Threshold activated coupling of chaotic elements

Be-heading local chaos

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Sinha and Biswas, Phys. Rev. Letts, 1993
Sinha, Phys. Rev. E, 1994
Sinha, Phys. Letts. A, 1995
Sinha, Int. Jour. Mod. Phys., 1995 (Review)

Threshold Activated Chaotic Networks

- Time : discrete ; labelled by n

- Space : discrete ;

For a one-dimensional chain the elements are labelled by i : $i = 1, 2 \ldots N$,

where N is system size

- State variable $x_n(i)$: continuous

In physical systems $x_n(i)$ could be quantities like energy, velocity, pressure or concentration

Each individual site in the lattice evolves under a chaotic map $f(x)$

Network updates as follows:

$$\boxed{x_{n+1}(i) \rightarrow f(\ x_n(i)\ )}$$

For instance, the local map $f(x)$ can chosen to be the prototypical logistic map

$$\boxed{f(x) = 1 - ax^2}$$

$x \in [-1, 1]$ Nonlinearity parameter $a = 2.0$

On this chaotic lattice we incorporate threshold activated coupling

Unidirectional Transport

The threshold mechanism is triggered when a site in the lattice exceeds the critical value x_c i.e. when a certain site $x_n(i) > x_c$

The supercritical site then relaxes (*topples*) transporting the excess $\delta x = x_n(i) - x_c$ to its neighbour :

$$\begin{aligned} x_n(i) &\rightarrow x_c \\ x_n(i+1) &\rightarrow x_n(i+1) + \delta x \end{aligned}$$

This induces a unidirectional nonlinear transport down the array (by initiating a domino effect)

The boundary is open so that the "excess" may be transported out of the system

Bidirectional Transport

If $x_n(i) > x_c$ the supercritical site relaxes to x_c by transporting the excess $(x_n(i) - x_c)$ equally to its two neighbours:

$$\begin{aligned} x_n(i) &\rightarrow x_c \\ x_n(i+1) &\rightarrow x_n(i+1) + \delta x \\ x_n(i-1) &\rightarrow x_n(i-1) + \delta x \end{aligned}$$

where $\delta x = (x_n(i) - x_c)/2$

Dynamics of the network depends on the time scales for autonomously updating each site and propagating threshold coupling between sites

First case: System is allowed to relax completely before each chaotic update

One threshold relaxation of all existing supercritical sites in the lattice constitutes one relaxation time step

(which may in turn create some other supercritical sites)

After Δ such relaxation steps, the system undergoes the next chaotic update

So time n associated with the chaotic dynamics is measured in units of Δ

When Δ is large enough ($\Delta > N$) chaotic updates take place after all $x(i) \leq x_c$, i.e. after the system is stationary

So the time scales of the intrinsic chaotic dynamics and the threshold relaxation are separable

System operates in the "dilute" perturbation limit

At the other end of the spectrum is the limit of very small Δ

Analogous to Coupled Map Lattice

Lowering Δ essentially allows us to move from a picture of adiabatically separated adaptive processes to one where these processes overlap, and disturbances do not die before the subsequent chaotic update

i.e. *the tails of the relaxation processes carry over to the next event and there are "interference" effects.*

Easily generalised to :

- Higher dimensions
- Anisotropic transport
- Different Boundary Conditions
- Hierarchical schemes

Reminiscent of the “sandpile” model

Significant differences :

• Threshold mechanism now occurs on a chaotic “substrate”

i.e. there is an intrinsic deterministic dynamics at each site

• State variable analogous to the integer “height” variable in sandpile models is continuous here

Local chaos : intrinsic or internal perturbation

Sandpile model : “external” perturbation effected by dropping “sand” from outside

- Enhanced complexity

This system is capable of exhibiting a wider repertoire of dynamics :

From spatiotemporal cycles of all orders and spatiotemporal chaos to scaling SOC-like regimes

Threshold coupling of local chaos:

• Local nonlinear dynamics : source of local instability/pertrubation

• Thresholding process : self regulatory; adaptive; can serve as a global organizing principle in large interactive systems

Relevant for certain mechanical systems like chains of nonlinear springs

As also for some biological systems, such as synapses of nerve tissue

(Individual neurons display complex chaotic behaviour and have step function-like reponses to stimuli)

Relevant parameters in the model are :

- Threshold level x_c
- Local nonlinearity parameter
- Relaxation time Δ

Interesting features to be examined are :

- Temporal evolution of local quantities, such as the individual sites $x_n(i)$

- Temporal evolution of globally defined quantities, such as "avalanches" (denoted by s):

Total number of "active" sites, i.e. sites that have "toppled" during the threshold relaxation

- Spatial distribution of $x(i)$;

Presence of clustering and coherence in space as indicated by the cluster distribution

- Dynamics and distribution of the "drop rate"

i.e. the "excess" transported out of the lattice edge during relaxation

Unidirectional Model: Phenomenology

Unlike sandpile models, this 1-d unidirectional transport model is not trivial : gives rise to many dynamical phases

- Local on-site dynamics : $f(x) = 1 - ax^2$, $x \in [-1, 1]$:

Operating in the chaotic regime,
i.e. $a = 2$

- Threshold range : $-1 < x_c < 1$

- Δ large : allows complete relaxation of the active sites before the subsequent chaotic update

Most significant parameter is the threshold level x_c : $-1 < x_c < 1$:

Dynamical phases :

- When $x_c < 0.5$: coherent state

All sites $x_n(i) = x_c$ for all times

Size and duration of the avalanches always equal to N

Threshold mechanism suppresses the underlying chaos in the lattice and yields spatiotemporal regularity

When $0.5 < x_c < 1.0$: Temporal evolution of the lattice sites (and avalanches) attracted to exact cycles

Periodicity (order) of the cycle depends on threshold

- For $0.5 < x_c \leq 0.809$: 2-cycles
- For $0.809 < x_c < 0.85$: 4-cycles
- For $x_c \sim 0.86$: 6-cycles
- For $x_c \sim 0.88$: 7-cycles
- For $x_c \sim 0.9$: 10-cycles
- For $x_c \sim 0.98$: 4-cycles

Using symbolic dynamics techniques one can analytically obtain:

- Spatial and Temporal characteristics of the various cycles

- Exact scaling relations for the position and width of periodic windows in threshold parameter space

(Sinha, 1993-94)

Possible engineering applications

Control and Synchronization schemes :

Sinha, Phys. Rev. E (2001)
Murali and Sinha (2002-2003)

- This strategy for control does not entail adjusting parameters

Principle : restricts available phase space; dynamic range limiter

- System does not have to be close to target for successful control

- Control transience extremely short

- Controller simple

- In higher-dimensional system :
only one variable needs to be thresholded in order to effect control

Even in hyper-chaotic systems with more than one positive lyapunov exponent

- Does not entail any run-time computation

Ideal as a look-up table

- Perturbations involved in threshold control are small

- Control (threshold event) infrequent for higher cycles

- Robust to noise and delays in thresholding action

Chaos-based Computing Architectures

Sinha and Ditto, Phys. Rev. Lett. (1998)
Sinha, Phys. Rev. E (1999)
Sinha, Munakata and Ditto, Phys.Rev. E (2002)
Munakata, Sinha and Ditto, IEEE Trans. on Circuits and Systems, (2002)

- Thresholding yields controlled I/O relations

- Wide range of periods obtainable from chaos

Exploit :

- The complex dynamics of the individual units

- The controlled response afforded by thresholding

Opportunities afforded by chaos:

- Determinism : can reverse engineer

- Richness of temporal patterns :

can be used to obtain flexible and versatile computing schemes

Demonstrated the flexible implementation of all fundamental logic gates

dynamic hardware : ingredients of a programmable general purpose machine

Threshold dynamics on a random lattice :
no regular dynamical phases

For these phases to occur we crucially require a deterministic dynamics, even if entirely chaotic

Difference between stochasticity and deterministic chaos is of vital importance

Variations of relaxation time Δ in unidirectional transport models

- Large Δ ($\Delta > N$) : exact cyles

- Intermediate Δ ($N/4 \leq \Delta \leq N/2$) :

pronounced $1/f$ noise

Phenomenology of

- Uni-directional transport under noise
- Bi-directional transport

- Yields well defined $1/f$ spectrum :

$$\boxed{S(f) \sim \frac{1}{f^{\phi}} \qquad\qquad \phi \sim 1}$$

Plus a sharp peak at a rough characteristic frequency, namely the frequency of the exact cycle associated with the thresholded chaotic map

• Scaling in distribution of avalanche sizes s :

Range of values from 0 to $s_{max} = \Delta \times N$

$$\boxed{P(s) \quad \sim \quad s^{-\phi_s}}$$

Plus a peak at large values of s ($s \to s_{max}$): related to large events which occur approximately periodically

• Size of active clusters k at any given instant of time

i.e. the number of consecutive sites that have "toppled" or become active during the course of the "avalanche" ($1 \leq c \leq N$):

$$\boxed{P(k) \quad \sim \quad s^{-\phi_k}}$$

- Spatial cluster sizes c ($1 \leq c \leq N$):

i.e. number of consecutive sites for which the inequality $|x_n(i+1) - x_n(i)| < \epsilon$ holds, (ϵ small)

$$\boxed{P(c) \quad \sim \quad s^{-\phi_c}}$$

These scaling realtionships indicate that threshold responses take place over a range of length and time scales :

From local re-arrangements

To large events where the activity involves nearly all elements of the network

Exponents are system dependent

Consequences of random connections on threshold coupled chaotic maps

Some degree of randomness in spatial coupling can be closer to physical reality than strict nearest neighbour scenarios

In fact many systems of biological, technological and physical significance are better described by randomising some fraction of the regular links

Try to determine what dynamical properties are significantly affected by the way connections are made between elements

Consider the threshold coupled system with its coupling connections rewired randomly in varying degrees

At every thresholding step we will transport the excess δ with probability p to randomly chosen sites in the lattice, and with probability $(1-p)$ to the nearest neighbour

$p = 0$: usual nearest neighbour interaction

$p = 1$: completely random coupling

Study the spatiotemporal dynamics of this network with respect to p : $0 \leq p \leq 1$

Numerical evidence obtained by sampling a large set of random initial conditions ($\sim 10^4$), and with lattice sizes ranging from 10 to 1000, indicate :

Randomness in spatial connections favours synchronised networks in threshold regimes which have multiple coexisting unsynchronised attractors under regular connections

- Average time required to reach synchronisation decreases as a power law with the fraction of random rewirings, for small p

- Average time required to reach synchronisation is minimum for a certain fraction of random rewirings

- Degree of synchronisation is maximum for a certain fraction of random rewirings

Optimum randomness in connections

The robust synchronisation obtained by random coupling has immediate relevance to controlling/synchronising extended chaotic systems

The regularising effect of random coupling may then help to :

- Design synchronisation methods for spatially extended interactive systems

- Suggest natural regularising mechanisms in physical and biological systems