Amalgamated free products of embeddable von Neumann algebras and sofic groups

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References

[BDJ] Nate Brown, K.D., Kenley Jung, "Free entropy dimension in amalgamated free products," *Proc. London Math. Soc.* (2008).

[CD] Benoit Collins, K.D., "Free products of sofic groups with amalgamation over amenable groups," preprint.

Approximation properties in finite von Neumann algebras

Hyperfiniteness

A von Neumann algebra $\mathcal{M}$ is *hyperfinite* if for all $x_1, \ldots x_n \in \mathcal{M}$ and all $\epsilon > 0$ there is a finite dimensional subalgebra $D \subseteq \mathcal{M}$ such that $\mathrm{dist}_{\|\cdot\|_2}(x_j, D) < \epsilon$ (for all j), where $\|a\|_2 = \tau(a^*a)^{1/2}$.

For example, the hyperfinite II$_1$–factor $R = \overline{\bigcup_{n\geq 1} M_{2^n}(\mathbf{C})}$ or $L(G)$ for G amenable [Connes, '76].

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Connes' Embedding Problem (CEP) [1976]

Do all finite von Neumann algebras $\mathcal{M}$ having separable predual embed into R^ω, (the ultrapower of the hyperfinite II_1–factor)?

A reformulation of Connes' embedding problem:

We take a finite von Neumann algebra $\mathcal{M}$ with a fixed trace $\tau : \mathcal{M} \to \mathbf{C}$, with $\tau(1) = 1$. Also, $\mathcal{M}_{s.a.} = \{x \in \mathcal{M} \mid x^* = x\}$.

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I.e., given $m \in \mathbf{N}$ and $\epsilon > 0$, are there $a_1, \ldots, a_n \in M_k(\mathbb{C})_{s.a.}$ for some $k \in \mathbf{N}$ whose mixed moments up to order m are ϵ–close to those of $x_1, \ldots, x_n$?

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$$\left| \mathrm{tr}_k(a_{i_1} a_{i_2} \cdots a_{i_p}) - \tau(x_{i_1} x_{i_2} \cdots x_{i_p}) \right| < \gamma$$

for all $p \le m$ and all $i_1, \ldots, i_p \in \{1, \ldots, n\}$? (The existence of such matricial microstates is equivalent to $\mathcal{M}$ embedding in R^ω, written $\mathcal{M} \hookrightarrow R^\omega$.)

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In fact, CEP $\Leftrightarrow$ the case $n = 2$ ([Collins, D. '08]).

Microstates free entropy dimension (Voiculescu)

$\Gamma_R(x_1, \ldots, x_n; m, k, \gamma)$ is the set of all n–tuples $(a_1, \ldots, a_n)$ of such approximating matricial microstates, having $\|a_i\| \leq R$.

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To save space, we will write X for the list (or set) $x_1, \ldots, x_n$, and also $\Gamma_R(X; m, k, \gamma)$, etc.

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By [Jung, '03]:

$$\mathbb{P}_\epsilon(X) = \sup_{R>0} \inf_{\substack{m\geq 1 \\ \gamma>0}} \limsup_{k\to\infty} k^{-2} \log P_\epsilon\big(\Gamma_R(X; m, k, \gamma)\big).$$

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$$\delta_0(X)=\limsup_{\epsilon\to 0}\frac{\mathbb{P}_\epsilon(X)}{|\log\epsilon|}.$$

Instead of taking $\sup_{R>0}$, fixing any $R>\max_i\|x_i\|$ will yield the same value for $\delta_0(X)$, and we can also take $R=+\infty$, in which case we write $\Gamma(X;m,k,\gamma)$.

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Proof:

$$\Gamma_R(X \cup Y; m, k, \gamma) \subseteq \Gamma_R(X; m, k, \gamma) \times \Gamma_R(Y; m, k, \gamma).$$

Open problems about matricial microstates

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2. W*–invariance

Does $W^*(x_1, \ldots, x_N) = W^*(y_1, \ldots, y_M)$ imply $\delta_0(x_1, \ldots, x_N) = \delta_0(y_1, \ldots, y_M)$?

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If, in Jung's formula for δ_0, the $\limsup_{k\to\infty}$ and $\limsup_{\epsilon\to 0}$ are replaced by $\liminf$, do we get the same number?

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(If "yes," then we say X is *microstates packing regular*.)

Recalling of the formula for $\delta_0(X)$

$$\mathbb{P}_\epsilon(X) = \sup_{R>0} \inf_{\substack{m\geq 1\\ \gamma>0}} \limsup_{k\to\infty} k^{-2} \log P_\epsilon\big(\Gamma_R(X;m,k,\gamma)\big).$$

$$\delta_0(X) = \limsup_{\epsilon\to 0} \frac{\mathbb{P}_\epsilon(X)}{|\log\epsilon|}.$$

Regarding W^*–invariance:

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This number can be written $\delta_0(B)$, and satisfies $0 \leq \delta_0(B) \leq 1$, with equality on the left if and only if $B = \mathbb{C}$ and equality on the right if and only if B is diffuse, i.e., has no minimal projections.

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$*$–algebra invariance [Voiculescu]

If $*\text{-alg}(x_1, \ldots, x_N) = *\text{-alg}(y_1, \ldots, y_M)$, then $\delta_0(x_1, \ldots, x_N) = \delta_0(y_1, \ldots, y_M)$.

Regarding regularity

Thm. [Voiculescu]

If $x_1, \ldots, x_n$ are free, then $\delta_0(X) = \delta_0(x_1, \ldots, x_n) = \sum_1^n \delta_0(x_j)$.

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If $X = \{x_1, \ldots, x_N\}$ and $Y = \{y_1, \ldots, y_M\}$ are free and if at least one is regular, then $\delta_0(X \cup Y) = \delta_0(X) + \delta_0(Y)$.

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Thm. [BDJ]

Let $\mathcal{M} = W^*(X)$. If either (a) $\mathcal{M}$ is diffuse, is embeddable in R^ω and $\delta_0(X) = 1$ or (b) $\mathcal{M}$ is hyperfinite, then X is regular.

Regarding Connes' embedding problem and free products

Without a regularity assumption, we do not know if $\delta_0(X \cup Y) = \delta_0(X) + \delta_0(Y)$ holds whenever X and Y are free sets of finitely many self–adjoints.

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However, if one assumes $\delta_0(X) \geq 0$ and $\delta_0(Y) \geq 0$, i.e., that $W^*(X) \hookrightarrow R^\omega$ and $W^*(Y) \hookrightarrow R^\omega$, then one can construct sufficiently many approximating microstates for $X \cup Y$ to prove that $W^*(X \cup Y) = W^*(X) * W^*(Y) \hookrightarrow R^\omega$, i.e., that $\delta_0(X \cup Y) \geq 0$.

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How?

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How? By a fundamental result of Voiculescu, given m, γ, there are m', γ' such that if

$$a = (a_1, \ldots, a_N) \in \Gamma_R(X; m', k, \gamma')$$
$$b = (b_1, \ldots, b_M) \in \Gamma_R(Y; m', k, \gamma'),$$

and if $u \in \mathcal{U}_k$ is a *randomly chosen* $k \times k$ unitary matrix, then with probability $P(R, m, \gamma, k)$, that approaches 1 as $k \to \infty$, $a \cup ubu^* \in \Gamma_R(X \cup Y; m, k, \gamma)$.

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If $B \subseteq A_i \subseteq A$ are subalgebras, then the A_i are *free* with respect to E (over B) if

$$E(a_1 \cdots a_n) = 0 \text{ whenever } a_j \in A_{i(j)} \cap \ker E \text{ and } i(j) \neq i(j+1) \text{ for all } j.$$

Amalgamated free products of von Neumann algebras [Voiculescu]

Given $E_i : A_i \to B$ conditional expectations (with faithful GNS construction), then their amaglamated free product is

$$(A, E) = \underset{i \in I}{*_B}(A_i, E_i),$$

with $A_i \hookrightarrow A$ so that the A_i are free over B and together generate A, and $E\restriction_{A_i} = E_i$.

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In this case, we say the amalgamated free product is *tracial*.

Example of an amalgamated free product of von Neumann algebras

Example: if $H \subseteq G_i$ and $G = G_1 *_H G_2$ is an amalgamated free product of groups, then

$$(L(G_1), E_1) *_{L(H)} (L(G_2), E_2) = (L(G), E),$$

where E_i and E are the cannonical–trace–preserving conditional expectations onto $L(H)$.

Free entropy dimension in amalg. free products [BDJ]

The setting: let $(\mathcal{M}, E) = (A_1, E) *_B (A_2, E)$ be a tracial amalgamated free product, where B is hyperfinite. Suppose $X_i \subseteq A_i$ and $Y \subseteq B$ are finite sets of self–adjoint elements, where $W^*(Y) = B$.

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By [Jung, '03],

$$\delta_0(X_1 \cup X_2 \cup Y) \leq \delta_0(X_1 \cup Y) + \delta_0(X_2 \cup Y) - \delta_0(Y).$$

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Theorem [BDJ]

If at least one of $X_1 \cup Y$ and $X_2 \cup Y$ is regular, then

$$\delta_0(X_1 \cup X_2 \cup Y) = \delta_0(X_1 \cup Y) + \delta_0(X_2 \cup Y) - \delta_0(Y),$$

while if both are regular then also $X_1 \cup X_2 \cup Y$ is regular.

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Let $a \cup c \in \Gamma_{R'}(X_1 \cup Y; m', k, \gamma')$ and $b \cup c' \in \Gamma_{R'}(X_2 \cup Y; m', k, \gamma')$. We may take $c' = c = \pi_k(Y)$.

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Now, choosing u randomly in $\mathcal{U}_k \cap \pi_k(B)'$ we have, with probability approaching 1 as $k \to \infty$,

$$a \cup ubu^* \cup c \in \Gamma_r(X_1 \cup X_2 \cup Y; m, k, \gamma).$$

Embeddings in R^ω

Also, even without assuming regularity, this argument is sufficient to construct at least some approximating microstates, enough to give R^ω–embeddability.

Theorem [BDJ]

If $(\mathcal{M}, E) = (A_1, E) *_B (A_2, E)$ is a tracial amalgamated free product with B hyperfinite, and if $A_i \hookrightarrow R^\omega$, $(i = 1, 2)$, then $\mathcal{M} \hookrightarrow R^\omega$.

Hyperlinear groups

Definition [Rădulescu]

A group Γ is hyperlinear if for all finite sets $F \subseteq \Gamma$ and all $\epsilon > 0$, there is a map $\phi : \Gamma \to \mathcal{U}_n$ (the $n \times n$ unitary matrices) for some n, such that

(i) $\forall g \in F \backslash \{e\}$, $\mathrm{dist}(\phi(g), \mathrm{id}) > 1 - \epsilon$

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Theorem [Rădulescu]

For a group Γ, TFAE:

(i) Γ is hyperlinear

(ii) Γ is isomorphic to a subgroup of the unitary group of R^ω

(iii) $L(\Gamma) \hookrightarrow R^\omega$

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Open Problem (part of Connes' Embedding Problem)

Are all groups hyperlinear?

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Examples

• amenable groups • residually finite groups • residually amenable groups • other recent examples by [A. Thom], [Y. Cornulier].

Sofic groups (2)

Some nice properties of every sofic group Γ

- satisfies Gottschalk's Surjunctivity Conjecture [Gromov '99].
- satisfies Kaplansky's Direct Finiteness Conjecture [Elek, Szabó, '04].
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Caveat [Gromov]

Any statement about all countable groups is either trivial or false.

Sofic groups (3)

Constructions

The class of sofic groups is closed under taking • subgroups • direct limits • direct products • inverse limits • extensions by amenable groups [Elek and Szabó, '06] • free products [Elek and Szabó, '06].

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If Γ_1 and Γ_2 are sofic groups and if $H \subseteq \Gamma_i$ is a subgroup that is either a finite group or infinite cyclic or $\ldots$, then the amalgamated free product $\Gamma_1 *_H \Gamma_2$ is sofic.

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We thought we had a proof for H amenable, but there are some problems $\ldots$.

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$$\phi_1(g_1)\big(U\phi_2(g_2)U^{-1}\big)\cdots\phi_1(g_{2m-1})\big(U\phi_2(g_{2m})U^{-1}\big)$$

is vanishingly small, (taking $g_{\text{odd}} \in F_1 \backslash \{e\}$, $g_{\text{even}} \in F_2 \backslash \{e\}$ and $\epsilon_n \to 0$).

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This shows: from quasi–actions ϕ_1 and ϕ_2 of G_1 and G_2, we get sufficiently many quasi–actions $\phi_1 * \big(U\phi_2(\,\cdot\,)U^{-1}\big)$ of $G_1 * G_2$.