first discuss the case of countable discrete groups and then proceed to discuss the general locally compact case.

EXAMPLE 5.8. 1. *Let Γ be a countable group, and let $\ell^2(\Gamma)$ be the Hilbert space with orthonormal asis $\{\xi_t : t \in \Gamma\}$ - which can alternatively be thought of $L^2(\Gamma, \mu)$ where μ is the 'counting measure' defined on any subset of Γ as the cardinality (in $\{0, 1, 2, \cdots, \infty\}$) of that set (with $\xi_t = 1_{\{t\}}$). The so-called* **left-regular representation** *of Γ is the unitary representation $\lambda^\Gamma : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ defined by the requirement that $\lambda^\Gamma(s)\xi_t = \xi_{st}$. One defines the so-called* **group von Neumann algebra** *by the equation $L\Gamma = \lambda^\Gamma(\Gamma)''$. It is an important fact that the equation*

$$tr(x) = \langle x\xi_1, \xi_1 \rangle$$

defines a faithful (obviously normal) tracial state on $L\Gamma$.

2. *Let G be a locally compact group. It is then known that there exists a regular Borel measure μ (the so-called* **left Haar measure on** *G) defined on the Borel sets of G which satisfies and is uniquely determined up to a positive multiplicative constant by the following properties:*

   (a) *$\mu(K) < \infty$ for all compact sets;*
   (b) *$\mu(E) = \sup\{\mu(K) : K \subset E, K$ compact$\}$;*
   (c) *$\mu(E) = \mu(tE)$ for all $t \in G$ - i.e., μ is invariant under 'left-translations'. It is a classic fact that while right Haar measures exist just as well as left-Haar measures, some groups exhibit the uncomfortable feature that no left Haar measure is also a right Haar measure. Nevertheless it turns out that any left Haar measure is mutually absolutely continuous with respect to any right Haar measure.*

In fact, there turns out to exist a continuous (so-called modular) homomorphism $\Delta : G \to \mathbb{R}_+^\times$ such that

$$\int_G \xi(st)d\mu(s) = \Delta(t^{-1}) \int_G \xi(s)d\mu(s) \ \forall t \in G, \xi \in C_c(G).$$

In fact Δ is the 'Radon-Nikodym derivative of the inversion map' meaning that $\int_G \xi(s^{-1})ds = \int_G \xi(s)\Delta(s^{-1})ds \ \forall \xi \in C_c(G)$.

*It turns out that the following equations define a *-algebra structure on $C_c(G)$:*

$$\begin{aligned} (\xi \cdot \eta)(s) &= int_G \xi(t)\eta(t^{-1}s)d\mu(t) \\ \xi^*(s) &= \Delta(s^{-1})\overline{\xi(s^{-1})} \end{aligned}$$

*and that there exists a unique *-algebra representation $\pi : C_c(G) \to \mathcal{B}(L^2(G,\mu))$ with the property that $\pi(\xi)\eta = \xi \cdot \eta$. In fact, the equation*

$$(\lambda^G(s)f)(t) = f(s^{-1}t)$$

defines a strongly continuous unitary representation (the **left-regular representation***) of G and we have*

$$\langle \pi(\xi)f, g\rangle = \int_G \xi(s)\langle \lambda^G(s)f, g\rangle d\mu(s).$$

We define $LG = \lambda^G(G)'' = \pi(C_c(G))''$. Finally, it is true that there is a unique faithful normal semifinite weight ϕ on LG such that

$$\phi(\pi(\eta^* \cdot \xi)) = \int_G \xi\bar{\eta}d\mu \ .$$

Finally, LG is in standard form on $L^2(G,\mu)$ as explained below!

In order to formulate the full generalisation of Corollary 4.8, one needs to observe that if $M \subset \mathcal{B}(\mathcal{H})$ admits a cyclic and separating vector ξ, and if j is Tomita's modular conjugation operator, then $P = [\{xJx\xi : x \in M\}]$ is a self-dual cone in the sense that

$$\langle \zeta, \eta\rangle \geq 0 \ \forall \eta \in P \Leftrightarrow \zeta \in P \ .$$

It is further true that

$$\begin{aligned} JMJ &= M'. \\ JzJ &= z^* \ \forall z \in Z(M). \\ J\xi &= \xi \ \forall \xi \in P. \\ xJxJ(P) &\subset P \ \forall x \in M. \end{aligned} \tag{5.14}$$

A quadruple $(M, \mathcal{H}, J, P)$ (consisting of a von Neumann algebra M acting on a Hilbert space $\mathcal{H}$, anti-unitary involution J and self-dual cone P) which satisfies the four conditions of 5.14, is called a von Neumann algebra in standard form.

The result generalising Corollary 4.8 that we wish to state is the following uniqueness result (see [H]) for the standard form of a von Neumann algebra:

THEOREM 5.9. *If $(M_i, \mathcal{H}_i, J_i, P_i), i = 1, 2,$ are two standard forms, and if $\pi : M_1 \to M_2$ is a *-isomorphism, there exists a unique unitary operator $u : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\begin{aligned} \pi(x_1) &= ux_1u^* \ \forall x_1 \in M_1. \\ J_2 &= uJ_1u^*. \\ P_2 &= u(P_1). \end{aligned}$$

6 Factors

It is a useful thing to realise that a concrete von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ may be alternatively described as the set of intertwiners of a unitary representation of a group; i.e., M is a von Neumann algebra if and only if there exists a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{B}(\mathcal{H}))$ such that $M = \pi(G)'$. While such an M is clearly a von Neumann algebra, the converse implication is verified by defining π to be the identity map of the group $G = \mathcal{U}(M')$.

This point of view helps understand much of the philosophy and structure of von Neumann algebras. If $M = \pi(G)'$ as above, it is clear that a subspace $\mathcal{M}$ of $\mathcal{H}$ is a stable subspace and therefore defineas a sub-representation $\pi_0(g) = \pi(g)|_{\mathcal{M}}$, if and only if the orthogonal projection p onto $\mathcal{M}$ satisfies $p \in M$. Thus, sub-representations are in 1-1 correspondence with $\mathcal{P}(M)$. The proof of the following Proposition is another easy exercise:

PROPOSITION 6.1. *Let $M = \pi(G)'$ be as above.*

1. *The following conditions are equivalent:*

(a) π *is irreducible, i.e.,* $\mathcal{P}(M) = \{0, 1\}$

(b) $M' = \mathbb{C}.1$.

(c) $M = \mathcal{B}(\mathcal{H}_\pi)$*, where* $\mathcal{H}_\pi$ *is the Hilbert space underlying the representation* π.

2. *Let* $p_i \in \mathcal{P}(M_i), \mathcal{M}_i =$ *ran* p_i *and let* π_i *be the subrepresentation of* π *corresponding to the* G*-stable subspace* $\mathcal{M}_i$*, for* $i = 1, 2$. *Then the following conditions are equivalent:*

   (a) *there exists* $u \in M$ *such that* $u^*u = p_1, uu^* = p_2$.

   (b) *the representations* π_i *of* G *are unitarily equivalent.*

When these conditions are satisfied, the projections p_1 *and* p_2 *are said to be '(Murray-von Neumann) equivalent' and we write* $p_1 \cong p_2(rel\ M)$.

It follows eaily from the Cartesian decomposition in a *-algebra, the functional calculus for self-adjoint operators, and the uniform approximability of bounded measurable functions by simple functions, that M is the norm-closed subaspace spanned by $\mathcal{P}(M)$ It is not surprising therefore that von Neumann algebras are 'determined' by their lattice of projections. Consider a hybrid of the usual order relation order relation and Murray-von Neumann equivalence defined on $\mathcal{P}(M)$ thus: For $p, q \in \mathcal{P}(M)$ write $p \preceq_M q$ if there exists $p_0 \leq q$ such that $p \cong p_0(rel\ M)$.

Exercise 6.2. 1. *Verify that Murray-von Neumann equivalence is an equivalence relation on* $\mathcal{P}(M)$.

2. *Show that the center* $Z(M) = M \cap M'$ *is a von Neumann algebra.*

3. *If* $p \in \mathcal{P}(M)$*, the projection* $z_M(p)$ *defined by the equation* $z_M(p) = \inf\{e \in \mathcal{P}(Z(M)) : p \leq e\}$ *is called the* central cover *of* p. *Show that* $z_M(p) = \sup\{upu^* : u \in \mathcal{U}(M)\}$ *(by observing that the collection on the right is invariant under conjuhation by any element of* $\mathcal{U}(M)$ *and hence belongs to* M'*).*

4. *Show that if* $p, q \in \mathcal{P}(M)$*, then* $z_M(p)z_M(q) = 0$ *if and only if* $qxp = 0\ \forall x \in M$.

The next proposition identifies an important notion.

Proposition 6.3. *The following conditions on a von Neumann algebra* M *are equivalent:*

1. *If $p, q \in \mathcal{P}(M)$, then either $p \preceq q$ or $q \preceq p$.*

2. *$Z(M)$ $(= M \cap M') = \mathbb{C}1$.*

Such a von Neumann algebra is called a **factor**.

Proof. (1) $\Rightarrow$ (2) If there exists a non-zero $z \in \mathcal{P}(Z(M))$, set $p = z, q = 1 - z$, and observe that trivially $z = z_M(p)$ and $1 - z = z_M(q)$. Deduce now that if $u \in M$ and $u^*u = p$, then $u = up = uz$ and hence $qu = (1 - z)uz = 0$ (as $zu = uz$), and hence it is not true that $p \preceq q$; and an identical proof shows that neither is it true that $q \preceq p$. $\square$

The finite-dimensional case is instructive. Thus, suppose $M = \pi(G)'$, for a finite group G. To say that M is a factor is, by the above proposition, the same as saying that any two non-zero sub-representations of π have further non-zero sub-representations which are unitarily equivalent - which in the presence of the finiteness hypothesis is the same as saing that π is a multiple of an irreducible representation; thus $\pi(G)'$ is a factor precisely when π is **isotypical**.

If $\hat{G} = \{\pi_1, \cdots, \pi_k\}$ is a lising of the distinct irreducible representations of G, then it is a fact that any unitary representation of G is unitarily equivalent to $\oplus_{i=1}^k (\pi_i \oplus \cdots_{m_i \ terms} \oplus \pi_i)$ for uniquely determined 'multiplicities' $m_i = \langle \pi, \pi_i \rangle$. This fact and the integers featuring above have the following natural interpretations in terms of the von Neumann algebra $M = \pi(G)'$.

PROPOSITION 6.4. *Any finite-dimensional von Neumann algebra M admits a canonical* **central decomposition** *$M = \oplus M_i$ where M_i are factors with dimensions m_i^2.*

A purely von Neumann algebraic proof of this proposition is outlined in the following exercise.

EXERCISE 6.5. *Suppose M is a finite-dimensional von Neumann algebra.*

1. *Every projection $p \in \mathcal{P}(M)$ is expressible as $p = \sum_{i=1}^m e_i$, where the e_i's are minimal projections in M (i.e, if $e \in \mathcal{P}(M)$ and $e \leq e_i$, then $e \in \{0, e_i\}$.*

2. *Show that e is a minimal projection in M if and only if eMe $(= \{exe : x \in M\})$ $= \{\lambda e : \lambda \in \mathbb{C}\}$. (*Hint *Use the Cartesian decomposition, the spectral theorem of self-adjoint elements of M and the fact that $N = \{y \in M : eye \in \mathbb{C}.e\}$ is a weakly closed algebra to show that $N = [\mathcal{P}(N)] = \mathbb{C}.e$.)*

3. *Show that if $A = W^*x$ is the von Neumann subalgebra determined by a normal element $x \in M$, there exist $\{p_1 : 1 \leq i \leq n\}$ in $\mathcal{P}(A)$ and $\{\lambda_1 : 1 \leq i \leq n\}$ in $\mathbb{C}$ such that $x = \sum_{i=1}^n \lambda_i p_i$.*

4. *Suppose that M is a (still finite-dimensional) factor.*

   (a) *Use part (1) of this problem to deduce that there exist minimal projections $e_i, 1 \leq i \leq n$ such that $1 = \sum_{i=1}^n e_i$.*

   (b) *For each $1 \leq j \leq n$, deduce from Exercise 6.2 (4) that there exists an $x_j \in M$ such that $e_j x_j e_1 \neq 0$; if $x_j = u_j|x_j$ is the polar decomposition of x_j; deduce from the minimality of the e_i's that $u_j u_j^* = e_1$ and $u_j u_j^* = e_j$.*

   (c) *With the notation of part (b) of this exercise, define $e_{ij} = u_i u_j^*$ and show that $\{e_{ij} : 1 \leq i, j \leq n\}$ is a* **system of matrix units** *meaning that*

      i. $e_{ij}^* = e_{ji}$

      ii. $e_{ij} e_{kl} = \delta_{jk} e_{il}$

      iii. $\sum_{i=1}^n e_{ii} = 1$

   (d) *Deduce from the minimality of the e_i's that if $u, v \in M$ satisfy $u^*u = v^*v = e_q$ and $uu^* = vv^* = e_p$ for some p, q, then $u = \omega v$ for some $\omega \in \mathbb{T}$.*

   (e) *If $a \in M$, show that there exists $\alpha_{ij} \in \mathbb{C}$ such that $e_i a e_j = \alpha_{ij} e_{ij}$.*

   (f) *Define $\pi : M \to M_n(\mathbb{C})$ by setting $\pi(a) = ((\alpha_{ij}))$, if a and α_{ij} are related as in part(e) of this exercise. Verify that $\pi(e_{ij})$ is the matrix with the only non-zero entry being a 1 in the (i, j)-th entry, and deduce from part(c) that π is a *-homomorphism and from the fact that any normal homomorphism of a factor is injective, that $M \cong M_n(\mathbb{C})$.*

REMARK 6.6. *It is a fact that if M is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ and if $p \in \mathcal{P}(M)$ then $M_p :=: pMp$ is a concrete von Neumann subalgebra of $\mathcal{B}(p\mathcal{H})$ and that*

1. $(M_p)' = M'p$ $(= \{x'p : x' \in M'\}$; *and*

2. $Z(M_p) = Z(M)p$.

EXERCISE 6.7. *Suppose again that M is a finite-dimensional von Neumann algebra.*

1. *Verify that an intersection of any family of von Neumann subalgebras of a von Neumann algebra is also a von Neumann algebra, and in particular, so is $Z(M)$.*

2. *Show that any two distinct minimal projections of $Z(M)$ are orthogonal, and hence that the set $\{z_i : 1 \leq i \leq k\} \subset \mathcal{P}(Z(M))$ of minimal central projections of M is finite, and satisfies $\sum_{i=1}^k z_i = 1$.*

3. *Use Remark 6.6 and Exercise 6.5 4 to deduce that Mz_i is a factor and hence there exists $d_i \in \mathbb{N}$ such that $Mz_i \cong M_{d_i}(\mathbb{C})$*

4. *Conclude that $M \cong \oplus_{i=1}^k M_{d_i}(\mathbb{C})$.*

5. *If $M = \pi(G)'$, deduce from Remark 6.6 that if $z_i = \sum_{j=1}^{d_i} p_j^{(i)}$ is a decomposition of z_i as a sum of minimal projections of M (and necessarily also of Mz_i), then the equations*

$$\pi_i^{(j)}(g) = \pi(g)p_i^{(j)}$$

define irreducible representations of G into $\mathcal{B}(p_i^{(j)}\mathcal{H})$ such that $p_i^{(j)}$ and $p_s^{(r)}$ are equivalent if and only if $i = s$, and hence that $\pi = \oplus_{i=1}^k \oplus_{j=1}^{d_i} \pi_i^{(j)}$ is a decomposition of π into irreducible decomposition, with $\oplus_{j=1}^{d_i} \pi_i^{(j)}$ being isotypical representations for each $1 \leq i \leq k$.

Murray and von Neumann realised early that factors are the building blocks of von Neumann algebras. More generally than in Exercise 6.7 4, they showed that any von Neumann was expressible as a 'direct integral' of factors. In fact, it would not be improper to say that one of the first big advances in the theory was their preliminary classification of factors, using their 'order relation' $\preceq$ $(rel\ M)$
.

Definition 6.8. *A projection $p \in \mathcal{P}(M)$ is said to be* **finite** *if*

$$p \cong p_0 \ (relM) \ \text{and} \ \ p_0 \leq p \Rightarrow p_0 = p$$

A von Neumann algebra is said to be **finite** *if 1 is a finite projection in M, while it is said to be* **semi-finite** *if*

$$1 = \sup\{p \in \mathcal{P}(M) : p \text{ is finite}\}.$$

Definition 6.9. *A factor M is said to be:*

1. *of* **Type I** *if it has a non-zero minimal projection*

2. *of* **type II** *if it has no minimal projections but has non-zero finite projections,*

3. of **type III** *if it has no non-zero finite projections,*

Here are some facts about factors whose proofs, while not very difficult, can be found in [S].

1. If M is a type I factor, there exists a family $\{e_i : i \in I\}$ of minimal projections in M such that $1 = \sum_{i \in I} p_i$, and the cardinality of I - which is in $\{1, 2, \cdots, \infty\}$ by our standing separability assumption (on the predual of M) - is determined uniquely by M; it is a fact that $M \cong \mathcal{B}(\mathcal{H})$ where $dim\ \mathcal{H} = |I|$ and one says that M is of type $I_{|I|}$.

2. Any type II factor is semi-finite and a finite type II factor is said to be of type II_1 or type II_∞ accordingl to whether it is finite or not.

3. Thus a factor is not semi-finite precisely when it is of type III.

Semi-finite factors are akin to unimodular groups, in view of the following fact, whose proof is less trivial than the ones stated above.

REMARK 6.10. *A factor is finite (resp., semi-finite) if and only if it admits a faithful normal tracial state (resp., semifinite weight). Thus, with respect to some faithful normal state (resp., weight), the modular operator* Δ *is trivial.*

We conclude with the so-called crossed-product construction which simultaneously produces examples of factors of all sorts.

7 References

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