In lieu of an apology for my constant cribbing in Vasanth's lectures

(A couple of comments before wading into the math discussion in defense of my constant heckling of Vasanth in his discussion of what he called $\lambda(G)$ and M. I want to use slightly different notation: while Vasanth used $\lambda(G)$ for a von Neumann algebra, I like to reserve Γ for countable groups and G for possibly non-discrete topological groups, and for me, $\lambda(\Gamma)$ will denote the group which is the image under the left regular representation of Γ and $L\Gamma$ will be the generated von Neumann algebra. Also, while Vasanth discussed the standard form only for II_1 factors, that notion and subsequent analysis ($JMJ = M'$, etc.) make perfectly good sense for any von Neumann algebra with a faithful normal tracial state. And finally, like Kunal, I will write $\{\delta_t : t \in \Gamma\}$ for the canonical ONB for $\ell^2(\Gamma)$.)

Let Γ be a countable discrete group, and let $\ell^2(\Gamma)$ be the (separable) Hilbert space with orthonormal basis given by $\{\delta_t : t \in \Gamma\}$. Let λ, ρ denote the **left-** and **right-regular** (unitary) representations of Γ on $\ell^2(\Gamma)$ defined by

$$\lambda(s)\delta_t = \delta_{st}, \rho(s)\delta_t = \delta_{ts^{-1}}.$$

Then $L\Gamma =: \lambda(\Gamma)''$ is a von Neumann algebra . Note that $\lambda(\Gamma) \subset \rho(\Gamma)'$, and so $L\Gamma \subset \rho(\Gamma)'$. Clearly δ_1 is a cyclic vector for $L\Gamma$ as well as for $\rho(\Gamma)''$ (since $\lambda(\Gamma)\delta_1 = \rho(\Gamma)\delta_1 = \{\delta_t : t \in \Gamma\}$) and hence also for $(L\Gamma)'$. Thus δ_1 is a cyclic and separating vector for $L\Gamma$. Also

$$\langle \lambda(s)^*\lambda(t)\delta_1, \delta_1\rangle = \delta_{s,t} = \langle \lambda(t)\lambda(s)^*\delta_1, \delta_1\rangle \ ,$$

so we find that and the equation $tr(x) = \langle x\delta_1, \delta_1\rangle$ defines a faithful normal tracial statetr on $L\Gamma$, which is thus a finite von Neumann algebra in standard form.

If J denotes the modular conjugation operator $J_{L\Gamma}$, we have, by definition,

$$J\delta_t = J\lambda(t)\delta_1 = \lambda(t)^*\delta_1 = \delta_{t^{-1}}$$

and so,

$$J\lambda(s)J\delta_t = J\lambda(s)\delta_{t^{-1}} = J\delta_{st^{-1}} = \delta_{ts^{-1}} = \rho(s)\delta_t$$

and $J\lambda(G)J = \rho(G)$.

We may deduce that $\rho(G)'' = \lambda(\Gamma)' = (L\Gamma)'$.

Note next that each $x \in B(\ell^2(\Gamma))$ has a natural representation as a matrix with respect to the orthonormal basis $\{\delta_t\}_t$. Thus $x = ((x(s,t)))$ where

$$x(s,t) = \langle x\delta_t, \delta_s \rangle.$$

Direct computation shows that for arbitrary $u, s, t \in \Gamma$ we have

$$\begin{aligned}(\rho(u)^* x \rho(u))(s,t) &= \langle \rho(u)^* x \rho(u)\delta_t, \delta_s \rangle \\ &= \langle x\rho(u)\delta_t, \rho(u)\delta_s \rangle \\ &= \langle x\delta_{tu^{-1}}, \delta_{su^{-1}} \rangle \\ &= x(su^{-1}, tu^{-1})\end{aligned}$$

and hence,

$$x \in (\rho(\Sigma)' \Leftrightarrow x(s,t) = x(su^{-1}, tu^{-1}) \; \forall s,t,u \Leftrightarrow x(s,t) = x(st^{-1}, 1) \; \forall s,t.$$

In other words, the following conditions on an $x \in B(\ell^2(\Gamma))$ are equivalent:

1. $x \in L\Gamma$.
2. $x \in \rho(\Gamma)'$.
3. The matrix coefficients of x satisfy $x(s,t) = x(st^{-1}, 1) \; \forall s,t$.

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The purpose of the following result is to show that - even when $\Gamma = \mathbb{Z}$ - one should be wary of hastily deducing from the previous assertion that one may write $x = \sum_{s\in\Gamma} x(s,1)\lambda(s)$ or interprest the 'series' on the right as an SOT (or WOT) limit of a sequence $\sum_{s\in G_n} x(s,1)\lambda(s)$ where G_n are finite sets increasing to Γ.

PROPOSITION 0.1. *There exists $f \in C(\mathbb{T})$ such that $sup_n \|S_n f\|_\infty = \infty$ and in particular it is not true that $\|(S_n f)g - fg\|_2 \to 0 \forall g \in L^2(\mathbb{T})$. (i.e., the Fourier series of elements of $C(\mathbb{T})$ need not converge in the strong operator topology of $B(L^2(\mathbb{T}, \frac{1}{2\pi}d\theta))$; here, of course, $S_n f$ denotes the n-th partial sum of the Fourier series of f.)*

Proof. [1] On the contrary, suppose $sup_n \|S_n f\|_\infty < \infty$ for all $f \in C(\mathbb{T})$. Then, in particular, we should have $sup_n \|(S_n f)(0)\| < \infty$ for all $f \in C(\mathbb{T})$. If we define $\phi_n \in C(\mathbb{T})^*$ by $\phi_n(f) = (S_n f)(0)$, we find

[1]I must thank Narayanan (IISc) for this neat proof.

that $\{\phi_n(f) : n \in \mathbb{N}\}$ is bounded for every $f \in C(\mathbb{T})$, and hence, by the uniform boundedness principle, that $sup_n \|\phi_n\|_{C(\mathbb{T})^*} < \infty$.

On the other hand $\phi_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})D_n(e^{i\theta})d\theta$, where D_n denotes the 'Dirichlet kernel' given by $D_n(e^{i\theta}) = \sum_{k=-n}^{n} e^{ik\theta}$; and so

$$\|\phi_n\|_{C(\mathbb{T})^*} = \frac{1}{2\pi}\int_0^{2\pi} |D_n(e^{i\theta})|d\theta$$

while it is a well-known fact - see $https://en.wikipedia.org/wiki/Dirichlet\_kernel$ - that $sup_n \int_0^{2\pi} |D_n(e^{i\theta})|d\theta = +\infty$. Thus we have arrived at a contradiction, and the proposition is proved. □

REMARK 0.2. The point is that any 'bad' $f \in C(\mathbb{T})$ as in Proposition 0.1 will be such that the inverse Fourier transform $f^\vee$ is an element of ℓ^2 which is a bounded 'convolver' - because $f = \widehat{(f^\vee)} \in C(\mathbb{T}) \subset L^\infty(\mathbb{T})$.