On extendability of endomorphisms and of E_0-semigroups on factors

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Abstract

We examine what it means to say that certain endomorphisms of a factor (which we call equi-modular) are extendable. We obtain several conditions on an equi-modular endomorphism, and single out one of them with a purely 'subfactor flavour' as a theorem. We then exhibit the obvious example of endomorphisms satifying the condition in this theorem. We use our theorem to determine when every endomorphism in an E_0-semigroup on a factor is extendable - which property is easily seen to be a cocycle-conjugacy invariant of the E_0-semigroup. We conclude by giving examples of extendable E_0-semigroups, and by showing that the Clifford flow on the hyperfinite II_1 factor is not extendable, neither is the free flow.

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1 Introduction

We begin this note with a von Neumann algebraic version of the elementary but extremely useful fact about being able to extend inner-product preserving maps from a total set of the domain Hilbert space to an isometry defined on the entire domain. This leads us to the notion of when a well-behaved (equi-modular, as we term it) endomorphism of a factorial probability space (M, ϕ) admits a natural extension to an endomorphism of $L^2(M, \phi)$. After deriving some equivalent conditions under which an endomorphism is extendable, we exhibit examples of such extendable endomorphisms.

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We then pass to E_0-semigroups $\alpha = \{\alpha_t : t \geq 0\}$ of factors, and observe that extendability of this semigroup (i.e., extendability of each α_t) is a cocycle-conjugacy invariant of the semigroup. We identify a necessary condition for extendability of such an E_0-semigroup, which we then use to show that the Clifford flow on the hyperfinite II_1 factor is not extendable.

Our notion of extendable E_0-semigroups is related to a notion called 'regular semigroups' in [ABS], where they erroneously claim to prove that the Clifford flow is extendable.

We start by setting up some notation. For any index set I, we write $I^* = \bigcup_{n=0}^{\infty} I^n$ where $I^0 = \emptyset$, and $\mathbf{i} \vee \mathbf{j} = (i_1, \cdots, i_m) \vee (j_1, \cdots, j_n) = (i_1, \cdots, i_m, j_1, \cdots, j_n)$ whenever $\mathbf{i} = (i_1, \cdots, i_m), \mathbf{j} = (j_1, \cdots, j_n) \in I^*$.

By a von Neumann probability space, we shall mean a pair (M, ϕ) consisting of a von Neumann algebra and a normal state. For such an (M, ϕ), and an $x \in M$, we shall write $\hat{x} = \lambda_M(x)\widehat{1_M}$ and $\widehat{1}_M$ for the cyclic vector for $\lambda_M(M)$ in $L^2(M, \phi)$.

Recall that the central support of the normal state ϕ is the central projection $z(=: z_\phi)$ such that $ker(\lambda_M) = M(1-z)$. Clearly $z_\phi = 1_M$ if M is a factor.

Finally, if $\{x_i : i \in I\} \subset M$, and $\mathbf{i} = (i_1, \cdots, i_n) \in I^n$, we shall write $x_{\mathbf{i}} = x_{i_1} x_{i_2} \cdots x_{i_n}$. We also use $[S]$ either to denote the norm (respectively strong) closure of the span, for $S \subseteq \mathcal{H}$ (respectively $S \subseteq \mathcal{L}(\mathcal{H})$), for any Hilbert space $\mathcal{H}$.

2 An existence result

PROPOSITION 2.1. *Let $(M_i, \phi_i), i = 1, 2$ be von Neumann probability spaces with $z_i = z_{\phi_i}$. Suppose $S^{(j)} = \{x_i^{(j)} : i \in I\}$ is a set of self-adjoint elements which generates M_j as a von Neumann algebra, for $j = 1, 2$. (Note the crucial assumption that both the $S^{(j)}$ are indexed by the same set.) Suppose*

$$\phi_1(x_{\mathbf{i}}^{(1)}) = \phi_2(x_{\mathbf{i}}^{(2)}) \ \forall \mathbf{i} \in I^* \ . \tag{2.1}$$

Then there exists a unique isomorphism $\theta : M_1 z_1 \to M_2 z_2$ such that $\phi_2 \circ \theta|_{M_1 z_1} = \phi_1|_{M_1 z_1}$ and $\theta(x_i^{(1)} z_1) = x_i^{(2)} z_2 \ \forall i \in I$.

Proof. The hypothesis implies that, for $j = 1, 2$, the set $\{x_{\mathbf{i}}^{(j)} : \mathbf{i} \in I^*\}$ linearly spans a *-subalgebra which is necessarily σ-weakly dense in M_j. Since $\langle \widehat{x_{\mathbf{i}}^{(1)}}, \widehat{x_{\mathbf{j}}^{(1)}} \rangle = \langle \widehat{x_{\mathbf{i}}^{(2)}}, \widehat{x_{\mathbf{j}}^{(2)}} \rangle \ \forall \mathbf{i}, \mathbf{j} \in I^*$, there exists a unique unitary operator $u : L^2(M_1, \phi_1) \to L^2(M_2, \phi_2)$ such that $u\widehat{x_{\mathbf{i}}^{(1)}} = \widehat{x_{\mathbf{i}}^{(2)}}$ $\forall \mathbf{i} \in I^*$.

Now observe that

$$\begin{aligned} u\lambda_{M_1}(x_{\mathbf{i}}^{(1)})u^*\widehat{x_{\mathbf{j}}^{(2)}} &= u\lambda_{M_1}(x_{\mathbf{i}}^{(1)})\widehat{x_{\mathbf{j}}^{(1)}} \\ &= u\widehat{x_{\mathbf{i}\vee\mathbf{j}}^{(1)}} \\ &= \widehat{x_{\mathbf{i}\vee\mathbf{j}}^{(2)}} \\ &= \lambda_{M_2}(x_{\mathbf{i}}^{(2)})\widehat{x_{\mathbf{j}}^{(2)}} \ ; \end{aligned}$$

and hence that $u\lambda_{M_1}(x_{\mathbf{i}}^{(1)})u^* = \lambda_{M_2}(x_{\mathbf{i}}^{(2)})$ $\forall \mathbf{i} \in I$.

On the other hand, $\{x \in M_1 : u\lambda_{M_1}(x)u^* \in \lambda_{M_2}(M_2)\}$ is clearly a von Neumann subalgebra of M_1; since this has been shown to contain each $x_{\mathbf{i}}^{(1)}$, we may deduce that this must be all of M_1. Now notice that $L^2(M_j, \phi_j) = L^2(M_j z_j, \phi_j|_{M_j z_j})$, that $\lambda_{M_j}(x) = \lambda_{M_j z_j}(x z_j)$ $\forall x \in M_j$, and that $\lambda_{M_j z_j}$ maps $M_j z_j$ isomorphically onto its image.

The proof is completed by defining

$$\theta(x) = \lambda_{M_2 z_2}^{-1}(u\lambda_{M_1}(x)u^*) \ \forall x \in M_1 z_1.$$

□

REMARK 2.2. *1. In the proposition, even if it is the case that $N := \{x_i^{(2)} : i \in I\}'' \subsetneq M_2$, we can still apply the result to $(N, \phi_2|_N)$ in place of (M_2, ϕ_2) and deduce the existence of a normal homomorphism of M_1 into M_2 which sends $x_i^{(1)}$ to $x_i^{(2)} z$ for each i (and 1_{M_1} to the projection $z = z_{\phi_2|_N} \in N$).*

2. In the special case that the N of the last paragraph is a factor, the z there is nothing but id_{M_2} and in particular, Proposition 2.1 can be strengthened as follows:

Let $(M_j, \phi_j), j = 1, 2$ be von Neumann probability spaces. Suppose $S^{(j)} = \{x_i^{(j)} : i \in I\} \subset M_j$ is a set of self-adjoint elements such that $S^{(1)''} = M_1$ and $S^{(2)''}$ is a factor $N \subset M_2$. Suppose

$$\phi_1(x_{\mathbf{i}}^{(1)}) = \phi_2(x_{\mathbf{i}}^{(2)}) \ \forall \mathbf{i} \in I^* \ . \tag{2.2}$$

Then there exists a unique normal $*$-homomorphism $\theta : M_1 \to N \subset M_2$ such that $\theta(x_i^{(1)}) = x_i^{(2)}$ for all $i \in I$.

Corollary 2.3. *1. If θ_i is a ϕ_i-preserving unital endomorphism of a von Neumann probability space (M_i, ϕ_i), for $i \in \Lambda$, then there exists:*

(a) a unique unital endomorphism $\otimes_{i\in\Lambda}\theta_i$ of the tensor product $(\otimes_{i\in\Lambda}M_i, \otimes_{i\in\Lambda}\phi_i)$ such that

$$(\otimes_{i\in\Lambda}\theta_i)(\otimes_{i\in\Lambda}x_i) = z(\otimes_{i\in\Lambda}\theta_i(x_i)) \; \forall x_i = x_i^* \in M_i;$$

(b) a unique unital endomorphism $*_{i\in\Lambda}\theta_i$ of the free product $(*_{i\in\Lambda}M_i, *_{i\in\Lambda}\phi_i)$ such that

$$(*_{i\in\Lambda}\theta_i)(\lambda(x_j)) = z\lambda(\theta_j(x_j)) \; \forall x_j \in M_j$$

where we simply write λ for each 'left-creation representation' $\lambda : M_j \to \mathcal{L}(*_{i\in\Lambda}L^2(M_i, \phi_i))$ for every $j \in I$.

In the above existence assertions, the symbol z represents an appropriate projection (= image of the identity of the domain of the endomorphism in question).

2. If each M_i above is a factor, then (the z in the above statement can be ignored, as it is the identity of the appropriate algebra) and all endomorphisms above are unital monomorphisms.

Proof. It is not hard to see that Remark 2.2(1) is applicable to $S^{(1)} = \{\otimes_i x_i : x_i = x_i^* \in M_i, x_i = 1_{M_i} \text{ for all but finitely many } i\}$ and $S^{(2)} = \{\otimes_i \theta_i(x_i) : x_i = x_i^* \in M_i, x_i = 1_{M_i} \text{ for all but finitely many } i\}$ (resp., $S^{(1)} = \{\lambda(x_i) : i \in \Lambda, x_i = x_i^* \in M_i, \phi_i(x_i) = 0\}$ and $S^{(2)} = \{\lambda(\theta_i(x_i)) : i \in \Lambda, x_i = x_i^* \in M_i, \phi_i(x_i) = 0\}$.

The second fact follows from Remark 2.2(2) because normal endomorphisms of factors are unital isomorphisms onto their images, and the tensor (resp., free) product of factors is a factor. □

For later reference, the next lemma identifies the central support z_ϕ of a normal state ϕ on a von Neumann algebra in the simple special case when ϕ is a vector-state.

Lemma 2.4. *Suppose $N \subset \mathcal{L}(\mathcal{H})$ is a von Neumann algebra, $\xi \in \mathcal{H}$ is a unit vector, and ϕ is the vector state defined on N by $\phi(x) = \langle x\xi, \xi\rangle$. If $\mathcal{H}_0 = \overline{N\xi}$, then a candidate for 'the GNS triple for (N, ϕ)' is given by $(\mathcal{H}_0, id_N|_{\mathcal{H}_0}, \xi)$. In particular, the central support of ϕ is given by the projection $z = \wedge\{p \in N : ran\ p \supset \mathcal{H}_0\}$ and $ran\ z = [N'N\xi]$.*

Proof. It is clear that ξ is a cyclic vector for $N|_{\mathcal{H}_0}$ and the assertion regarding GNS triples follows. Hence if $z \in \mathcal{P}(Z(N))$ is such that $N(1-z) = ker\ id_N|_{\mathcal{H}_0}$, then $z = \wedge\{p \in \mathcal{P}(N) : p|_{\mathcal{H}_0} = (1_N)|_{\mathcal{H}_0}\}$, i.e., $z = \wedge\{p \in \mathcal{P}(N) : ran\ p \supset \mathcal{H}_0\}$. As z is the smallest projection in $(N \cap N')$ whose range contains $N\xi$, or equivalently the smallest subspace containing $[N\xi]$ which is invariant under $(N \cap N')'$, equivalently invariant under $N'N$, the last assertion follows. □

3 Extendable endomorphisms

For the remainder of this paper, we make the standing assumption that ϕ is a faithful normal state on a factor M. We identify $x \in M$ with $\lambda_M(x)$, and simply write J and Δ for the modular conjugation operator J_ϕ and the modular operator Δ_ϕ respectively. Recall, thanks to the Tomita-Takesaki theorem that $j = J(\cdot)J$ is a *-preserving conjugate-linear isomorphism of $\mathcal{L}(L^2(M, \phi))$ onto itself, which maps M and M' onto one another, and that $\widehat{1_M}$ is also a cyclic and separating vector for M'. We shall assume that θ is a normal unital *-endomorphism which preserves ϕ. The invariance assumption $\phi \circ \theta = \phi$ implies that there exists a unique isometry u_θ on $L^2(M, \phi)$ such that $u_\theta x\widehat{1_M} = \theta(x)\widehat{1_M}$ and equivalently, that $u_\theta x = \theta(x)u_\theta\ \forall x \in M$ and $u_\theta\widehat{1_M} = \widehat{1_M}$.

Definition 3.1. *If M, ϕ, θ are as above, and if the associated isometry u_θ of $L^2(M, \phi)$ commutes with the modular conjugation operator $J(= J_\phi)$, we shall simply say θ is a* **equi-modular** *(as this is related to endomorphisms commuting with the modular automorphism group) endomorphism of the factorial non-commutative probability space (M, ϕ).*

Remark 3.2. *It is true that if θ is an equi-modular endomorphism of a factor M as above, then there always exists a ϕ-preserving faithful normal conditional expectation $E : M \to \theta(M)$, and in fact $u_\theta u_\theta^*$ is the Jones projection associated to this conditional expectation. For this, notice to start with, that as θ is a *-homomorphism, u_θ commutes with the conjugate-linear Tomita operator S (which has $M\widehat{1_M}$ as a core and maps $x\widehat{1_M}$ to $x^*\widehat{1_M}$ for $x \in M$). More precisely, we have $Su_\theta \supset u_\theta S$, meaning that whenever ξ is in the domain of S, so is $u_\theta\xi$ and $Su_\theta\xi = u_\theta S\xi$ holds. Since u_θ commutes with J and $S = J\Delta^{1/2}$, we have $\Delta^{1/2}u_\theta \supset u_\theta\Delta^{1/2}$, and so Δ^{it} commutes with u_θ*

for any $t \in \mathbb{R}$. Hence, conclude that for $x \in M$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}\theta(\sigma_t^\phi(x))\widehat{1_M} &= u_\theta \Delta^{it} x \Delta^{-it} \widehat{1_M} = u_\theta \Delta^{it} x \widehat{1_M} \\ &= \Delta^{it} u_\theta x \widehat{1_M} = \Delta^{it}\theta(x)\widehat{1_M} \\ &= \sigma_t^\phi(\theta(x))\widehat{1_M} \ .\end{aligned}$$

As $\widehat{1_M}$ is a separating vector for M, deduce that

$$\theta(\sigma_t^\phi(x)) = \sigma_t^\phi(\theta(x)) \ \forall x \in M, t \in \mathbb{R} \ .$$

Hence $\sigma_t^\phi(\theta(M)) = \theta(M) \ \forall t \in \mathbb{R}$ and it follows from Takesaki's theorem (see [Ta, Section 4]) that there exists a unique ϕ-preserving conditional expectation E of M onto the subfactor $P = \theta(M)$. It is true, as the definition shows, that $e_\theta = u_\theta(u_\theta)^*$ is the orthogonal projection onto $[P\widehat{1_M}]$ and $E(x)e_\theta = e_\theta x e_\theta \ \forall x \in M$.

THEOREM 3.3. *Suppose θ is an equi-modular endomorphism of a factorial non-commutative probability space (M, ϕ). Then,*

1. *The equation $\theta' = j \circ \theta \circ j$ defines a unital normal *-endomorphism of M' which preserves $\phi' = \overline{\phi \circ j}$; and*

2. *We have an identification*

$$\begin{aligned} L^2(M', \phi') &= L^2(M, \phi) \\ \widehat{1_{M'}} &= \widehat{1_M} \\ u_{\theta'} &= u_\theta \end{aligned}$$

3. *there exists a unique endomorphism $\theta^{(2)}$ of $\mathcal{L}(L^2(M,\phi))$ satisfying*

$$\theta^{(2)}(xj(y)) = \theta(x)j(\theta(y))z, \ \forall x, y \in M$$

where $z = \wedge\{p \in (\theta(M) \cup \theta'(M'))'' : ran(p) \supset \{\widehat{\theta(x)} : x \in M\}\}$.

Proof. 1. It is clear that $\theta' = j \circ \theta \circ j$ is a unital normal linear *-endomorphism of M' and that

$$\overline{\phi' \circ \theta'} = \overline{\phi'} \circ \theta' = (\phi \circ j) \circ (j \circ \theta \circ j) = (\phi \circ \theta) \circ j = \phi \circ j = \overline{\phi'} \ ,$$

thereby proving (1).

2. This follows from the facts that $\widehat{1_M}$ is a cyclic and separating vector for M and hence also for M', the definition of ϕ' which guarantees that

$$\begin{aligned}
\langle j(x)\widehat{1_{M'}}, j(y)\widehat{1_{M'}}\rangle &= \phi'(j(y)^*j(x)) \\
&= \phi'(j(y^*x)) \\
&= \overline{\phi(y*x)} \\
&= \phi(x^*y) \\
&= \langle y\widehat{1_M}, x\widehat{1_M}\rangle \\
&= \langle Jx\widehat{1_M}, Jy\widehat{1_M}\rangle \\
&= \langle JxJ\widehat{1_M}, JyJ\widehat{1_M}\rangle \\
&= \langle j(x)\widehat{1_M}, j(y)\widehat{1_M}\rangle
\end{aligned}$$

and the definitions of the 'implementing isometries', which show that

$$\begin{aligned}
u_{\theta'}(j(x)\widehat{1_{M'}}) &= \theta'(j(x))\widehat{1_{M'}} \\
&= j(\theta(x))\widehat{1_{M'}} \\
&= J\theta(x)J\widehat{1_M} \\
&= J\theta(x)\widehat{1_M} \\
&= Ju_\theta x\widehat{1_M} \\
&= u_\theta Jx\widehat{1_M} \\
&= u_\theta JxJ\widehat{1_M} \\
&= u_\theta j(x)\widehat{1_{M'}}\ .
\end{aligned}$$

3. Notice that if $x, y \in M$, then

$$\begin{aligned}
\langle \theta(x)J\theta(y)J\widehat{1_M}, \widehat{1_M}\rangle &= \langle \theta(x)J\theta(y)\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle \theta(x)Ju_\theta y\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle \theta(x)u_\theta Jy\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle u_\theta xJy\widehat{1_M}, \widehat{1_M}\rangle \\
&= \langle u_\theta xJy\widehat{1_M}, u_\theta\widehat{1_M}\rangle \\
&= \langle xJyJ\widehat{1_M}, \widehat{1_M}\rangle\ ,
\end{aligned}$$

where we have used the fact that θ is equi-modular.

Set $S^1 = \{xj(y) : x = x^*, y = y^*, x, y \in M\}$, and $S^{(2)} = \{\theta(x)j(\theta(y)) : xj(y) \in S^{(1)}\}$, and deduce from the factoriality of M that $S^{(1)''} = \mathcal{L}(L^2(M, \phi))$.

Now we wish to apply Remark 2.2(1) with $N = S^{(2)''} = \theta(M) \vee j(\theta(M))$ (where, both here and in the sequel, we write $A \vee B = (A \cup B)''$ for the von Neumann algebra generated by von Neumann algebras A and B) and $\phi_1 = \phi_2 = \langle (\cdot)\hat{1}_M, \hat{1}_M \rangle$. For this, deduce from Lemma 2.4 that

$$\begin{aligned} z &= \wedge\{p \in \mathcal{P}(N) : ran\ p \supset N\hat{1_M}\} \\ &= \wedge\{p \in \mathcal{P}(N) : ran\ p \supset \{\theta(x)\hat{1_M}, \theta'(j(x))\hat{1_M} : x \in M\}\} \\ &= \wedge\{p \in \mathcal{P}(N) : ran\ p \supset \{\widehat{\theta(x)} : x \in M\}\} \end{aligned}$$

and the proof of the Theorem is complete.

□

Remark 3.4. *It must be observed that the projection z of Theorem 3.3 is nothing but the central support of the projection $e_\theta = u_\theta u_\theta^*$ in $P' \cap P_1$ where $P = \theta(M) \subset M \subset P_1$ is Jones' basic construction (thus, $P_1 = JP'J$) since, by Lemma 2.4, we have:*

$$ran\ z = [(P \vee JPJ)'(P \vee JPJ)\widehat{1_M}] = [(P' \cap P_1)e_\theta L^2(M, \phi)]\ .$$

This is because

$$\begin{aligned} [(P \vee JPJ)\widehat{1_M}] &= [PJPJ\widehat{1_M}] = [PJP\widehat{1_M}] \\ &= [PJu_\theta M\widehat{1_M}] = [Pu_\theta JM\widehat{1_M}] \\ &= [Pu_\theta M\widehat{1_M}]\ = [P\widehat{1_M}]\ . \end{aligned}$$

In particular, since e_θ is a minimal projection in $P' \cap P_1$, its central support z in $P' \cap P_1$ is 1 if and only if $P' \cap P_1$ is a type I factor. In the following corollary, we continue to use the symbols P and P_1 with the meaning attributed to them here.

The following corollary is an immediate consequence of Lemma 2.4, Theorem 3.3 and Remark 3.4.

Corollary 3.5. *Let θ be a equi-modular endomorphism of a factorial non-commutative probability space (M, ϕ) in standard form (i.e., viewed as embedded in $\mathcal{L}(L^2(M, \phi))$ as above). The following conditions on θ are equivalent:*

1. *there exists a unique unital normal $*$-endomorphism $\theta^{(2)}$ of $\mathcal{L}(L^2(M, \phi))$ such that $\theta^{(2)}(x) = \theta(x)$ and $\theta^{(2)}(j(x)) = j(\theta(x))$ for all $x \in M$.*

2. $P \vee JPJ$ *is a factor; and in this case, it is necessarily a type I factor.*

3. $(P \vee JPJ)' = P' \cap P_1$ *is a factor; and in this case, it is necessarily a type I factor.*

4. $\{x\widehat{y} : x \in P' \cap P_1, y \in P\}$ *is total in* $L^2(M, \phi)$.

An endomorphism of a factor which satisfies the equivalent conditions above will be said to be **extendable**.

REMARK 3.6. *It should be noted that extendability is not a property of just an endomorphism* θ *but it is also dependent on a state which is not only left invariant under the enomorphism but must also satisfy the requirement we have called equi-modular. Strictly speaking, we should probably talk of* ϕ*-extendability, but shall not do so in the interest of notational convenience.*

THEOREM 3.7. *Let the notation be as above. Then the following conditions are equivalent:*

1. θ *is extendable.*

2. $M = (M \cap \theta(M)') \vee \theta(M)$. *(Note that the right-hand side is naturally identified with the von Neumann algebra tensor product* $(M \cap \theta(M)') \otimes \theta(M)$ *in this case.)*

Proof. Recall that $P = \theta(M)$ is globally preserved by the modular automorphism group $\{\sigma_t^\phi\}_{t\in\mathbb{R}}$ and there exists a ϕ-preserving faithful normal conditional expectation E from M onto P. Thus $(M \cap P') \vee P$ is naturally identified with the von Neumann algebra tensor product $(M \cap P') \otimes P$ (see [Ta, Corollary 1]). If we assume the second condition in the statement, the basic construction for $P \subset M$ essentially comes from that of $\mathbb{C} \subset (M \cap P')$, and so $P' \cap P_1$ is a type I factor. This means that θ is extendable.

Assume that θ is extendable now. We will show that $Q := (M \cap P') \vee P$ coincides with M. For this, it suffices to show $[Q\widehat{1_M}] = L^2(M, \phi)$. Indeed, since P is globally preserved by σ_t^ϕ, so is Q, and there exists a ϕ-preserving faithful normal conditional expectation from M onto Q thanks to Takesaki's theorem. Thus if Q were a proper subalgebra of M, $[Q\widehat{1_M}]$ would be a proper subspace of $L^2(M, \phi)$.

Let $\hat{E}$ be the dual operator valued weight from P_1 to M (see [Ko] for the definition of $\hat{E}$ and its properties). Since $E \circ \hat{E}(e_\theta) = 1 < \infty$ and $P' \cap P_1$ is a factor, the restriction of $E \circ \hat{E}$ to the type I factor

$P' \cap P_1$ is a faithful normal semifinite weight (see [ILP, Lemma 2.5]). Thus there exists a (not necessarily bounded) non-singular positive operator ρ affiliated to $P' \cap P_1$ satisfying $\sigma_t^{E\circ\hat{E}} = \mathrm{Ad}\rho^{it}$ and

$$E \circ \hat{E}(a) = \lim_{n\to\infty} \mathrm{Tr}(\rho(1+\frac{1}{n}\rho)^{-1}a), \quad \forall a \in (P' \cap P_1)_+,$$

where $\{\sigma_t^{E\circ\hat{E}}\}_{t\in\mathbb{R}}$ is the relative modular automorphism group (the restriction of $\{\sigma_t^{\phi\circ\hat{E}}\}_{t\in\mathbb{R}}$ to $P' \cap P_1$). Note that the trace Tr makes sense as $P' \cap P_1$ is a type I factor.

From the above argument we see that there exists a partition of unity $\{e_i\}_{i\in I}$ consisting of minimal projections $e_i \in P' \cap P_1$ with $E \circ \hat{E}(e_i) < \infty$. Since e_θ is a minimal projection in $P' \cap P_1$ satisfying $\sigma_t^{E\circ\hat{E}}(e_\theta) = e_\theta$ and $E \circ \hat{E}(e_\theta) = 1$, we may assume $0 \in I$ and $e_0 = e_\theta$. Let $\{e_{ij}\}_{i,j\in I}$ be a system of matrix units in $P' \cap P_1$ satisfying $e_{ii} = e_i$. Then we can apply the push down lemma [ILP, Proposition 2.2] to e_{0i}, and we have $e_{0i} = e_\theta q_i$, where $q_i = \hat{E}(e_{0i}) \in P' \cap M$. Now for any $x \in M$, we have

$$x\widehat{1_M} = \sum_{i\in I} e_{ii}x\widehat{1_M} = \sum_{i\in I} q_i^* e_\theta q_i x\widehat{1_M} = \sum_{i\in I} q_i^* E(q_i x)\widehat{1_M},$$

which shows $[Q\widehat{1_M}] = L^2(M, \phi)$. □

4 Examples of Extendable Endomorphisms

Note that any automorphism on a factor is extendable, since the conditions in Corollary 3.5 are satisfied.

Let $\mathcal{R}$ denote the hyperfinite II_1 factor and M be any II_1 factor which is also a McDuff factor; i.e., $M \otimes \mathcal{R} \cong M$. Let $\alpha : M \otimes \mathcal{R} \mapsto M$ be an isomorphism and $\beta : M \mapsto M \otimes \mathcal{R}$ be the monomorphism defined by $\beta(m) = m \otimes 1$, for $m \in M$. Let us write $\theta = \beta \circ \alpha$. so θ is an endomorphism of $M \otimes \mathcal{R}$ such that $\theta(M \otimes \mathcal{R}) = M \otimes 1$. As $M \otimes \mathcal{R}$ is a II_1 factor, the endomorphism θ is necessarily equi-modular. Now by corollary 3.5, showing that θ is extendable is equivalent to showing that $\{\theta(M \otimes \mathcal{R}) \vee J\theta(M \otimes \mathcal{R})J\}$ is a type I factor, where J is the modular conjugation of $M \otimes \mathcal{R}$, which, of course, is $J_M \otimes J_R$. Note that

$$\begin{aligned}
\{\theta(M \otimes \mathcal{R}) \vee J\theta(M \otimes \mathcal{R})J\} &= \{M \otimes 1 \vee J(M \otimes 1)J\} \\
&= \{M \otimes 1 \vee J_M M J_M \otimes 1)\} \\
&= \mathcal{L}(L^2(M) \otimes 1
\end{aligned}$$

So $\{\theta(M \otimes \mathcal{R}) \vee J\theta(M \otimes \mathcal{R})J\}$ is a type I factor. That is θ is extendable.

5 Extendability for E_0-semigroups

Definition 5.1. *$\{\alpha_t : t \geq 0\}$ is said to be an E_0-semigroup on a von Neumann probability space (M, ϕ) if:*

1. *α_t is a ϕ-preserving normal unital *-homomorphism of M for each $t \geq 0$;*
2. *$\alpha_0 = id_M$ and $\alpha_s \circ \alpha_t = \alpha_{s+t}$; and*
3. *$[0, \infty) \ni t \mapsto \rho(\alpha_t(x))$ is continuous for each $x \in M, \rho \in M_*$.*

Suppose α_t is (equi-modular and) extendable for each t, then we say that the E_0-semigroup α is extendable.

Remark 5.2. *A remark along the lines of Remark 3.6, with endomorphism replaced by E_0-semigroup, is in place here.*

Proposition 5.3. *Suppose $\alpha = \{\alpha_t : t \geq 0\}$ is an E_0-semigroup on a factorial non-commutative probability space (M, ϕ).*

1. *The equation $\alpha'_t(x') = j(\alpha_t(j(x'))$ defines an E_0-semigroup on (M', ϕ'), where $\phi'(x') = \omega_{\widehat{1_M}}(x') = \langle x'\widehat{1_M}, \widehat{1_M}\rangle$;*
2. *If α is extendable for each t, then there exists a unique E_0-semigroup $\{\alpha_t^{(2)} : t \geq 0\}$ on $(\mathcal{L}(L^2(M, \phi)), \omega_{\widehat{1_M}})$ such that $\alpha_t^{(2)}(xx') = \alpha_t(x)\alpha'_t(x') \; \forall x \in M, x' \in M'$.*

Proof. Existence of the endomorphisms α'_t and $\alpha_t^{(2)}$ is guaranteed by Corollary 3.5, The equation $\alpha'_t = j \circ \alpha_t \circ j$ shows that $\{\alpha'_t : t \geq 0\}$ inherits the property of being an E_0-semigroup from that of $\{\alpha_t : t \geq 0\}$. The corresponding property for $\{\alpha_t^{(2)} : t \geq 0\}$ is now seen to follow easily from the uniqueness assertion in Corollary 3.5(1). $\square$

By using standard arguments from the theory of E_0-semigroups on type I factors, we can strengthen Corollary 3.5 in the case of E_0-semigroups thus:

Proposition 5.4. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on a factorial non-commutative probability space (M, ϕ), and suppose α_t is equi-modular for each t. Suppose M is acting standardly on $\mathcal{H} = L^2(M)$. Consistent with the notation of Remark 3.4, we shall write $P(t) = \alpha_t(M) \subset M \subset P_1(t)$ for Jones' basic construction.*

The following conditions on α are equivalent.

1. *α is extendable.*

2. *$P'(t) \cap P_1(t)$ is either: (i) a factor of type I_1 (i.e., is isomorphic to $\mathbb{C}$) and α_t is an automorphism for all t; or (ii) a factor of type I_∞ for all t and no α_t is an automorphism.*

Proof. (1) $\Rightarrow$ (2) If each α_t is extendable, then $P'(t) \cap P_1(t)$ is a factor of type I_{n_t}, say, by Corollary 3.5. The first fact to be noted is that if $\{\mathcal{E}_{\alpha^{(2)}}(t) : t \geq 0\}$ is the product system associated to the E_0-semigroup $\alpha^{(2)}$ on $\mathcal{L}(L^2(M))$, then n_t is the dimension of the Hilbert space $\mathcal{E}_{\alpha^{(2)}}(t)$. Hence $\{n_t : t \geq 0\}$ is a multiplicative semi-group of integers. Hence either n_t is constant in t (identically 1 or identically infinity).

(2) $\Rightarrow$ (1) follows from Corollary 3.5. $\square$

Remark 5.5. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on a factorial non-commutative probability space (M, ϕ), and suppose α_t is equi-modular for each t. If α_t is an extendable endomorphism for some $t > 0$, then α_s is also extendable for all $0 \leq s < t$. Indeed, α_t being extendable means that M as a $P(t) - P(t)$ bimodule is a direct sum of copies of $P(t)$, and hence $P(t-s)$ as a $P(t) - P(t)$ bimodule is also a direct sum of copies of $P(t)$. This means that $\alpha_{t-s}(M)$ is generated by $\alpha_{t-s}(M) \cap \alpha_t(M)'$ and $\alpha_t(M)$, which means α_s is extendable. Now, since the compositions of extendable endomorphisms are extendable, the E_0-semigroup α itself is extendable.*

Now let us consider the following spaces;

$$E^{\alpha_t} = \{T \in \mathcal{L}(L^2(M)) : \alpha_t(x)T = Tx, \text{ forall } x \in M\};$$

$$E^{\alpha'_t} = \{T \in \mathcal{L}(L^2(M)) : \alpha'_t(x')T = Tx', \text{ forall } x' \in M'\}.$$

For every $t \geq 0$, we write $H(t) = E^{\alpha_t} \cap E^{\alpha'_t}$. Then we have the following Lemma.

Proposition 5.6. *Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on a factorial non-commutative probability space (M, ϕ) and suppose α_t is equi-modular for each t. If α is extendable then*

$$H = \{(t, T) : t \in (0, \infty), T \in H(t)\}$$

is a product system (in the sense of [Arv]) with the family of unitary maps $u_{st} : H(t) \otimes H(s) \mapsto H(s+t)$, given by

$$u_{st}(T \otimes S) = TS \ \forall T \in H(t), S \in H(s).$$

Proof. Let $\alpha^{(2)} = \{\alpha_t^{(2)} : t > 0\}$ be the extension of α on $\mathcal{L}(L^2(M))$. For $t > 0$, consider

$$\mathcal{E}(t) = \{T \in \mathcal{L}(L^2(M)) : \alpha_t^{(2)}(x)T = Tx, \text{ for all } x \in \mathcal{L}(L^2(M))\}.$$

We shall write $\mathcal{E} = \{(t,T) : T \in \mathcal{E}(t)\}$; then $\mathcal{E}$ is a product system (see [Arv]), and $H(t) = \mathcal{E}(t)$ for every $t > 0$. Indeed, if $T \in H(t) = E^{\alpha_t} \cap E^{\alpha'_t}$, then $\alpha_t(m)T = Tm$ for all $m \in M$ and $\alpha'_t(m')T = Tm'$ for all $m' \in M'$. So it is clear that $\alpha_t^{(2)}(x)T = Tx$ for all $x \in M \cup M'$ and hence also for all $x \in (M \vee M') = \mathcal{L}(L^2(M))$. So, $T \in \mathcal{E}(t)$, and $H(t) \subset \mathcal{E}(t)$. The reverse inclusion is immediate from the definition $\alpha_t^{(2)}$. So we have $H(t) = \mathcal{E}(t)$ and clearly H is a product system. $\square$

Now recall that an E_0-semigroup $\{\beta_t : t \geq 0\}$ of a von Neumann probability space (M, ϕ) is said to be a **cocycle perturbation** of an E_0-semigroup $\{\alpha_t : t \geq 0\}$ if there exists a weakly continuous family $\{u_t : t \geq 0\}$ of unitary elements of M such that

1. $u_{t+s} = u_s \alpha_s(u_t)$; and
2. $\beta_t(x) = u_t \alpha_t(x) u_t^*$ for all $x \in M$ and $s, t \geq 0$.

In such a case, we shall simply write

$$\{u_t : t \geq 0\} : \{\alpha_t : t \geq 0\} \simeq \{\beta_t : t \geq 0\}.$$

Proposition 5.7. *Suppose $\beta = \{\beta_t : t \geq 0\}$ is an E_0 semigroup on a factorial probability space (M, ϕ), which is a cocycle perturbation of another E_0 semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on (M, ϕ) with $\{u_t : t \geq 0\} : \{\alpha_t : t \geq 0\} \simeq \{\beta_t : t \geq 0\}$. Then*

1. *$\{j(u_t) : t \geq 0\} : \{\alpha'_t : t \geq 0\} \simeq \{\beta'_t : t \geq 0\}$.*
2. *If each α_t is extendable, as is each β_t, then*

$$\{u_t j(u_t) : t \geq 0\} : \{\alpha_t^{(2)} : t \geq 0\} \simeq \{\beta_t^{(2)} : t \geq 0\}.$$

Proof. The verifications are elementary and a routine computation. For example, once (1) has been verified, the verification of (2) involves such straightforward computations as: if we let $U_t = u_t u'_t$, where we write $u'_t = j(u_t)$, and if $x \in M, x' \in M'$, then

$$\begin{aligned}
U_t\alpha_t(x)U_t^* &= u_t\alpha_t(x)\overline{u}_t^* = \beta_t(x) \\
U_t\alpha'_t(x')U_t^* &= u'_t\alpha'_t(x')u'^*_t = \beta'_t(x') \\
\beta_t(M) \vee \beta'_t(M') &= U_t(\alpha_t(M) \vee \alpha'_t(M'))U_t^* \\
U_{s+t} &= u_{s+t}u'_{s+t} \\
&= u_s\alpha_s(u_t)u'_s\alpha'_s(u'_t) \\
&= U_s\alpha_s^{(2)}(U_t)
\end{aligned}$$

and

$$\begin{aligned}
\beta_t^{(2)}(xx') &= \beta_t(x)\beta'_t(x') \\
&= u_t\alpha_t(x)u_t^*u'_t\alpha'_t(x')u'^*_t \\
&= U_t\alpha_t^{(2)}(xx')U_t^*.
\end{aligned}$$

□

Recall that two E_0-semigroups $\{\alpha_t : t \geq 0\}$ and $\{\beta_t : t \geq 0\}$ on a von Neumann algebra M are said to be **conjugate** if there exists an automorphism θ of M such that $\beta_t \circ \theta = \theta \circ \alpha_t \ \forall t$, while they are said to be **cocycle conjugate** if each is conjugate to a cocycle perturbation of the other.

REMARK 5.8. *While the index of E_0-semigroups of type I_∞ factors has been well-defined, we may now define the index of an extendable E_0 semigroup α of an arbitrary factor as the index of $\alpha^{(2)}$; and we may infer from Proposition 5.7 that the index of an extendable E_0-semigroup of an arbitrary factor is invariant under cocycle conjugacy - in the restricted sense that cocycle conjugate extendable E_0-semigroups have the same index. (One has to exercise some caution here in that there is a problem with invariance of equimodularity under cocycle conjugacy!)*[4] *It is to be noted from Corollary 3.5 that the extendability of an E_0-semigroup, each of whose endomorphisms is equi-modular, is a property which is invariant under cocycle conjugacy within the class of such E_0-semigroups.*

PROPOSITION 5.9. *If $\alpha = \{\alpha_t : t \geq 0\}$ (resp., $\beta = \{\beta_t : t \geq 0\}$) is an extendable E_0-semigroup of a factor M (resp., N), then $\alpha \otimes \beta =$*

[4]We wish to thank the referee for pointing this out, which also led to the insertion of the Remarks 3.6 and 5.2.

$\{\alpha_t \otimes \beta_t : t \geq 0\}$ *is an extendable* E_0*-semigroup of the factor* $M \otimes N$*, and in fact,*

$$(\alpha \otimes \beta)^{(2)} = \alpha^{(2)} \otimes \beta^{(2)}.$$

Proof. The hypothesis is that $\alpha_t(M) \vee J\alpha_t(M)J$ and $\beta_t(N) \vee J_N\beta_t(N)J_N$ are factors, for each $t \geq 0$, while the conclusions follow from the definition of $\alpha \otimes \beta$. $\square$

6 Examples

First we give examples of extendable E_0-semigroups. Throughout this section, let $\mathcal{H} = L^2(0,\infty)\otimes\mathcal{K}$, be the real Hilbert space of square integrable functions taking values in is a real Hilbert space $\mathcal{K}$. We always denote by $(\cdot)_{\mathbb{C}}$ the complexification of $(\cdot)$. Let S_t be the shift semigroup on $\mathcal{H}_{\mathbb{C}}$ defined by

$$\begin{aligned} (S_t f)(s) &= 0, \quad s < t, \\ &= f(s-t), \quad s \geq t. \end{aligned}$$

Thus $(S_t : t \geq 0)$ is a semigroup of isometries, and we denote its restriction to $\mathcal{H}$ also by $\{S_t\}$.

For the first set of examples, given by 'canonical commutation relations', we only need complex Hilbert spaces. Let $A \geq 1$ be a complex linear operator on $\mathcal{H}_{\mathbb{C}}$ such that $T = \frac{1}{2}(A-1)$ is injective. Consider the the quasi free state on the CCR algebra over $\mathcal{H}_{\mathbb{C}}$ given by

$$\varphi_A(W(f)) = e^{-\frac{1}{2}\langle Af,f\rangle} = e^{-\frac{1}{2}\|\sqrt{1+2T}f\|^2} \ \ \forall \ \ f \in \mathcal{H}_{\mathbb{C}}.$$

The space underlying the corresponding GNS representation may be identified with $\Gamma_s(\mathcal{H}_{\mathbb{C}})\otimes\Gamma_s(\mathcal{H}_{\mathbb{C}})$, with the GNS representation being described by

$$\pi(W(f)) = W_0(\sqrt{1+T}f) \otimes W_0(j\sqrt{T}f) \ \ \forall \ \ f \in \mathcal{H}_{\mathbb{C}},$$

where $\Gamma_s(\cdot)$ is the symmetric Fock space, $W_0(\cdot)$ is the Weyl operator on $\Gamma_s(\mathcal{H}_{\mathbb{C}})$ and j is an anti-unitary on $\mathcal{H}_{\mathbb{C}}$ induced by an anti-unitary operator on $\mathcal{K}_{\mathbb{C}}$. The vacuum vector $\Omega \otimes \Omega \in \Gamma_s(\mathcal{H}_{\mathbb{C}}) \otimes \Gamma_s(\mathcal{H}_{\mathbb{C}})$ is the cyclic and separating vector for $M_A = \{\pi(W(f)) : f \in \mathcal{H}_{\mathbb{C}}\}''$ (see [Ark]).

Example 6.1. *Let* $A = \frac{1+\lambda}{1-\lambda}$ *with* $\lambda \in (0,1)$*, then it is well-known that* $M_\lambda = M_A$ *is a type* III_λ *factor. There exists a unique* E_0-*semigroup* β_t^λ *on* M_λ *satisfying*

$$\beta_t^\lambda(\pi(W(f))) = \pi(W(S_t f)) \ \ \forall \ f \in \mathcal{H}_{\mathbb{C}}.$$

Further, $\{\beta_t^\gamma ; t \geq 0\}$ is equi-modular and the relative commutant is given by

$$\beta_t^\lambda(M_\lambda)' \cap M_\lambda = \left\{\pi(W(f)) : f \in \left(L^2(0,t) \otimes \mathcal{K}\right)_{\mathbb{C}}\right\}''.$$

Now theorem 3.7 imply that all these E_0-semigroups on type III_λ factors are extendable. (See [MS1], where these examples are discussed in more detail.)

We will write $\mathcal{F}(\mathcal{H})$ for the anisymmetric Fock space; thus

$$\mathcal{F}(\mathcal{H}_{\mathbb{C}}) = \mathbb{C}\Omega \oplus \mathcal{H}_{\mathbb{C}} \oplus (\mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}}) \oplus (\mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}} \wedge \mathcal{H}_{\mathbb{C}}) \oplus \cdots ,$$

where Ω is a fixed complex number with modulus 1.

Recall the left creation operator on $\mathcal{F}(\mathcal{H})$ (corresponding to $f \in \mathcal{H}_{\mathbb{C}}$ given by

$$\begin{aligned} a(f)\Omega &= f \\ a(f)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) &= f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \ , \ \xi_i \in \mathcal{H}_{\mathbb{C}} \ . \end{aligned}$$

These operators obey the *canonical anticommutation relations*:

$$a(f)a(g) + a(g)a(f) = 0 \ , \ a(f)a(g)^* + a(g)^*a(f) = \langle f \ , \ g\rangle id_{\mathcal{F}(\mathcal{H})}$$

for all $f \ , \ g \in \mathcal{H}_{\mathbb{C}}$,

For any $f \in \mathcal{H}$, let $u(f) = a(f) + a(f)^*$. It is well-known that the von Neumann algebra

$$\{u(f) : f \in \mathcal{H}\}'' \subseteq \mathcal{L}(\mathcal{F}(\mathcal{H}_{\mathbb{C}}))$$

is the hyperfinite II_1 with cyclic and separating (trace)vector Ω.

Example 6.2. *For every $t \geq 0$ there exist a unique normal, unital $*$-endomorphism $\alpha_t : \mathcal{R} \mapsto \mathcal{R}$ satisfying*

$$\alpha_t(u(f)) = u(S_t f)) \ \forall f \in \mathcal{H}_{\mathbb{C}} \ .$$

(Although this is a well-known fact, we remark that this is in fact a consequence of Remark 2.2 (2).) Then $\alpha = \{\alpha_t : t \geq 0\}$ is an E_0-semigroup on $\mathcal{R}$, called the **Clifford flow of rank dim $\mathcal{K}$**.

It is known from [Alev] that

$$\alpha_t(M)' \cap M = \{u(f)u(g) : spt(f), spt(g) \subset [0,t]\}'' \ . \tag{6.3}$$

It follows from equation 6.3 that if $spt(f) \subset [0,t]$, then $u(f)\Omega \perp \{(\alpha_t(\mathcal{R})' \cap \mathcal{R})\Omega \cup \alpha_t(\mathcal{R})\Omega\}$; (in fact the same assertion holds for any $a(f_1) \cdots a(f_{2n+1}\Omega$ for any n and any $f_1, \cdots, f_{2n+1}$ with support in $[0,t]$.) Consequently, in view of Ω being a separating vector for $\mathcal{R}$, it is an easy consequence of Theorem 3.7 that the **Clifford flow on $\mathcal{R}$ (of any rank) is not extendable**.

The Clifford flows of the hyperfinite II_1 factor are closely related to another family of E_0-semigroups, called the CAR flows. (We should remember these are CAR flows on type II_1 factors, not to be confused with the usual CAR flows on the type I factor of all bounded operators on the antisymmetric Fock space.) We recall the definition of CAR algebra and some facts regarding the GNS representations of CAR algebras given by quasi-free states.

For a complex Hilbert space K, the associated CAR algebra $CAR(K)$ is the universal C^*-algebra generated by a unit 1 and elements $\{b(f): f \in K\}$, subject to the following relations

(i) $b(\lambda f) = \lambda b(f)$,

(ii) $b(f)b(g) + b(g)b(f) = 0$,

(iii) $b(f)b^*(g) + b^*(g)b(f) = \langle f, g\rangle 1$,

for all $\lambda \in \mathbb{C}$, $f, g \in K$, where $b^*(f) = b(f)^*$.

Given a positive contraction A on K, there exists a unique quasi-free state ω_A on $CAR(K)$ satisfying

$$\omega_A(b(x_n)\cdots b(x_1)b(y_1)^*\cdots b(y_m)^*) = \delta_{n,m} det(\langle Ax_i, y_j\rangle),$$

where $det(\cdot)$ denotes the determinant of a matrix. Let (H_A, π_A, Ω_A) be the corresponding GNS triple. Then $M_A = \pi_A(CAR(K))''$ is a factor.

Here onwards we fix the contraction with $A = \frac{1}{2}$, then $M_A = \mathcal{R}$ is the hyperfinite type II_1 factor and ω_A is a tracial state. We define the CAR flow on $\mathcal{R}$ as follows.

Now let $K = \mathcal{H}_{\mathbb{C}}$. Then there exists a unique E_0-semigroup $\{\alpha_t\}$ on $\mathcal{R}$ satisfying

$$\alpha_t(\pi(b(f))) = \pi(b(S_t f)) \ \ \forall\ f \in \mathcal{H}_{\mathbb{C}}.$$

This α is called as the CAR flow of index n on $\mathcal{R}$.

We recall the following proposition from [Alev] (see proposition 2.6).

Proposition 6.3. *The CAR flow of rank n on $\mathcal{R}$ is conjugate to the Clifford flow of rank $2n$.*

We point out an error in [ABS] in the following remark.

Remark 6.4. *In section 5, [ABS], it is claimed that CAR flows of any given rank are extendable. In fact a 'proof' is given, for any*

$\lambda \in (0, \frac{1}{2}]$ *with* $A = \lambda$, *that the corresponding* E_0-*semigroup on* M_A *is extendable. (When* $\lambda \neq \frac{1}{2}$ *they are type III factors.) But we have seen that Clifford flows are not extendable. This consequently implies, thanks to proposition 6.3, and the invariance of extendability of* E_0*-semigroups of* II_1 *factors (where equimodularity - with respect to the trace - comes for free), that CAR flows on the hyperfinite type* II_1 *factor* $\mathcal{R}$ *are not extendable.*

In fact it has been proved in [BK] that CAR flows flows on any type III_λ *factors (considered in [ABS]) are also not extendable.)*

Let $\Gamma_f(\mathcal{H}_{\mathbb{C}})$ be the full Fock space associated with a Hilbert space K. For $f \in \mathcal{H}$, define $s(f) = \frac{l(f)+l(f)^*}{2}$ where

$$l(f)\xi = \left\{ \begin{array}{ll} f & \text{if } \xi = \Omega, \\ f \otimes \xi & \text{if } \langle \xi, \Omega \rangle = 0. \end{array} \right\}.$$

The von Neumann algebra $\Phi(\mathcal{K}) = \{s(f) : f \in \mathcal{H}\}''$, is isomorphic to the free group factor $L(F_\infty)$ and the vacuum is cyclic and separating with $\langle \Omega, x\Omega \rangle = \tau(x)$ (see [DV]) a tracial state on $\Phi(\mathcal{K})$.

Example 6.5. *There exists a unique* E_0-*semigroup* γ *on* $\Phi(\mathcal{K})$ *satisfying*

$$\gamma_t(s(f)) := s(S_t f) \qquad (f \in \mathcal{H},\ t \geq 0).$$

This is called the free flow of rank $dim(\mathcal{K})$.

Let γ *be a free flow of any rank. It is known - see [Popa] - that* $\gamma_t(\Phi(\mathcal{K}))' \cap \Phi(\mathcal{K}) = \mathbb{C}1$. *So it follows from Theorem 3.7 that free flow is not extendable.*

It is proved in [MS] that

$$H(t) = (E^{\gamma_t} \cap E^{\gamma'_t}) = \mathbb{C}S_t.$$

So $H = \{(t, \eta) : \eta \in H(t)\}$ *is a product system. This means that free flows provide examples to show that the converse of the Corollary 5.6 is not true: for free flows, the family* $\{(t, (E^{\gamma_t} \cap E^{\gamma'_t})\ ) : t \geq 0\}$ *forms a product system, but still they are not extendable.*

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