Classification of noncommutative 2-spheres

joint with
Wolfgang Lück, Chris Phillips, Samuel Walters

Satellite conference to ICM 2010 on Operator Algebras
Chennai, August 11, 2010.

Siegfried Echterhoff

Westfälische Wilhelms-Universität Münster

Overview of the lecture

- Introduce non-commutative 2-spheres as crossed products $A_\theta \rtimes F$ of the irrational rotation algebra A_θ by finite subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$.

Overview of the lecture

- Introduce non-commutative 2-spheres as crossed products $A_\theta \rtimes F$ of the irrational rotation algebra A_θ by finite subgroup $F \subseteq \mathrm{SL}(2, \mathbb{Z})$.
- Realize these algebras as twisted group algebras $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.

Overview of the lecture

- Introduce non-commutative 2-spheres as crossed products $A_\theta \rtimes F$ of the irrational rotation algebra A_θ by finite subgroup $F \subseteq \mathrm{SL}(2, \mathbb{Z})$.
- Realize these algebras as twisted group algebras $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Compute K-theory (with generators) with the help of the Baum-Connes conjecture.

Overview of the lecture

- Introduce non-commutative 2-spheres as crossed products $A_\theta \rtimes F$ of the irrational rotation algebra A_θ by finite subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$.
- Realize these algebras as twisted group algebras $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Compute K-theory (with generators) with the help of the Baum-Connes conjecture.
- Obtain complete classification of these algebras.

Overview of the lecture

- Introduce non-commutative 2-spheres as crossed products $A_\theta \rtimes F$ of the irrational rotation algebra A_θ by finite subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$.
- Realize these algebras as twisted group algebras $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Compute K-theory (with generators) with the help of the Baum-Connes conjecture.
- Obtain complete classification of these algebras.

This lecture is based on the paper

The structure of crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}(2,\mathbb{Z})$*.*
J. reine angew. Math. (Crelle’s Journal) 639 (2010), 173–221.
by S. E. , Wolfgang Lück, Chris Phillips, Sam Walters.

The commutative 2-sphere (as orbifold)

Consider the standard action of $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{T}^2$ given by

$$\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^{n_{11}} v^{n_{12}} \\ u^{n_{21}} v^{n_{22}} \end{pmatrix}.$$

The commutative 2-sphere (as orbifold)

Consider the standard action of $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ given by

$$\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^{n_{11}} v^{n_{12}} \\ u^{n_{21}} v^{n_{22}} \end{pmatrix}.$$

Restrict this action to the finite subgroups $F \subseteq \mathrm{SL}(2,\mathbb{Z})$.
Up to conjugacy, these are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with generators

$$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right).$$

The commutative 2-sphere (as orbifold)

Consider the standard action of $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$ given by

$$\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^{n_{11}} v^{n_{12}} \\ u^{n_{21}} v^{n_{22}} \end{pmatrix}.$$

Restrict this action to the finite subgroups $F \subseteq \mathrm{SL}(2,\mathbb{Z})$. Up to conjugacy, these are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ with generators

$$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right).$$

Then for all choices of $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ one gets

$$F \backslash \mathbb{T}^2 \cong S^2 \quad \text{hence} \quad C(\mathbb{T}^2)^F \cong C(F \backslash \mathbb{T}^2) \cong C(S^2).$$

Noncommutative 2-spheres

Let $A_\theta = C^*(u_\theta, v_\theta)$ the non-commutative 2-torus generated by unitaries u_θ, v_θ with relation $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. Then $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ acts on A_θ by

$$N \cdot u_\theta := e^{\pi i n_{11} n_{21}\theta} u_\theta^{n_{11}} v_\theta^{n_{21}} \qquad N \cdot v_\theta := e^{\pi i n_{12} n_{22}\theta} u_\theta^{n_{12}} v_\theta^{n_{22}}$$

Noncommutative 2-spheres

Let $A_\theta = C^*(u_\theta, v_\theta)$ the non-commutative 2-torus generated by unitaries u_θ, v_θ with relation $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. Then $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ acts on A_θ by

$$N \cdot u_\theta := e^{\pi i n_{11} n_{21}\theta} u_\theta^{n_{11}} v_\theta^{n_{21}} \qquad N \cdot v_\theta := e^{\pi i n_{12} n_{22}\theta} u_\theta^{n_{12}} v_\theta^{n_{22}}$$

Restrict this action to any of the finite subgroups $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \subseteq \mathrm{SL}(2, \mathbb{Z})$. We define the non-commutative 2-sphere as

$$A_\theta^F := \{a \in A_\theta : N \cdot a = a \; \forall N \in F\}.$$

Noncommutative 2-spheres

Let $A_\theta = C^*(u_\theta, v_\theta)$ the non-commutative 2-torus generated by unitaries u_θ, v_θ with relation $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. Then $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ acts on A_θ by

$$N \cdot u_\theta := e^{\pi i n_{11} n_{21} \theta} u_\theta^{n_{11}} v_\theta^{n_{21}} \qquad N \cdot v_\theta := e^{\pi i n_{12} n_{22} \theta} u_\theta^{n_{12}} v_\theta^{n_{22}}$$

Restrict this action to any of the finite subgroups $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \subseteq \mathrm{SL}(2,\mathbb{Z})$. We define the non-commutative 2-sphere as

$$A_\theta^F := \{a \in A_\theta : N \cdot a = a \; \forall N \in F\}.$$

If $\theta \in [0,1] \setminus \mathbb{Q}$, then: $\qquad A_\theta^F \sim_M A_\theta \rtimes F$.

Hence in this case we may also regard $A_\theta \rtimes F$ as a non-commutative version of S^2!

Noncommutative 2-spheres

Non-commutative 2-spheres have been studied (among others) by: **Bratteli, Elliott, Evans, Farsi, Kishimoto, Kumjian, C. Phillips, Polishchuk, Walters, Watling**

Noncommutative 2-spheres

Non-commutative 2-spheres have been studied (among others) by: **Bratteli, Elliott, Evans, Farsi, Kishimoto, Kumjian, C. Phillips, Polishchuk, Walters, Watling**

Theorem (Kumjian 90)

$$K_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6 \quad \text{and} \quad K_1(A_\theta \rtimes \mathbb{Z}_2) = \{0\} \quad \forall \theta$$

Noncommutative 2-spheres

Non-commutative 2-spheres have been studied (among others) by: **Bratteli, Elliott, Evans, Farsi, Kishimoto, Kumjian, C. Phillips, Polishchuk, Walters, Watling**

Theorem (Kumjian 90)

$$K_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6 \quad \text{and} \quad K_1(A_\theta \rtimes \mathbb{Z}_2) = \{0\} \quad \forall\theta$$

Theorem (Bratteli-Kishimoto 92)

$A_\theta^{\mathbb{Z}_2} \sim_M A_\theta \rtimes \mathbb{Z}_2$ is an AF-algebra for all $\theta \in [0,1]\backslash\mathbb{Q}$.

Noncommutative 2-spheres

Non-commutative 2-spheres have been studied (among others) by: **Bratteli, Elliott, Evans, Farsi, Kishimoto, Kumjian, C. Phillips, Polishchuk, Walters, Watling**

Theorem (Kumjian 90)

$$K_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6 \quad \text{and} \quad K_1(A_\theta \rtimes \mathbb{Z}_2) = \{0\} \quad \forall\theta$$

Theorem (Bratteli-Kishimoto 92)

$$A_\theta^{\mathbb{Z}_2} \sim_M A_\theta \rtimes \mathbb{Z}_2 \text{ is an AF-algebra for all } \theta \in [0,1]\backslash\mathbb{Q}.$$

Farsi-Watling 92–93

K-theory groups of $A_\theta \rtimes F$ for rational θ and all F.

Noncommutative 2-spheres

Non-commutative 2-spheres have been studied (among others) by: **Bratteli, Elliott, Evans, Farsi, Kishimoto, Kumjian, C. Phillips, Polishchuk, Walters, Watling**

Theorem (Kumjian 90)

$$K_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6 \quad \text{and} \quad K_1(A_\theta \rtimes \mathbb{Z}_2) = \{0\} \quad \forall \theta$$

Theorem (Bratteli-Kishimoto 92)

$A_\theta^{\mathbb{Z}_2} \sim_M A_\theta \rtimes \mathbb{Z}_2$ is an AF-algebra for all $\theta \in [0,1] \backslash \mathbb{Q}$.

Farsi-Watling 92–93

K-theory groups of $A_\theta \rtimes F$ for rational θ and all F.

Theorem (S. Walters 04)

$A_\theta^{\mathbb{Z}_4} \sim_M A_\theta \rtimes \mathbb{Z}_4$ is an AF-algebra for "almost all" $\theta \in [0,1] \backslash \mathbb{Q}$.

Noncommutative 2-spheres

Non-commutative 2-spheres have been studied (among others) by: **Bratteli, Elliott, Evans, Farsi, Kishimoto, Kumjian, C. Phillips, Polishchuk, Walters, Watling**

Theorem (Kumjian 90)

$$K_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6 \quad \text{and} \quad K_1(A_\theta \rtimes \mathbb{Z}_2) = \{0\} \quad \forall\theta$$

Theorem (Bratteli-Kishimoto 92)

$A_\theta^{\mathbb{Z}_2} \sim_M A_\theta \rtimes \mathbb{Z}_2$ is an AF-algebra for all $\theta \in [0,1]\setminus\mathbb{Q}$.

Farsi-Watling 92–93

K-theory groups of $A_\theta \rtimes F$ for rational θ and all F.

Theorem (S. Walters 04)

$A_\theta^{\mathbb{Z}_4} \sim_M A_\theta \rtimes \mathbb{Z}_4$ is an AF-algebra for "almost all" $\theta \in [0,1]\setminus\mathbb{Q}$.

Question: Is it true that $A_\theta \rtimes F$ is an AF-Algebra for all F and all irrational θ? Can we give a complete classification?

Problems (for irrational θ)

Problem 1 Show that $A_\theta \rtimes F$ is simple and classifiable with respect to the Elliott-programme!

Problem 2 Compute all relevant invariants (ordered K_0-groups, the K_1-group, traces).

Problems (for irrational θ)

Problem 1 Show that $A_\theta \rtimes F$ is simple and classifiable with respect to the Elliott-programme!

Problem 2 Compute all relevant invariants (ordered K_0-groups, the K_1-group, traces).

Theorem (ELPW, but due to Phillips) The action of F on A_θ satsifies the tracial Rokhlin property and (therefore) $A_\theta \rtimes F$ is simple with unique normalized trace τ and it is tracially AF.

Problems (for irrational θ)

Problem 1 Show that $A_\theta \rtimes F$ is simple and classifiable with respect to the Elliott-programme!

Problem 2 Compute all relevant invariants (ordered K_0-groups, the K_1-group, traces).

Theorem (ELPW, but due to Phillips) The action of F on A_θ satsifies the tracial Rokhlin property and (therefore) $A_\theta \rtimes F$ is simple with unique normalized trace τ and it is tracially AF.

Theorem ELPW (based on ideas of Tu) $A_\theta \rtimes F$ is KK-equivalent to a type I C*-algebra and therefore satisfies the UCT.

Problems (for irrational θ)

Problem 1 Show that $A_\theta \rtimes F$ is simple and classifiable with respect to the Elliott-programme!

Problem 2 Compute all relevant invariants (ordered K_0-groups, the K_1-group, traces).

Theorem (ELPW, but due to Phillips) The action of F on A_θ satsifies the tracial Rokhlin property and (therefore) $A_\theta \rtimes F$ is simple with unique normalized trace τ and it is tracially AF.

Theorem ELPW (based on ideas of Tu) $A_\theta \rtimes F$ is KK-equivalent to a type I C*-algebra and therefore satisfies the UCT.

Theorem (Huaxin Lin 2005) The above results imply that $A_\theta \rtimes F$ is classifiable.

Strategy for the solution of Problem 2

- Show that $A_\theta \rtimes F$ is isomorphic to a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.

Strategy for the solution of Problem 2

- Show that $A_\theta \rtimes F$ is isomorphic to a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Show that the K-theory of these twisted group algebras only depends on the homotopy class of the cocycle.

Strategy for the solution of Problem 2

- Show that $A_\theta \rtimes F$ is isomorphic to a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Show that the K-theory of these twisted group algebras only depends on the homotopy class of the cocycle.
- Show that all cocycles ω_θ are homotopic to the trivial cocycle.

Strategy for the solution of Problem 2

- Show that $A_\theta \rtimes F$ is isomorphic to a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Show that the K-theory of these twisted group algebras only depends on the homotopy class of the cocycle.
- Show that all cocycles ω_θ are homotopic to the trivial cocycle.
- Compute K-theory groups of $C^*(\mathbb{Z}^2 \rtimes F) \cong C(\mathbb{T}^2) \rtimes F$ with generators.

Strategy for the solution of Problem 2

- Show that $A_\theta \rtimes F$ is isomorphic to a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Show that the K-theory of these twisted group algebras only depends on the homotopy class of the cocycle.
- Show that all cocycles ω_θ are homotopic to the trivial cocycle.
- Compute K-theory groups of $C^*(\mathbb{Z}^2 \rtimes F) \cong C(\mathbb{T}^2) \rtimes F$ with generators.
- Use a refined version of item 2 to get generators for the K-theory groups of $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.

Strategy for the solution of Problem 2

- Show that $A_\theta \rtimes F$ is isomorphic to a twisted group algebra $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Show that the K-theory of these twisted group algebras only depends on the homotopy class of the cocycle.
- Show that all cocycles ω_θ are homotopic to the trivial cocycle.
- Compute K-theory groups of $C^*(\mathbb{Z}^2 \rtimes F) \cong C(\mathbb{T}^2) \rtimes F$ with generators.
- Use a refined version of item 2 to get generators for the K-theory groups of $C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta)$.
- Compute the image of $K_0(C^*(\mathbb{Z}^2 \rtimes F, \omega_\theta))$ by the unique trace τ.

Twisted group algebras

A 2-cocycle on the (discrete) group G is a map $\omega : G \times G \to \mathbb{T}$ such that

$$\omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r) \quad \forall s,t,r \in G.$$

Twisted group algebras

A 2-cocycle on the (discrete) group G is a map $\omega : G \times G \longrightarrow \mathbb{T}$ such that

$$\omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r) \quad \forall s,t,r \in G.$$

Define a representation $L_\omega : l^1(G) \to \mathcal{B}(l^2(G))$ via twisted convolution

$$(L_\omega(f)\xi)(g) = (f *_\omega \xi)(g) = \sum_{h \in G} f(h)g(h^{-1}g)\omega(h,h^{-1}g)$$

Twisted group algebras

A 2-cocycle on the (discrete) group G is a map $\omega : G \times G \to \mathbb{T}$ such that

$$\omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r) \quad \forall s,t,r \in G.$$

Define a representation $L_\omega : l^1(G) \to \mathcal{B}(l^2(G))$ via twisted convolution

$$(L_\omega(f)\xi)(g) = (f *_\omega \xi)(g) = \sum_{h \in G} f(h)g(h^{-1}g)\omega(h,h^{-1}g)$$

The (reduced) twisted group algebra of G with respect to ω is defined as

$$C^*_r(G,\omega) = \overline{L_\omega(l^1(G))} \subseteq \mathcal{B}(l^2(G)).$$

Twisted group algebras

A 2-cocycle on the (discrete) group G is a map $\omega : G \times G \to \mathbb{T}$ such that

$$\omega(s,t)\omega(st,r) = \omega(s,tr)\omega(t,r) \quad \forall s,t,r \in G.$$

Define a representation $L_\omega : l^1(G) \to \mathcal{B}(l^2(G))$ via twisted convolution

$$(L_\omega(f)\xi)(g) = (f *_\omega \xi)(g) = \sum_{h \in G} f(h)g(h^{-1}g)\omega(h,h^{-1}g)$$

The (reduced) twisted group algebra of G with respect to ω is defined as

$$C^*_r(G,\omega) = \overline{L_\omega(l^1(G))} \subseteq \mathcal{B}(l^2(G)).$$

(A similar construction works for locally compact groups G.)

Noncommutative Tori (the case $G = \mathbb{Z}^2$)

For $\theta \in \mathbb{R}$ define $\omega_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by

$$\omega_\theta\big((n,m),(n',m')\big) = \exp(\pi i \theta (nm' - mn')),$$

Noncommutative Tori (the case $G = \mathbb{Z}^2$)

For $\theta \in \mathbb{R}$ define $\omega_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by

$$\omega_\theta\big((n,m),(n',m')\big) = \exp(\pi i\theta(nm' - mn')),$$

Write $\delta_{n,m} \in l^1(\mathbb{Z}^2)$ for the Dirac-function at (n,m). Then

$$\delta_{(1,0)} *_{\omega_\theta} \delta_{(0,1)} = e^{2\pi i\theta}\big(\delta_{(0,1)} *_{\omega_\theta} \delta_{(1,0)}\big)$$

Noncommutative Tori (the case $G = \mathbb{Z}^2$)

For $\theta \in \mathbb{R}$ define $\omega_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by

$$\omega_\theta\big((n,m),(n',m')\big) = \exp(\pi i\theta(nm' - mn')),$$

Write $\delta_{n,m} \in l^1(\mathbb{Z}^2)$ for the Dirac-function at (n,m). Then

$$\delta_{(1,0)} *_{\omega_\theta} \delta_{(0,1)} = e^{2\pi i\theta}\big(\delta_{(0,1)} *_{\omega_\theta} \delta_{(1,0)}\big)$$

Therefore $C^*_r(\mathbb{Z}^2, \omega_\theta) = A_\theta$ (non-commutative 2-torus)

Noncommutative Tori (the case $G = \mathbb{Z}^2$)

For $\theta \in \mathbb{R}$ define $\omega_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by

$$\omega_\theta\big((n,m),(n',m')\big) = \exp(\pi i\theta(nm' - mn')),$$

Write $\delta_{n,m} \in l^1(\mathbb{Z}^2)$ for the Dirac-function at (n,m). Then

$$\delta_{(1,0)} *_{\omega_\theta} \delta_{(0,1)} = e^{2\pi i\theta}\big(\delta_{(0,1)} *_{\omega_\theta} \delta_{(1,0)}\big)$$

Therefore $C_r^*(\mathbb{Z}^2, \omega_\theta) = A_\theta$ (non-commutative 2-torus)

For $\theta = 0$, we get $A_0 \cong C_r^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$!

Noncommutative Tori (the case $G = \mathbb{Z}^2$)

For $\theta \in \mathbb{R}$ define $\omega_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by

$$\omega_\theta\big((n,m),(n',m')\big) = \exp(\pi i\theta(nm' - mn')),$$

Write $\delta_{n,m} \in l^1(\mathbb{Z}^2)$ for the Dirac-function at (n,m). Then

$$\delta_{(1,0)} *_{\omega_\theta} \delta_{(0,1)} = e^{2\pi i\theta}\big(\delta_{(0,1)} *_{\omega_\theta} \delta_{(1,0)}\big)$$

Therefore $C_r^*(\mathbb{Z}^2, \omega_\theta) = A_\theta$ (non-commutative 2-torus)

For $\theta = 0$, we get $A_0 \cong C_r^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$!

(Analogously for $n > 2$: $C_r^*(\mathbb{Z}^n, \omega) =$ non-commutative n-torus.)

Noncommutative 2-spheres as twisted group algebras

Consider the canonical action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$. For any (finite) subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$ form the semidirect product $\mathbb{Z}^2 \rtimes F$ with respect to this action. Then we get

Noncommutative 2-spheres as twisted group algebras

Consider the canonical action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$. For any (finite) subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$ form the semidirect product $\mathbb{Z}^2 \rtimes F$ with respect to this action. Then we get

Lemma For each $\theta \in \mathbb{R}$ we get a cocycle $\tilde{\omega}_\theta$ of $\mathbb{Z}^2 \rtimes F$ by

$$\tilde{\omega}_\theta \left(\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right), N \right), \left(\left(\begin{smallmatrix} n' \\ m' \end{smallmatrix} \right), N' \right) \right) = \omega_\theta \left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right), N \cdot \left(\begin{smallmatrix} n' \\ m' \end{smallmatrix} \right) \right)$$

and then $C^*_r(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta) \cong C^*_r(\mathbb{Z}^2, \omega_\theta) \rtimes F \cong A_\theta \rtimes F$.

Noncommutative 2-spheres as twisted group algebras

Consider the canonical action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$. For any (finite) subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$ form the semidirect product $\mathbb{Z}^2 \rtimes F$ with respect to this action. Then we get

Lemma For each $\theta \in \mathbb{R}$ we get a cocycle $\tilde{\omega}_\theta$ of $\mathbb{Z}^2 \rtimes F$ by

$$\tilde{\omega}_\theta\left(\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), N\right), \left(\left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right), N'\right)\right) = \omega_\theta\left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right), N \cdot \left(\begin{smallmatrix} n' \\ m' \end{smallmatrix}\right)\right)$$

and then $C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta) \cong C_r^*(\mathbb{Z}^2, \omega_\theta) \rtimes F \cong A_\theta \rtimes F$.

Of course, if $\theta = 0$, we get the trivial cocycle $\tilde{\omega}_0 \equiv 1$ and therefore

$$C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_0) = C_r^*(\mathbb{Z}^2 \rtimes F) \cong C(\mathbb{T}^2) \rtimes F$$

Homotopy of cocycles

Definition: Two cocycles $\omega_0, \omega_1 \in Z^2(G, \mathbb{T})$ are *homotopic*, if there exists a cocycle $\Omega : G \times G \to C([0,1], \mathbb{T})$ such that

$$\omega_0(g,h) = \Omega(g,h)(0) \quad \text{and} \quad \omega_1(g,h) = \Omega(g,h)(1) \quad \forall s,t \in G$$

Homotopy of cocycles

Definition: Two cocycles $\omega_0, \omega_1 \in Z^2(G, \mathbb{T})$ are *homotopic*, if there exists a cocycle $\Omega : G \times G \to C([0,1], \mathbb{T})$ such that

$$\omega_0(g,h) = \Omega(g,h)(0) \quad \text{and} \quad \omega_1(g,h) = \Omega(g,h)(1) \quad \forall s,t \in G$$

Example:

- $\Omega : \mathbb{Z}^2 \times \mathbb{Z}^2 \to C([0,1], \mathbb{T}); \Omega(\cdot,\cdot)(s) = \omega_{s \cdot \theta}$ is homotopy between $1 = \omega_0$ and ω_θ.
- Similarly $\tilde{\Omega} : (\mathbb{Z}^2 \rtimes F) \times \mathbb{Z}^2 \rtimes F) \to C([0,1], \mathbb{T})$ given by $\tilde{\Omega}(\cdot,\cdot)(s) := \tilde{\omega}_{s \cdot \theta}$ is a homotopy between $1 = \tilde{\omega}_0$ and $\tilde{\omega}_\theta$

The Baum-Connes conjecture

We say that G satisfies the Baum-Connes conjecture with coefficients if for every G-algebra A a certain map

$$K_*^{top}(G;A) = \lim_{X \subset \underline{E}G} KK_*^G(C_0(X), A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

The Baum-Connes conjecture

We say that G satisfies the Baum-Connes conjecture with coefficients if for every G-algebra A a certain map

$$K_*^{top}(G;A) = \lim_{X \subset \underline{EG}} KK_*^G(C_0(X), A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Theorem (Higson-Kasparov, 2001) Every a-T-menable (hence every amenable) group G satisfies the conjecture for all G-algebras A.

The Baum-Connes conjecture

We say that G satisfies the Baum-Connes conjecture with coefficients if for every G-algebra A a certain map

$$K_*^{top}(G;A) = \lim_{X\subset \underline{EG}} KK_*^G(C_0(X),A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Theorem (Higson-Kasparov, 2001) Every a-T-menable (hence every amenable) group G satisfies the conjecture for all G-algebras A.

Theorem (E-Chabert-Oyono-Oyono, 2004) Suppose that G satisfies the conjecture for all G-algebras A. Then, if $\alpha : A \to B$ is a G-equivariant $*$-homomorphism which induces an isomorphism

$$(\alpha \rtimes L)_* : K_*(A \rtimes L) \to K_*(B \rtimes L)$$

for all compact subgroups $L \subseteq G$, then we also have

$$(\alpha \rtimes G)_* : K_*(A \rtimes G) \xrightarrow{\cong} K_*(B \rtimes G).$$

Homotopy-invariance of K-theory

Let $\omega : G \times G \to \mathbb{T}$ be a cocycle on G. Then we get an action $\alpha_\omega : G \to \mathrm{Aut}(\mathcal{K}(l^2(G))); \alpha_\omega(g)(T) = L_\omega(\delta_g)TL_\omega(\delta_g)^*$ and

$$C_r^*(G,\omega) \otimes \mathcal{K} \cong \mathcal{K} \rtimes_{\alpha_\omega,r} G; f \otimes T \mapsto (g \mapsto TL_\omega(\delta_g)^*) \in C_c(G,A).$$

Homotopy-invariance of K-theory

Let $\omega : G \times G \to \mathbb{T}$ be a cocycle on G. Then we get an action $\alpha_\omega : G \to \mathrm{Aut}(\mathcal{K}(l^2(G))); \alpha_\omega(g)(T) = L_\omega(\delta_g) T L_\omega(\delta_g)^*$ and

$$C^*_r(G,\omega) \otimes \mathcal{K} \cong \mathcal{K} \rtimes_{\alpha_\omega, r} G; f \otimes T \mapsto (g \mapsto T L_\omega(\delta_g)^*) \in C_c(G, A).$$

Similarly: A homotopy $\Omega : G \times G \to C([0,1], \mathbb{T})$ between cocycles induces a fiber-wise action

$$\alpha_\Omega : G \to \mathrm{Aut}(C[0,1], \mathcal{K}); \alpha_\Omega(\varphi)(s) = \alpha_{\Omega(s)}(\varphi(s)).$$

Homotopy-invariance of K-theory

Let $\omega : G \times G \to \mathbb{T}$ be a cocycle on G. Then we get an action $\alpha_\omega : G \to \mathrm{Aut}(\mathcal{K}(l^2(G))); \alpha_\omega(g)(T) = L_\omega(\delta_g) T L_\omega(\delta_g)^*$ and

$$C_r^*(G,\omega) \otimes \mathcal{K} \cong \mathcal{K} \rtimes_{\alpha_\omega, r} G; f \otimes T \mapsto (g \mapsto T L_\omega(\delta_g)^*) \in C_c(G, A).$$

Similarly: A homotopy $\Omega : G \times G \to C([0,1], \mathbb{T})$ between cocycles induces a fiber-wise action

$$\alpha_\Omega : G \to \mathrm{Aut}(C[0,1], \mathcal{K}); \alpha_\Omega(\varphi)(s) = \alpha_{\Omega(s)}(\varphi(s)).$$

Theorem (E.-Williams, 98) Any fiber-wise action of a compact group L on $C([0,1], \mathcal{K})$ is equivalent to a constant action, i.e., there exists a single action $\beta : G \to \mathrm{Aut}(\mathcal{K})$ such that $\alpha \sim \mathrm{id}_{C[0,1]} \otimes \beta$. As a consequence, evaluation at any $\theta \in [0,1]$ induces an isomorphism

$$\mathrm{ev}_\theta : K_*(C([0,1], \mathcal{K}) \rtimes_\alpha L) \xrightarrow{\cong} K_*(\mathcal{K} \rtimes_{\alpha_\theta} L).$$

(use $K_*(C\big([0,1], \mathcal{K}) \rtimes_{\alpha_\theta} L) \cong K_*(C([0,1], \mathcal{K} \rtimes_\beta L))$

Homotopy invariance of K-theory

Theorem (E-L-P-W): Suppose that G satisfies the Baum-Connes conjecture (with coefficients). Then $K_*(C_r^*(G,\omega))$ only depends on the homotopy class of ω.

Homotopy invariance of K-theory

Theorem (E-L-P-W): Suppose that G satisfies the Baum-Connes conjecture (with coefficients). Then $K_*(C^*_r(G,\omega))$ only depends on the homotopy class of ω.

Corollary: For all $\theta \in [0,1]$, $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, we have

$$K_*(C^*_r(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta) \cong K_*(C^*_r(\mathbb{Z}^2 \rtimes F)) \quad \big(\cong K_*(C(\mathbb{T}^2) \rtimes F)\big).$$

Homotopy invariance of K-theory

Theorem (E-L-P-W): Suppose that G satisfies the Baum-Connes conjecture (with coefficients). Then $K_*(C_r^*(G,\omega))$ only depends on the homotopy class of ω.

Corollary: For all $\theta \in [0,1]$, $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, we have

$$K_*(C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta) \cong K_*(C_r^*(\mathbb{Z}^2 \rtimes F)) \quad \big(\cong K_*(C(\mathbb{T}^2) \rtimes F)\big).$$

$$K_0(C_r^*(\mathbb{Z}^2 \rtimes F)) = \begin{Bmatrix} \mathbb{Z}^6 & \text{for } F = \mathbb{Z}_2 \\ \mathbb{Z}^8 & \text{for } F = \mathbb{Z}_3 \\ \mathbb{Z}^9 & \text{for } F = \mathbb{Z}_4 \\ \mathbb{Z}^{10} & \text{for } F = \mathbb{Z}_6 \end{Bmatrix}, \quad K_1(C_r^*(\mathbb{Z}^2 \rtimes F)) = 0$$

Homotopy invariance of K-theory

Theorem (E-L-P-W): Suppose that G satisfies the Baum-Connes conjecture (with coefficients). Then $K_*(C_r^*(G,\omega))$ only depends on the homotopy class of ω.

Corollary: For all $\theta \in [0,1]$, $F = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$, we have

$$K_*(C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta) \cong K_*(C_r^*(\mathbb{Z}^2 \rtimes F)) \quad \big(\cong K_*(C(\mathbb{T}^2) \rtimes F)\big).$$

$$K_0(C_r^*(\mathbb{Z}^2 \rtimes F)) = \begin{Bmatrix} \mathbb{Z}^6 & \text{for } F = \mathbb{Z}_2 \\ \mathbb{Z}^8 & \text{for } F = \mathbb{Z}_3 \\ \mathbb{Z}^9 & \text{for } F = \mathbb{Z}_4 \\ \mathbb{Z}^{10} & \text{for } F = \mathbb{Z}_6 \end{Bmatrix}, \quad K_1(C_r^*(\mathbb{Z}^2 \rtimes F)) = 0$$

There is an alternative proof for the corollary by Skandalis!

The generators of $K_0(A_\theta \rtimes F)$

Let $\Omega : G \times G \to C([0,1], \mathbb{T})$ be a cocycle homotopy and write $\omega_\theta := \Omega(\cdot,\cdot)(\theta)$. Put $l^1(G,\Omega) := l^1(G, C([0,1]))$ with convolution

$$\varphi * \psi(g,\theta) := \sum_{h \in G} \varphi(h,\theta)\psi(h^{-1}g,\theta)\Omega(h,h^{-1}g)(\theta)$$

Define $C_r^*(G,\Omega) := \overline{l^1(G,\Omega)}^{C^*}$. One checks that

$$C([0,1],\mathcal{K}) \rtimes_{\alpha_\Omega} G \cong \mathcal{K} \otimes C_r^*(G,\Omega)$$

The generators of $K_0(A_\theta \rtimes F)$

Let $\Omega : G \times G \to C([0,1], \mathbb{T})$ be a cocycle homotopy and write $\omega_\theta := \Omega(\cdot,\cdot)(\theta)$. Put $l^1(G,\Omega) := l^1(G, C([0,1]))$ with convolution

$$\varphi * \psi(g,\theta) := \sum_{h \in G} \varphi(h,\theta)\psi(h^{-1}g,\theta)\Omega(h,h^{-1}g)(\theta)$$

Define $C_r^*(G,\Omega) := \overline{l^1(G,\Omega)}^{C^*}$. One checks that

$$C([0,1],\mathcal{K}) \rtimes_{\alpha_\Omega} G \cong \mathcal{K} \otimes C_r^*(G,\Omega)$$

Therefore, if G satsisfies the Baum-Connes conjecture, the canonical evaluation maps

$$q_\theta : C_r^*(G,\Omega) \to C_r^*(G,\omega_\theta); \varphi \mapsto \varphi(\cdot,\theta)$$

induce isomorphisms in K-theory.

Generators for $K_0(C_r^*(\mathbb{Z}^2 \rtimes F))$ for $F = \mathbb{Z}_3$.

Generators for $K_0(C_r^*(\mathbb{Z}^2 \rtimes F))$ for $F = \mathbb{Z}_3$.

$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ is generated by three elements u, v, w subject to

$$uv = vu, \quad w^3 = 1, \quad wuw^{-1} = u^{-1}v, \quad wvw^{-1} = u^{-1}.$$

Generators for $K_0(C_r^*(\mathbb{Z}^2 \rtimes F))$ for $F = \mathbb{Z}_3$.

$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ is generated by three elements u, v, w subject to

$$uv = vu, \quad w^3 = 1, \quad wuw^{-1} = u^{-1}v, \quad wvw^{-1} = u^{-1}.$$

There are three conjugacy classes of maximal finite subgroups:

$$M_1 =< w >, \quad M_2 :=< uw >, \quad M_3 :=< u^2 w >$$

Generators for $K_0(C_r^*(\mathbb{Z}^2 \rtimes F))$ for $F = \mathbb{Z}_3$.

$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ is generated by three elements u, v, w subject to

$$uv = vu, \quad w^3 = 1, \quad wuw^{-1} = u^{-1}v, \quad wvw^{-1} = u^{-1}.$$

There are three conjugacy classes of maximal finite subgroups:

$$M_1 =< w >, \quad M_2 :=< uw >, \quad M_3 :=< u^2w >$$

They provide 7 projections: $1, p_0, p_1, q_0, q_1, r_0, r_1 \in C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_3)$

$$p_0 = \tfrac{1}{3}(1 + w + w^2) \qquad q_0 = \tfrac{1}{3}(1 + uw + (uw)^2) \qquad r_0 = \tfrac{1}{3}(1 + u^2w + (u^2w)^2)$$
$$p_1 = \tfrac{1}{3}(1 + \zeta w + (\zeta w)^2) \qquad q_0 = \tfrac{1}{3}(1 + \zeta uw + (\zeta uw)^2) \qquad r_0 = \tfrac{1}{3}(1 + \zeta u^2w + (\zeta u^2w)^2)$$

$\zeta = e(\frac{1}{3})$ when $e(t) = \exp(2\pi i t)$.

Generators for $K_0(C_r^*(\mathbb{Z}^2 \rtimes F))$ for $F = \mathbb{Z}_3$.

$\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ is generated by three elements u, v, w subject to

$$uv = vu, \quad w^3 = 1, \quad wuw^{-1} = u^{-1}v, \quad wvw^{-1} = u^{-1}.$$

There are three conjugacy classes of maximal finite subgroups:

$$M_1 =< w >, \quad M_2 :=< uw >, \quad M_3 :=< u^2 w >$$

They provide 7 projections: $1, p_0, p_1, q_0, q_1, r_0, r_1 \in C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_3)$

$$p_0 = \tfrac{1}{3}(1 + w + w^2) \qquad q_0 = \tfrac{1}{3}(1 + uw + (uw)^2) \qquad r_0 = \tfrac{1}{3}(1 + u^2w + (u^2w)^2)$$
$$p_1 = \tfrac{1}{3}(1 + \zeta w + (\zeta w)^2) \qquad q_0 = \tfrac{1}{3}(1 + \zeta uw + (\zeta uw)^2) \qquad r_0 = \tfrac{1}{3}(1 + \zeta u^2w + (\zeta u^2w)^2)$$

$\zeta = e(\frac{1}{3})$ when $e(t) = \exp(2\pi i t)$. Bott class: $\mathcal{E} = \overline{\mathcal{S}(\mathbb{R})}$ w.r.t action

$$(\xi \cdot u)(s) = \xi(s+1), \quad (\xi \cdot v)(s) = e(s)\xi(s), \quad (\xi \cdot w)(s) = e(\frac{6s^2 - \pi}{12}) \int_{-\infty}^{\infty} \xi(x)e(sx)\,dx.$$

Application to non-commutative 2-spheres ($F = \mathbb{Z}_3$)

Let $a > 0, \quad \tilde{\Omega} \in Z^2(\mathbb{Z}^2 \rtimes F, C([a,1], \mathbb{T})), \quad \tilde{\Omega}(\cdot,\cdot)(\theta) = \tilde{\omega}_\theta$.

Application to non-commutative 2-spheres ($F = \mathbb{Z}_3$)

Let $a > 0$, $\tilde{\Omega} \in Z^2(\mathbb{Z}^2 \rtimes F, C([a,1],\mathbb{T}))$, $\tilde{\Omega}(\cdot,\cdot)(\theta) = \tilde{\omega}_\theta$.
Consider

$$(\theta \mapsto u_\theta, v_\theta, w_\theta) \in C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\Omega})$$

images of the generators u, v, w of $\mathbb{Z}^2 \rtimes F$ in $C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta)$.

Application to non-commutative 2-spheres ($F = \mathbb{Z}_3$)

Let $a > 0$, $\tilde{\Omega} \in Z^2(\mathbb{Z}^2 \rtimes F, C([a,1], \mathbb{T}))$, $\tilde{\Omega}(\cdot,\cdot)(\theta) = \tilde{\omega}_\theta$.
Consider

$$(\theta \mapsto u_\theta, v_\theta, w_\theta) \in C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\Omega})$$

images of the generators u, v, w of $\mathbb{Z}^2 \rtimes F$ in $C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta)$.

$$K_0(C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta)) = < [1], [p_0^\theta], [p_1^\theta], [q_0^\theta], [q_1^\theta], [r_0^\theta], [r_1^\theta], [\mathcal{E}(\theta)] >$$

$$p_0^\theta = \tfrac{1}{3}(1 + w_\theta + w_\theta^2), \qquad p_1^\theta = \tfrac{1}{3}(1 + e(\tfrac{2}{3})w_\theta + (e(\tfrac{2}{3})w_\theta)^2),$$

$$q_0^\theta = \tfrac{1}{3}(1 + e(\tfrac{2+\theta}{6})u_\theta w_\theta + (e(\tfrac{2+\theta}{6})u_\theta w_\theta)^2), \qquad q_1^\theta = \tfrac{1}{3}(1 + e(\tfrac{\theta}{6})u_\theta t_\theta + (e(\tfrac{\theta}{6})u_\theta t_\theta)^2),$$

$$r_0^\theta = \tfrac{1}{3}(1 + u_\theta^2 t_\theta + (u_\theta^2 t_\theta)^2), \qquad r_1^\theta = \tfrac{1}{3}(1 + e(\tfrac{2}{3})u_\theta^2 t_\theta + (e(\tfrac{2}{3})u_\theta^2 t_\theta)^2),$$

Application to non-commutative 2-spheres ($F = \mathbb{Z}_3$)

Let $a > 0$, $\tilde{\Omega} \in Z^2(\mathbb{Z}^2 \rtimes F, C([a,1], \mathbb{T}))$, $\tilde{\Omega}(\cdot,\cdot)(\theta) = \tilde{\omega}_\theta$.
Consider

$$(\theta \mapsto u_\theta, v_\theta, w_\theta) \in C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\Omega})$$

images of the generators u, v, w of $\mathbb{Z}^2 \rtimes F$ in $C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta)$.

$$K_0(C_r^*(\mathbb{Z}^2 \rtimes F, \tilde{\omega}_\theta)) = < [1], [p_0^\theta], [p_1^\theta], [q_0^\theta], [q_1^\theta], [r_0^\theta], [r_1^\theta], [\mathcal{E}(\theta)] >$$

$$p_0^\theta = \tfrac{1}{3}(1 + w_\theta + w_\theta^2), \qquad p_1^\theta = \tfrac{1}{3}(1 + e(\tfrac{2}{3})w_\theta + (e(\tfrac{2}{3})w_\theta)^2),$$

$$q_0^\theta = \tfrac{1}{3}(1 + e(\tfrac{2+\theta}{6})u_\theta w_\theta + (e(\tfrac{2+\theta}{6})u_\theta w_\theta)^2), \qquad q_1^\theta = \tfrac{1}{3}(1 + e(\tfrac{\theta}{6})u_\theta t_\theta + (e(\tfrac{\theta}{6})u_\theta t_\theta)^2),$$

$$r_0^\theta = \tfrac{1}{3}(1 + u_\theta^2 t_\theta + (u_\theta^2 t_\theta)^2), \qquad r_1^\theta = \tfrac{1}{3}(1 + e(\tfrac{2}{3})u_\theta^2 t_\theta + (e(\tfrac{2}{3})u_\theta^2 t_\theta)^2),$$

$\mathcal{E}(\theta) = \overline{\mathcal{S}(\mathbb{R})}$ with actions of generators

$$(\xi \cdot u_\theta)(s) = \xi(s+\theta), \quad (\xi \cdot v_\theta)(s) = e(s)\xi(s) \quad (\xi \cdot w_\theta)(s) = \frac{i^{-1/6}}{\sqrt{\theta}} e(\frac{s^2}{2\theta}) \int_{-\infty}^{\infty} \xi(x) e(\frac{sx}{\theta})\, dx.$$

The main theorem

Theorem (E., Lück, Phillips, Walters) Suppose $\theta \in (0,1]$ is an irrational number and that $F \subseteq \mathrm{SL}(2,\mathbb{Z})$ is finite subgroup. Then $A_\theta \rtimes F$ is always an AF-algebra. For all $\theta \in \mathbb{R}$ we have

$$K_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6, \quad K_0(A_\theta \rtimes \mathbb{Z}_3) \cong \mathbb{Z}^8$$

$$K_0(A_\theta \rtimes \mathbb{Z}_4) \cong \mathbb{Z}^9, \quad \text{and} \quad K_0(A_\theta \rtimes \mathbb{Z}_6) \cong \mathbb{Z}^{10}$$

If $F = \mathbb{Z}_k$, $k = 2,3,4,6$, then the image of $K_0(A_\theta \rtimes \mathbb{Z}_k)$ under the canonical (and unique) trace is $\frac{1}{k}(\mathbb{Z} + \theta\mathbb{Z})$. As a consequence,

$$A_\theta \rtimes \mathbb{Z}_k \cong A_{\theta'} \rtimes \mathbb{Z}_l \Leftrightarrow k = l \text{ and } \theta = \pm\theta' \mod \mathbb{Z}$$

.