

Inducing Order in a Network of Chaotic Elements

Sudeshna Sinha

The Institute of Mathematical Sciences, Chennai



homepage: <http://www.imsc.res.in/~sudeshna>

- **Lattices of coupled maps** were introduced as simple models capturing the essential features of nonlinear dynamics of extended systems
- Such models yield suggestive conceptual frameworks for spatiotemporal phenomena, in fields ranging from biology to engineering
- The ubiquity of distributed complex systems made lattices of nonlinear maps a focus of sustained research interest

Then along came the **network boom!**

Various studies showed that some degree of randomness in spatial coupling is closer to physical reality than strictly regular scenarios

Many systems of biological, technological and physical significance are better described by **randomising some fraction of the regular links**

What is the spatiotemporal dynamics of a collection of elemental dynamical units with varying degrees of randomness in its spatial connections?

Consider a one-dimensional ring of coupled nonlinear maps

The **sites** (nodes) are denoted by integers $i = 1, \dots, N$
where N is the size of the lattice (network)

On each site is defined a continuous state variable denoted
by $x_n(i)$

Corresponds to the physical variable of interest

The evolution of this lattice in discrete time n
under standard **nearest neighbour interactions** :

$$x_{n+1}(i) = (1 - \epsilon)f(x_n(i)) + \frac{\epsilon}{2}\{x_n(i+1) + x_n(i-1)\}$$

Strength of coupling : ϵ

The **local on-site map** could, for instance, be the fully
chaotic logistic map:

$$f(x) = 4x(1 - x)$$

Now consider the system with its coupling connections rewired randomly in varying degrees

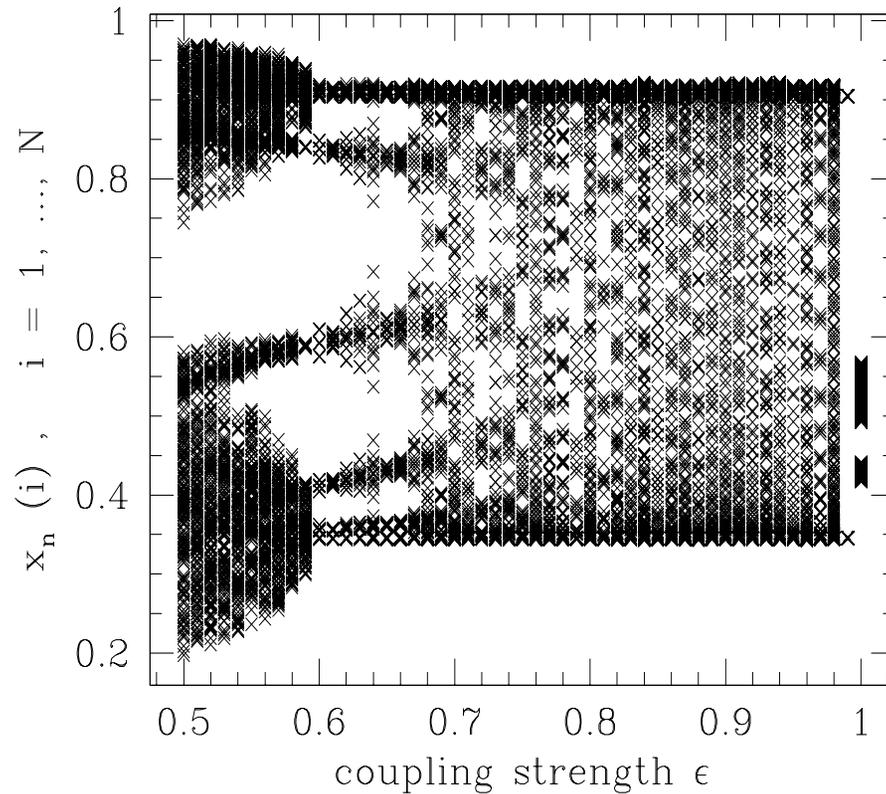
At every update we will connect a fraction p of randomly chosen sites in the lattice to two random sites

That is, we will **replace a fraction p of nearest neighbour links by random connections**

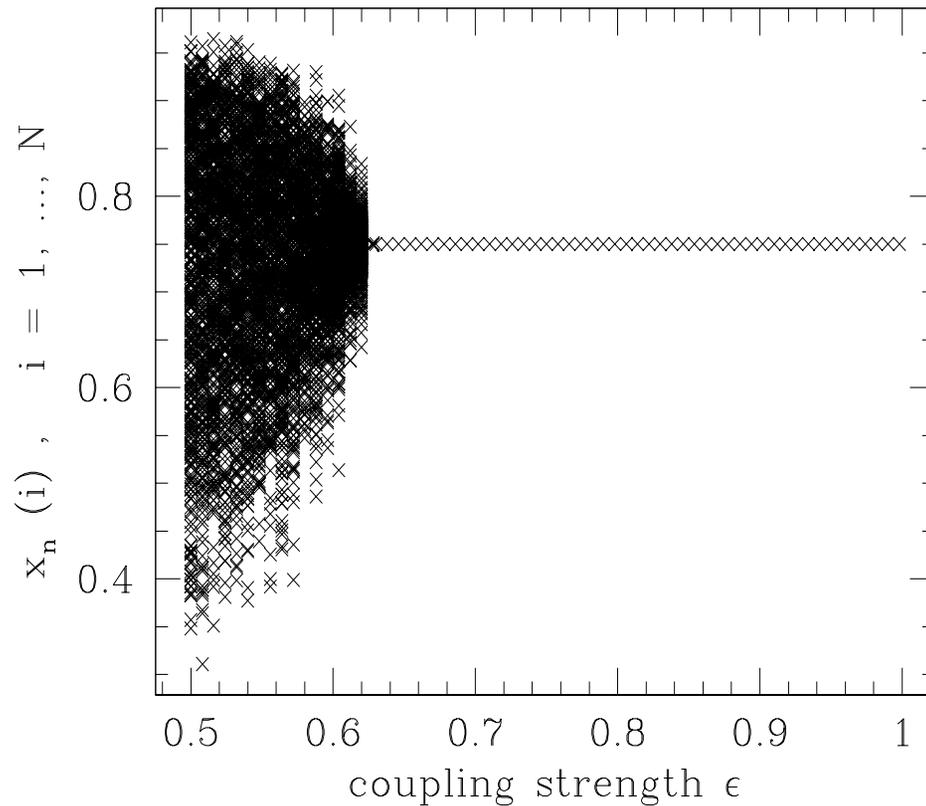
$p = 0$: corresponds to the usual nearest neighbour interaction

$p = 1$: corresponds to completely random coupling

- Dynamic Rewiring : random links are switched
- Static Rewiring (Quenched) : fixed random links



Coupled logistic maps with **regular nearest neighbour connections**



Coupled **strongly chaotic** logistic maps with **completely random connections**

(Sinha, 2002)

Dynamic Random Links create a large window in coupling parameter space where a **spatiotemporal fixed point state gains stability**

Onset of spatiotemporal fixed point : ϵ^*

For completely random coupling $p = 1 : \epsilon^* \sim 0.6$

For all $p > 0$: there is a **stable region of synchronized fixed points**

i.e. for all finite p , $\epsilon^* < 1$

In the stable region of synchronized fixed points
namely, in the parameter interval $\epsilon^* \leq \epsilon \leq 1$:

All lattice sites i are synchronized at $x_n(i) = x^* = 3/4$

Where $x^* = f(x^*)$ is the fixed point solution of the individual
chaotic maps

x^* : strongly unstable in the local chaotic map

Analyse this system to account for the much enhanced stability of the homogeneous phase under random connections

Only possible solution for a spatiotemporally synchronized state :

All $x_n(i) = x^*$ only when $x^* = f(x^*)$

For the case of the logistic map at $r = 4$:

Fixed point solution of the local map $x^* = 4x^*(1 - x^*) = 3/4$

To calculate the stability of the lattice with all sites at x^* we will construct an **average probabilistic evolution rule** for the sites :

mean field version of the dynamics

Some effects due to fluctuations are lost, but as a first approximation we have found this approach qualitatively right, and quantitatively close to to the numerical results as well

All sites have probability p of being coupled to random sites, and probability $(1 - p)$ of being wired to nearest neighbours

Then the averaged evolution equation of a site j is

$$x_{n+1}(j) = (1 - \epsilon)f(x_n(j)) + (1 - p)\frac{\epsilon}{2} \{x_n(j + 1) + x_n(j - 1)\} + p\frac{\epsilon}{2} \{x_n(\xi) + x_n(\eta)\}$$

where ξ and η are random integers between 1 and N

Linear Stability Analysis of the coherent state:

Replacing $x_n(j) = x^* + h_n(j)$, and expanding to first order gives

$$h_{n+1}(j) = (1 - \epsilon)f'(x^*)h_n(j) + (1 - p)\frac{\epsilon}{2} \{h_n(j + 1) + h_n(j - 1)\} \\ + p\frac{\epsilon}{2} \{h_n(\xi) + h_n(\eta)\}$$

$$\approx (1 - \epsilon)f'(x^*)h_n(j) + (1 - p)\frac{\epsilon}{2} \{h_n(j + 1) + h_n(j - 1)\}$$

i.e., to a first approximation one can consider the sum over the fluctuations of the random neighbours to be zero

This approximation is clearly more valid for small p

For stability considerations one can diagonalize the above expression using a Fourier transform

$$h_n(j) = \sum_q \phi_n(q) \exp(ijq)$$

where q is the wavenumber and j is the site index

This finally leads us to the following growth equation:

$$\frac{\phi_{n+1}}{\phi_n} = f'(x^*)(1 - \epsilon) + \epsilon(1 - p) \cos q$$

with q going from 0 to π

Clearly the stabilization condition will depend on the nature of the local map $f(x)$ through the term $f'(x)$

Considering the fully chaotic logistic map with $f'(x^*) = -2$, one finds that the growth coefficient that appears in this formula is **smaller than one in magnitude** if and only if

$$\frac{1}{1+p} < \epsilon < 1$$

i.e.

$$\epsilon^* = \frac{1}{1+p}$$

The range of stability $\mathcal{R} = 1 - \epsilon^*$ is

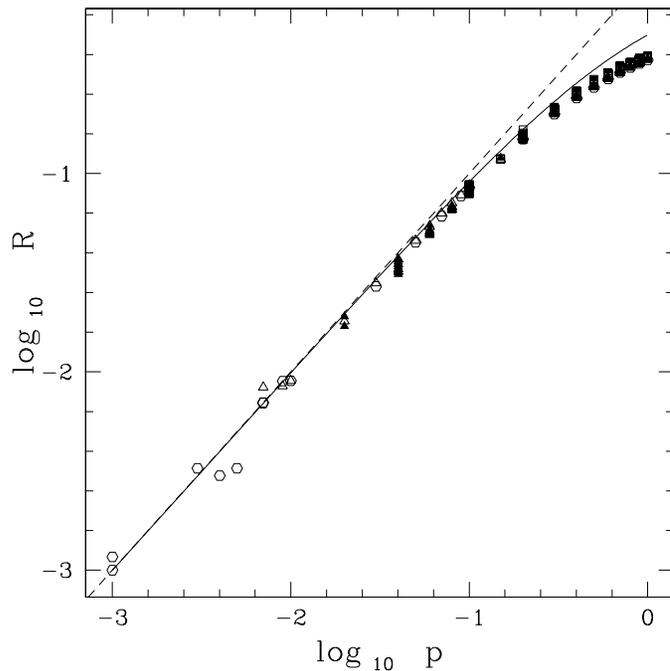
$$\mathcal{R} = 1 - \frac{1}{1+p} = \frac{p}{1+p}$$

For small p ($p \ll 1$) standard expansion gives

$$\mathcal{R} \sim p$$

- Regular nearest neighbour couplings ($p = 0$) gives a **null range**
- Fully random connections ($p = 1$) yields the **largest stable range** : $R \sim 1/2$

Synchronized Fixed Point Range \mathcal{R} vs. random rewiring probability p



Solid line : analytical result; Dotted line: $\mathcal{R} = p$

Points : Simulations (with different lattice sizes $N = 10, 50, 100, 500$)

(Sinha, 2002)

- So stability analysis clearly dictates that enhanced stability of the spatiotemporally homogeneous phase must occur under random connections, just as numerical evidence shows
- So any degree of randomness in spatial coupling connections opens up a synchronized fixed point window
- Dependence of the stability on p : **monotonic**
- No enhancement at dilute rewiring ('small world limit')

Results from Other Models

- Coupled tent maps, with the local map given as:

$$f(x) = 1 - 2|x - 1/2|$$

Unstable fixed point : $x^* = 2/3$

Local slope : $f'(x^*) = -2$

- Coupled circle map networks, where the local map is

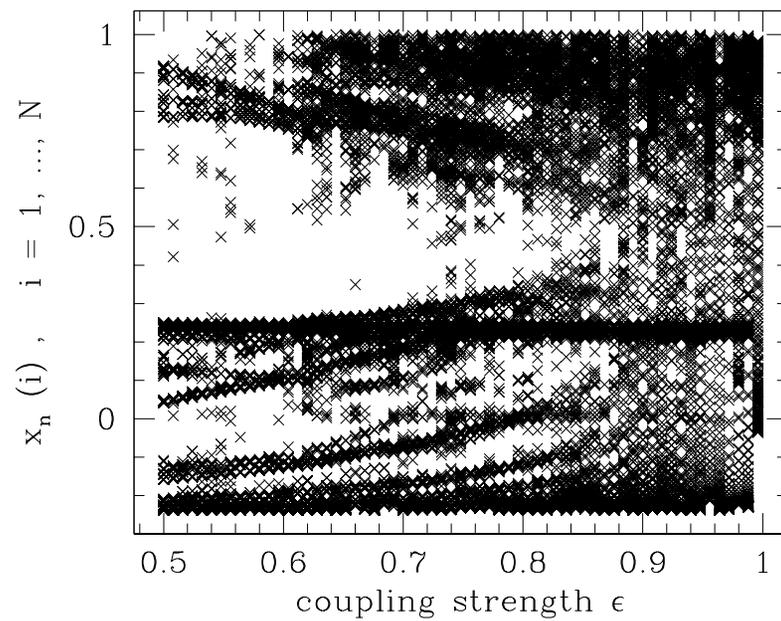
$$f = x + \Omega - \frac{K}{2\pi} \sin(2\pi x)$$

A representative example: $\Omega = 0, K = 3$

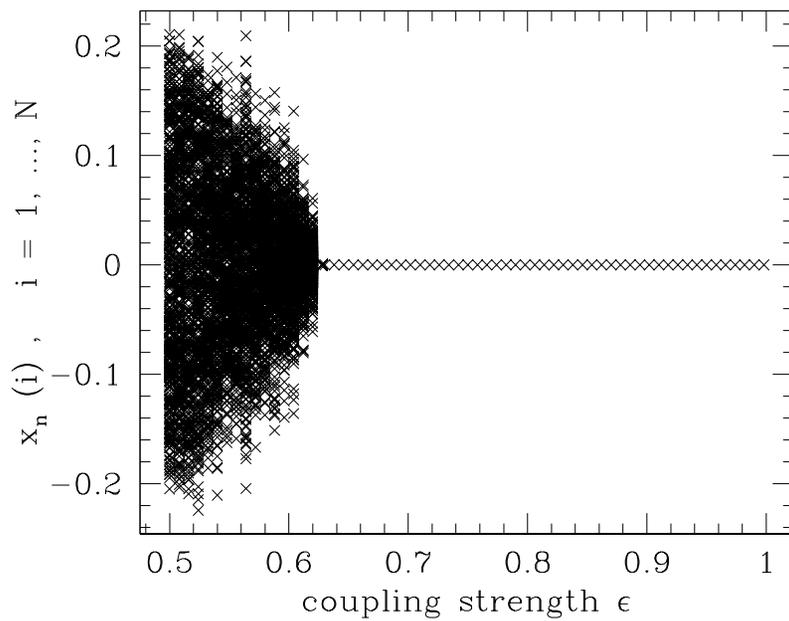
Unstable fixed point : $x^* = \frac{1}{2\pi} \sin^{-1}(\Omega/K)$

Local slope : $f'(x^*) = -2$

In both systems random rewiring yields stable spatiotemporal synchronisation



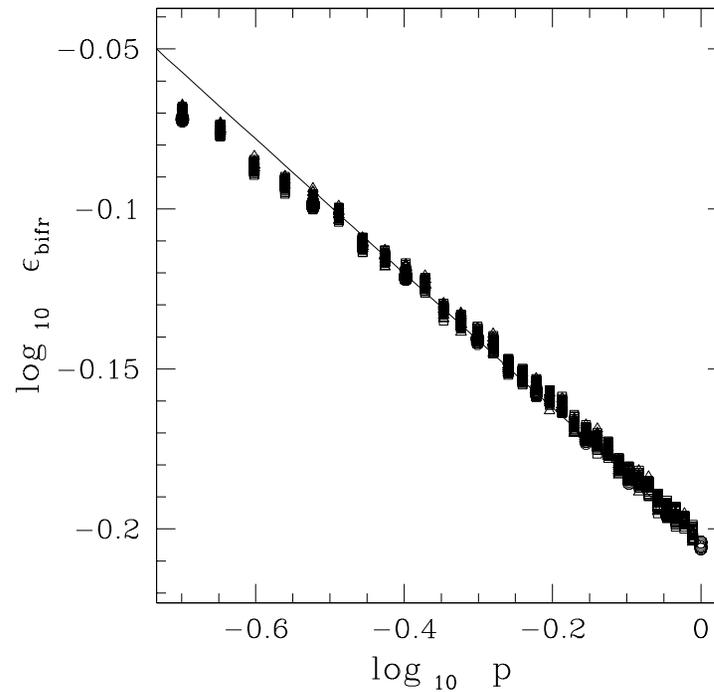
Coupled sine circle maps with strictly regular nearest neighbour connections



Coupled sine circle maps with completely random connections

- Since $f'(x^*) = -2$ for both the tent map and the circle map, we expect from our analysis that their ϵ^* and \mathcal{R} will be the same as for the logistic map
- This agrees with simulations :

Numerically obtained ϵ^* values for ensembles of coupled tent, circle and logistic maps fall indistinguishably around each other, even for high p where the analysis is expected to be less accurate



Coupled tent maps (open squares)
Coupled circle maps (open triangles)
Coupled logistic maps (open circles)

Static Rewiring

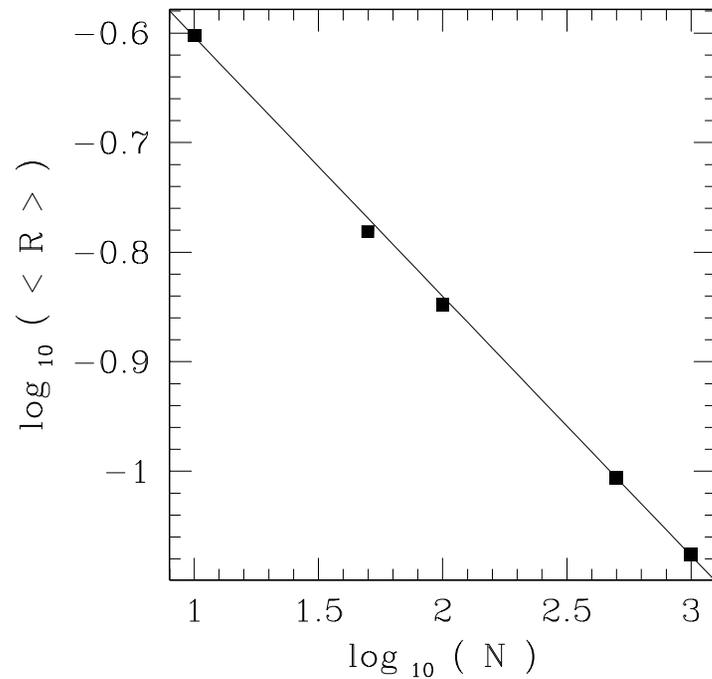
For dynamical rewiring : the synchronization range \mathcal{R} is independent of the size and initial network connections

On the other hand, for static rewiring there is a spread in the values of \mathcal{R} obtained from different realisations of the random connections

The distribution of \mathcal{R} is dependent on the size of network,
with average \mathcal{R} scaling with N as :

$$\langle \mathcal{R} \rangle \sim N^{-\nu}$$

with $\nu \sim 0.24$



Average range $\langle \mathcal{R} \rangle$ of spatiotemporal synchronization obtained in the case of **static random connections** with respect to network size N ($p = 1$)

- This behaviour can be understood by examining the linear stability of the homogeneous solution: $x_n(j) = x^*$ for all sites j at all times n
- Considering the dynamics of small perturbations over the network, one obtains the **transfer matrix** connecting the perturbation vectors at successive times to be a **sum** of :
 - $N \times N$ **diagonal matrix** : with entries $(1 - \epsilon)f'(x^*)$
 - $\epsilon/2 \times \mathbf{C}$: where \mathbf{C} is the connectivity matrix

For example: for $p = 1$, \mathbf{C} is a $N \times N$ **sparse non-symmetric matrix** with two random entries of 1 on each row

- The **minimum of the real part of the eigenvalues** of C , λ_{\min} , crucially governs the stability
- Typically $\epsilon^* = 2/\{\lambda_{\min} + 4\}$ when $f'(x^*) = -2$
- Now the values of λ_{\min} obtained from different static realisations of the connectivity matrix C are distributed differently for different sizes N
- For small N this distribution is broad and has less negative averages (~ -1)
- On the other hand for large N the distribution gets narrower and tends towards the limiting value of -2
- This implies that the range of stability tends to zero for large enough networks under static rewiring

Same Fraction of Random Links at all times

Dynamic Rewiring (Switched Random Links) :
Spatiotemporal Fixed Point

Static Rewiring (Frozen Random Links) : Spatiotemporal
Chaos

Static to Dynamic Transition

Poster of **Arghya Mondal**

How fast does a network have to rewire in order to induce spatiotemporal order?

Coupled Model Neurons

with Maruthi Pradeep, Abhijit Sonawane, Prashant Gade

Nodal Dynamics: Maps that reproduce some of the basic features of the firing dynamics of neurons

For instance:

$$x_{t+1} = x_t^2 \exp(y_t - x_t) + k$$

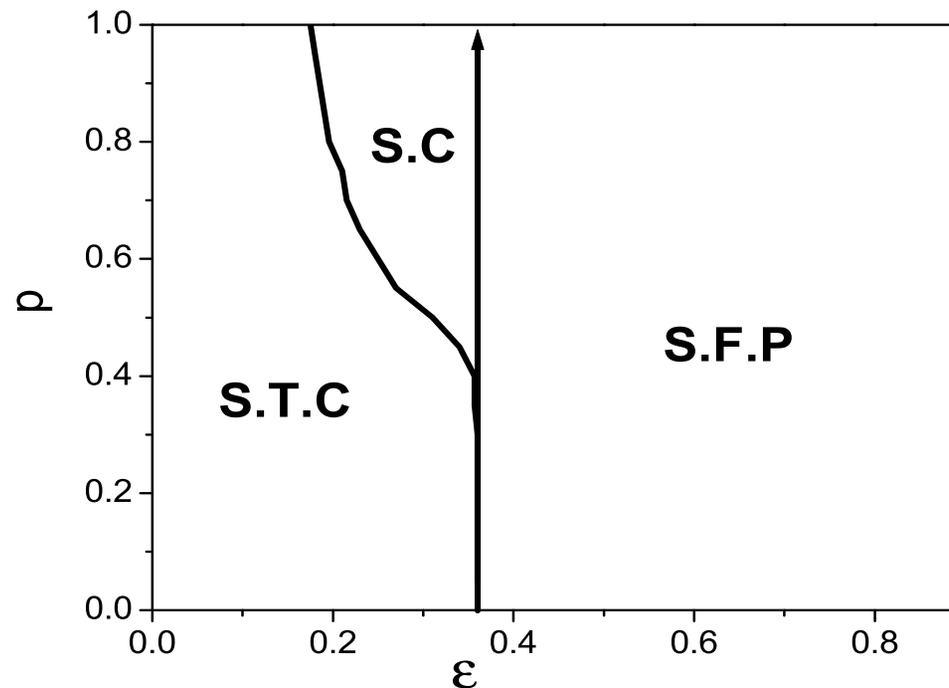
$$y_{t+1} = ay_t - bx_t + c$$

- x : related to an instantaneous membrane potential of the neuron
- y : equivalent to a recovery current

- parameter a : determines the time constant of reactivation
- parameter b : the activation dependence of the recovery process
- parameter c : the maximum amplitude of the recovery current
- parameter k : can be viewed either as a constant bias or as a time-dependent external stimulation

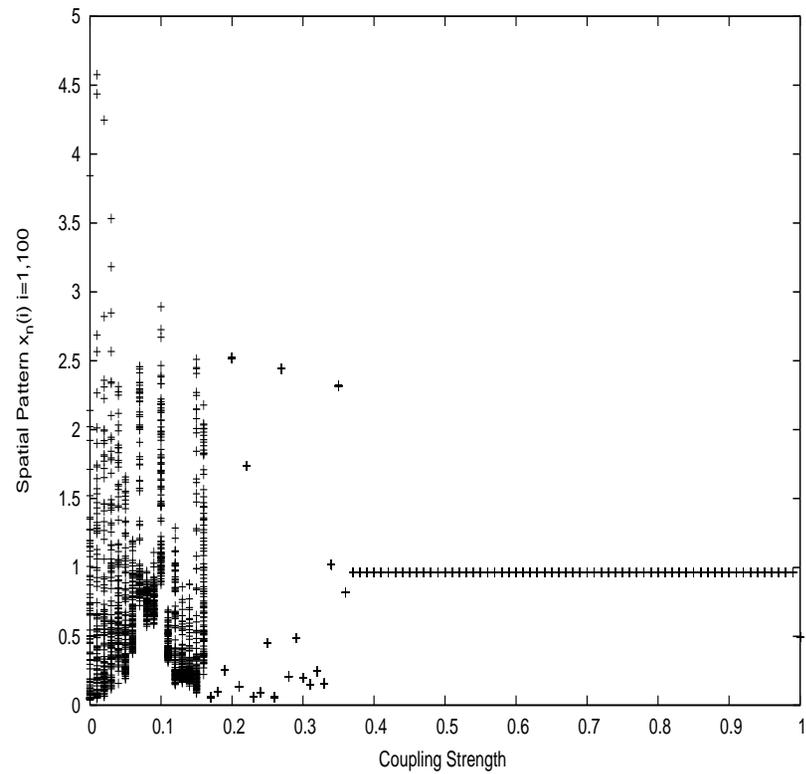
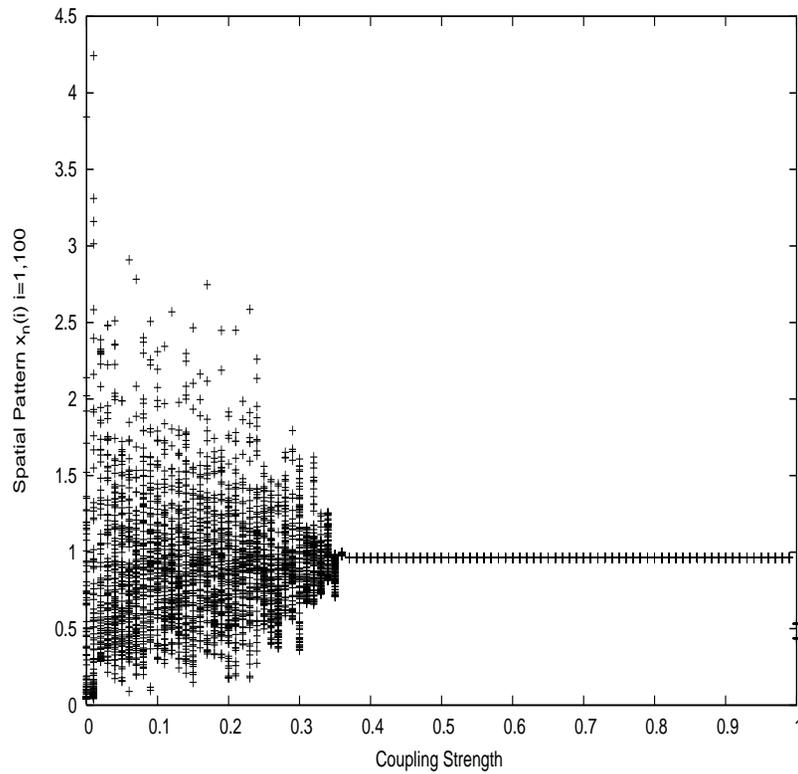
- The parameters here are chosen so as to make the dynamics completely **chaotic**

Phase diagram of coupled model neurons in the space of coupling strength ϵ and rewiring probability p

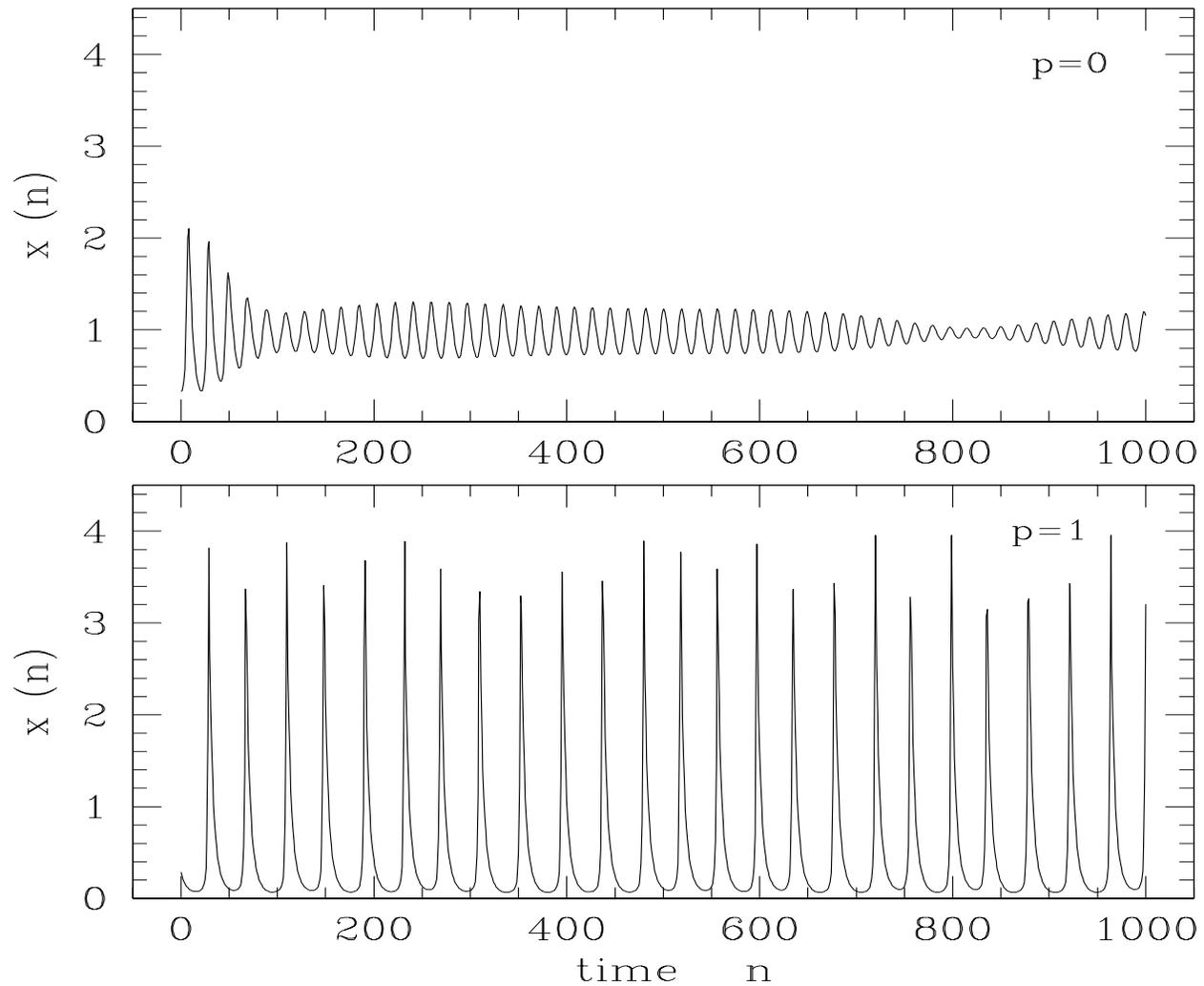


S.C.: synchronized chaos **S. T. C.:** spatiotemporal chaos
S. F. P. : synchronized fixed point

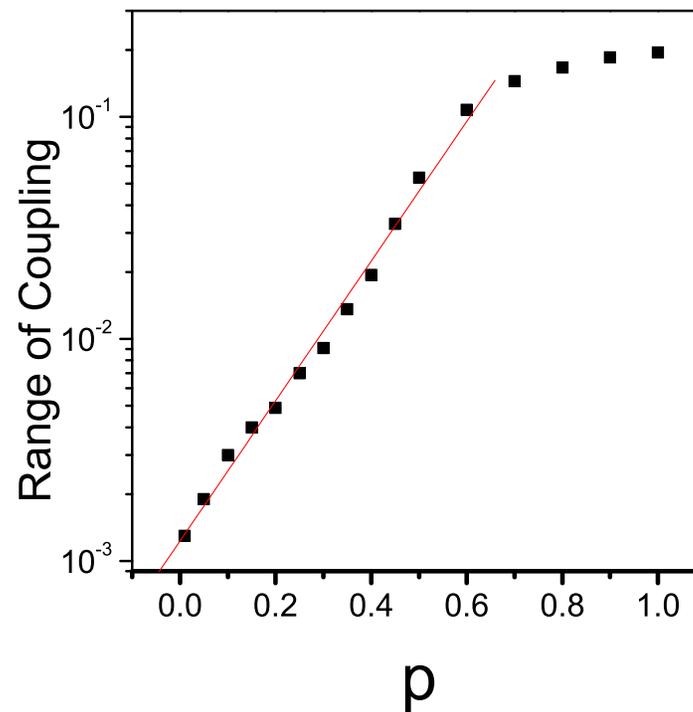
Instantaneous spatial pattern in the network : $p = 0$ and $p = 1$



Time series of of a representative model neuron



Range of **spatial synchronisation** increases with p



Exact analytic calculation : matches numerics

- Network of **Non-identical Neurons** : synchronization is **robust**
- Dynamic (Switched) random links gives rise to stronger synchronization than quenched (static) random links
- Quenched random coupling reduces synchronization error but does not yield complete synchronization as obtained by dynamic random coupling

Neuronal Model II:

$$x_{n+1} = f(x_n, y_n + \beta)$$

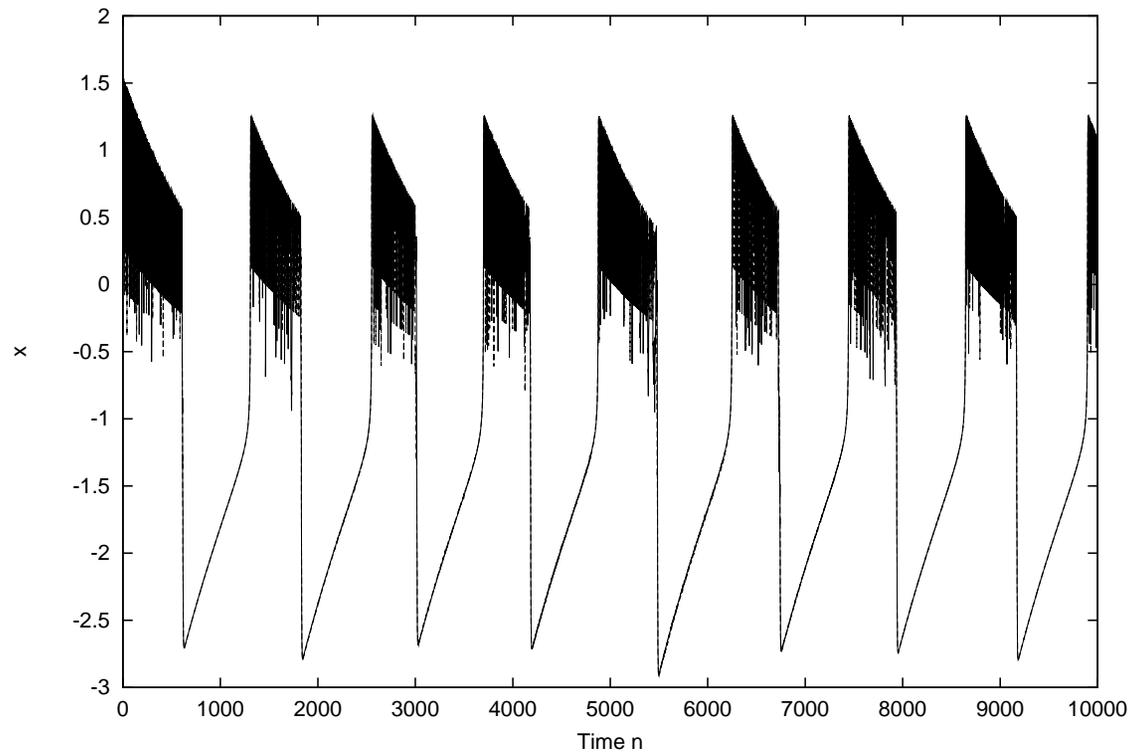
$$y_{n+1} = y_n - \mu(x_n + 1) + \mu\sigma$$

where x_n is the fast and y_n is the slow variable, and

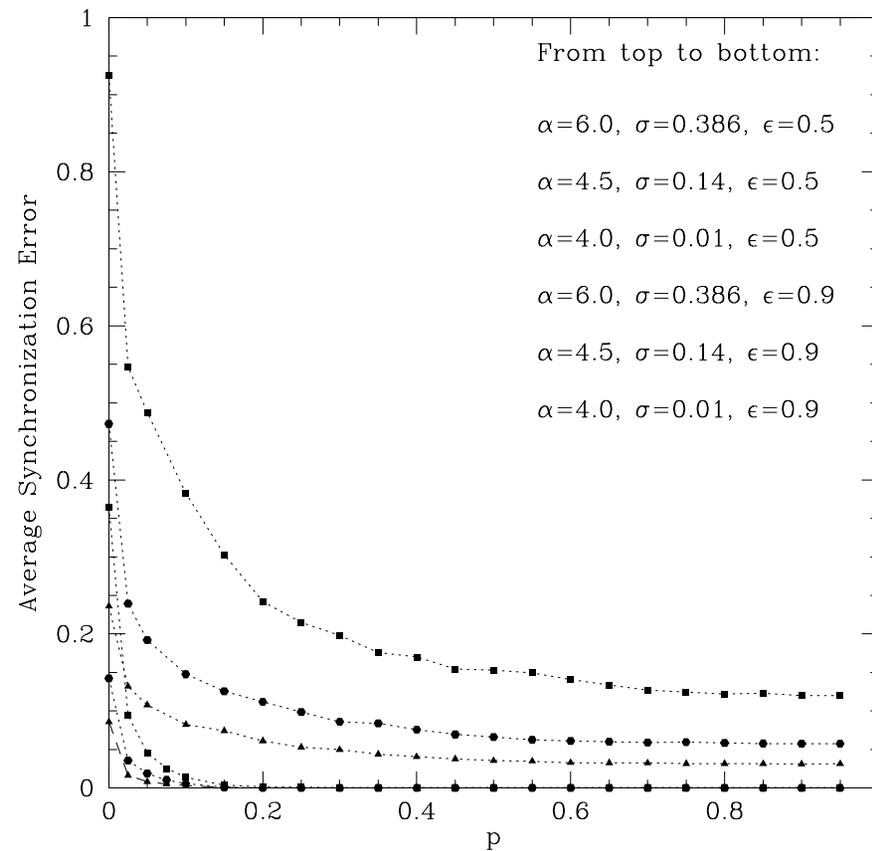
$$\begin{aligned} f(x, y) &= \alpha/(1 - x) + y && \text{when } x \leq 0 \\ f(x, y) &= \alpha + y && \text{when } 0 < x < \alpha + y \\ f(x, y) &= 1 && \text{when } x \geq \alpha + y \end{aligned}$$

Many different activity patterns occur at varying time scales:
including both **spiking** and **bursting**

Phase Synchronization: **Bursting pattern in-phase**



For Spiking, Bursting and Spiking-Bursting Patterns :
synchronization is enhanced by increasingly random links



Ring of Coupled Cells

The single cell consists of a **minimal biochemical pathway of three-step reaction sequence**

It is a general model of a large variety of functional dynamics observed in cellular systems

Somdatta Sinha, Rajesh

Here, a substrate S1 is converted to another substrate S2 and it is then converted to S3 through a reaction mediated by an enzyme

It is assumed that two different feedback loops governs the regulatory process of the pathway

The first one is a negative feedback, which is provided by the end-product inhibition of S2 by S3 and the other is due to the autocatalytic production of S2 by S3 by the enzyme

$$\frac{dx}{dt} = F(z) - kx$$

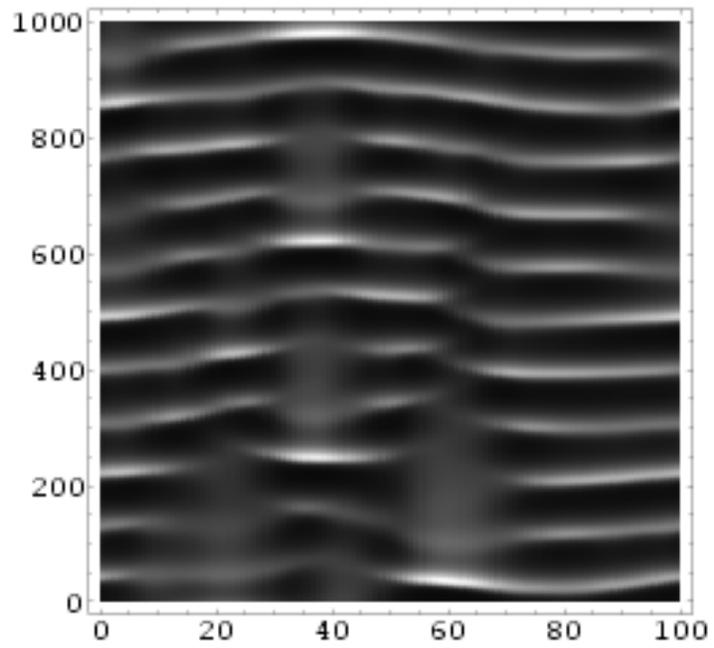
$$\frac{dy}{dt} = x - G(y, z)$$

$$\frac{dz}{dt} = G(y, z) - qz$$

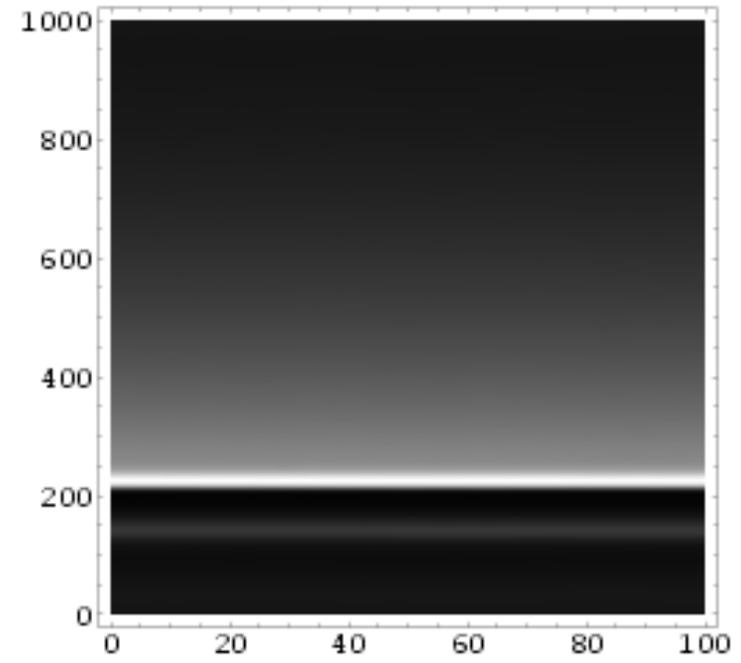
$$F(z) = \frac{1}{1+z^4}; \quad G(y, z) = \frac{Ty(1+y)(1+z)^2}{L+(1+y)^2(1+z)^2}$$

Coupling : **Diffusion of end-product z**

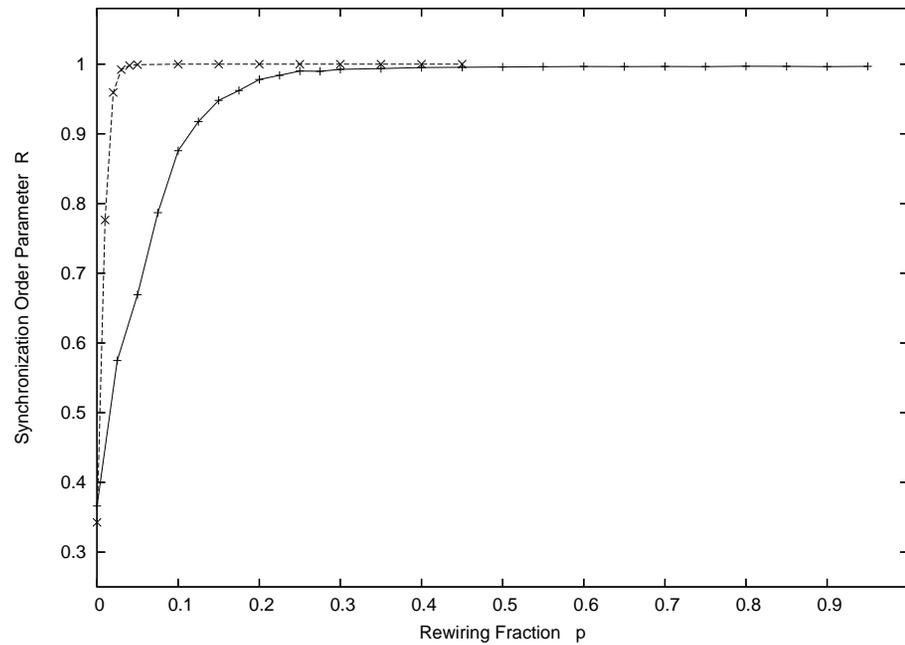
$p = 0$



$p = 0.05$



Efficiency of Dynamic Rewiring vis-a-vis Static Disorder



$p \rightarrow 0$ transition to synchronization for Dynamic Rewiring

Epidemic Modelling

Basic question: How does the dynamics of an infectious disease depend on the **structure and connectivity of a population** ?

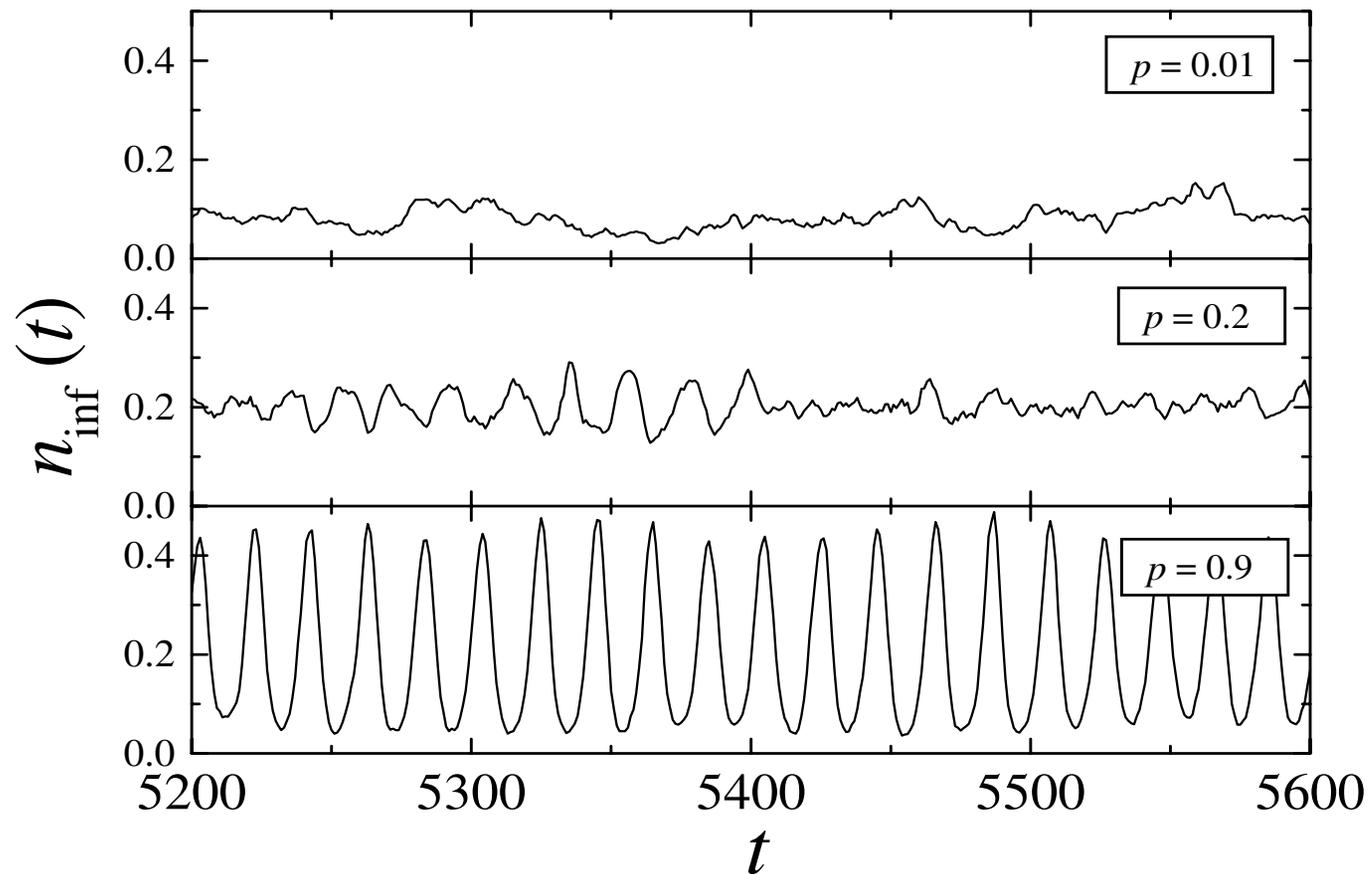
Role of the underlying network structure on the temporal dynamics of the epidemic

How does the **dynamics of the disease at the individual level** (at the nodes) affect the spread of infection ?

- Broad **qualitative** features of the spread of infectious disease can be captured by a **simple model** : SIRS (Susceptible-Infected-Refractory-Susceptible)
- Crucial ingredient : **connection network** :

Yields transition from **low quasi-fixed state** (analogous to endemic infection) to **self-sustained oscillations** (analogous to periodic outbreak of disease) as random links increases
- The formation of persistent oscillations in increasingly random networks corresponds to a **spontaneous synchronization of a significant fraction of the elements in the system**

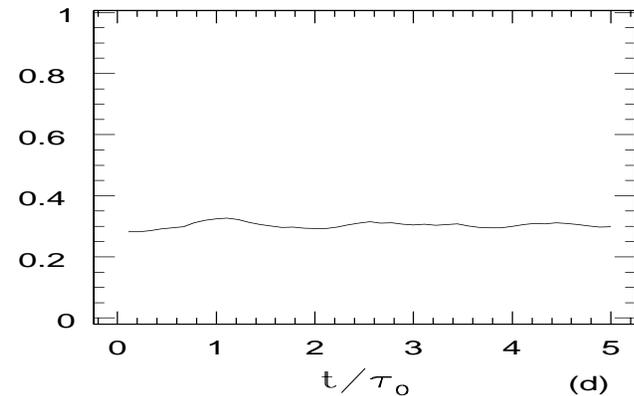
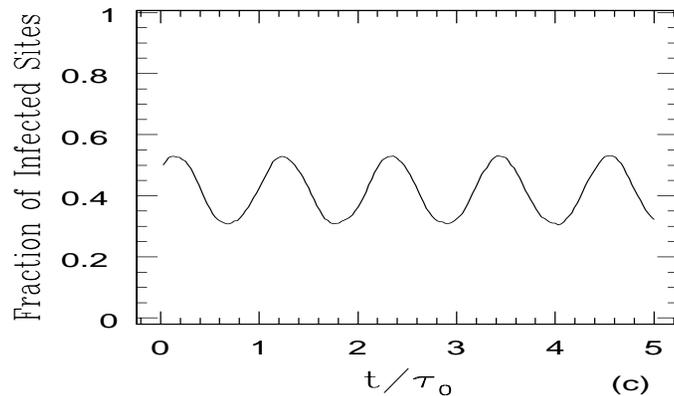
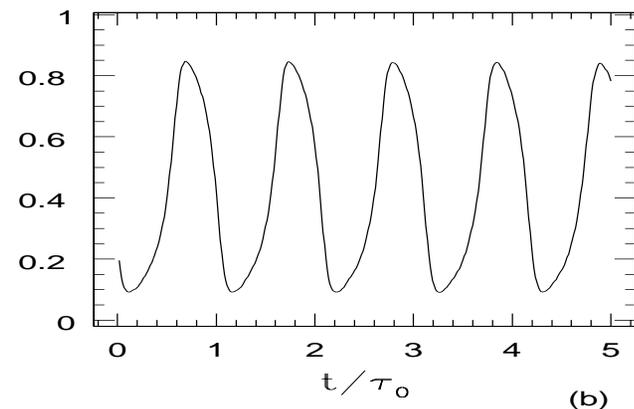
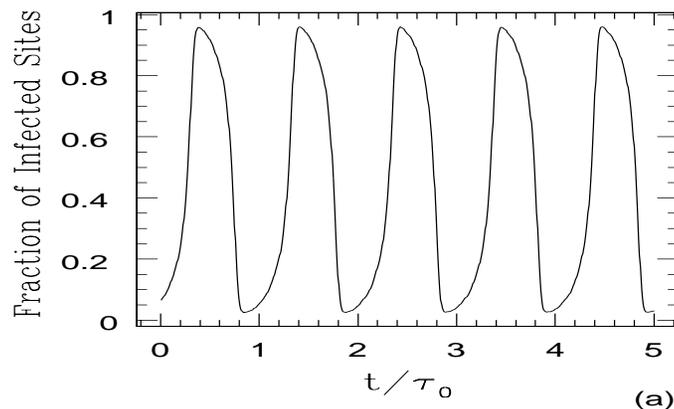
Nonlocal connections important in the spread of disease:
outbreaks can affect locations far apart geographically



Kuperman and Abramson

But timescales of the intrinsic disease cycle (nodal dynamics) also play a role

Longer disease cycle leads to large periodic oscillations arising from synchronized disease outbreaks



Fraction of infected sites vs time for different lengths of the disease cycle for fixed randomness

Gade, Sinha, 2005

An intuitive reason for this : For longer cycles the information that a given site is infected can propagate more, since the site stays infected for a longer time

Effective information transfer leads to collective phenomena like periodic excitations appearing spontaneously in the system

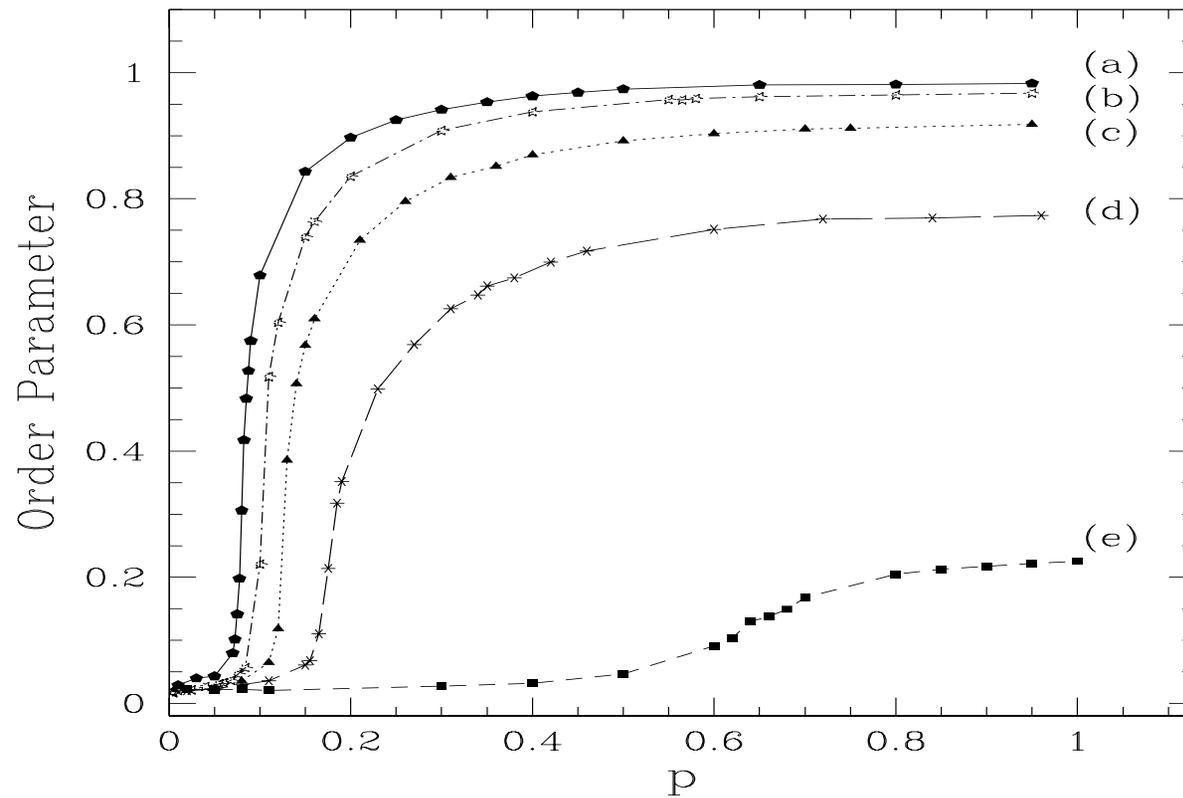
For instance, Ebola is far more deadly virus than HIV and kills the host much faster as it has a much shorter incubation period

However, due to the very fact that it kills so swiftly, Ebola outbreaks are contained very soon

The people infected by Ebola die very quickly, and so the virus has less time to jump to a new host and spread the disease

If no new victims come in contact with the body fluids of infected people in their short lifetime, the epidemic stops

On the other hand, HIV remains a problem worldwide since victim lives longer and has longer time to infect others



Transition point decreases as the disease progresses more slowly

Thus there is a clear interplay between the probability of nonlocal connections p and the timescales in the system

This suggests that in an extended parameter space one can find synchronization transitions at infinitesimal p in the very slow driving parameter limit

Asynchronous Updating enhances Spatiotemporal Regularity

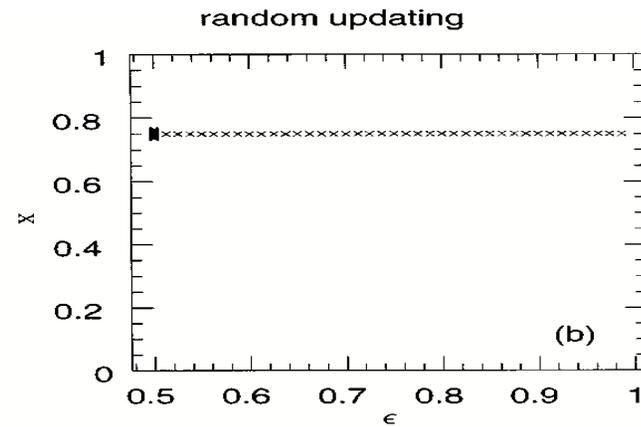
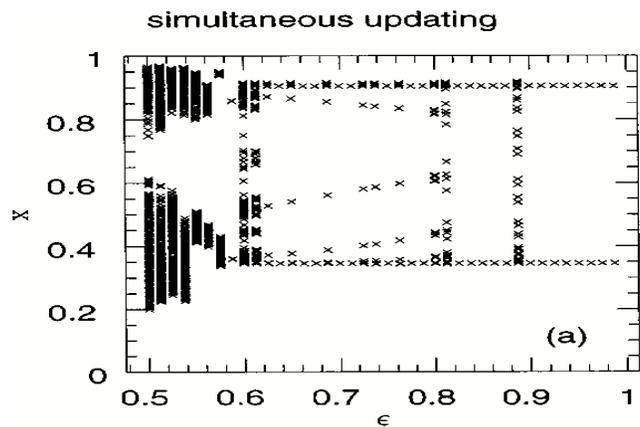
with Mitaxi Mehta (2000)

with Manish Shrimali & Kazu Aihara (2007)

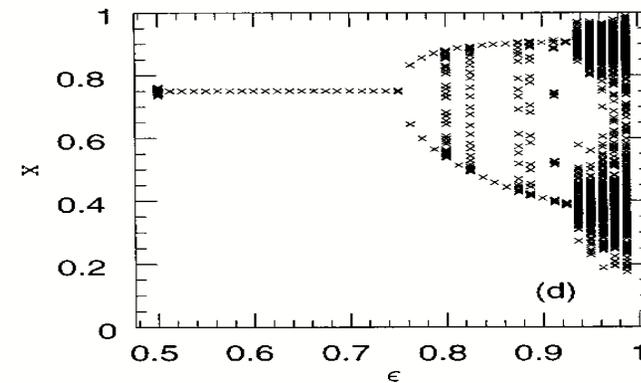
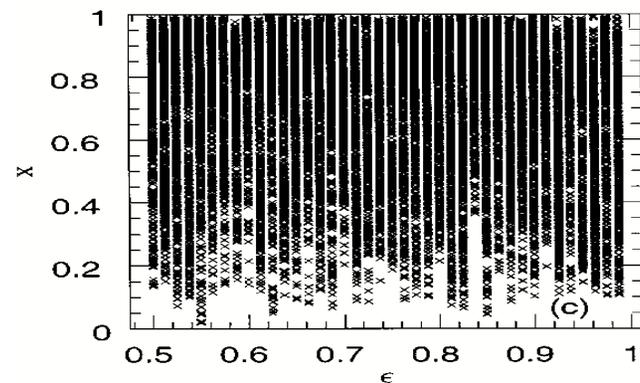
2-dimensional lattices of coupled chaotic maps

352

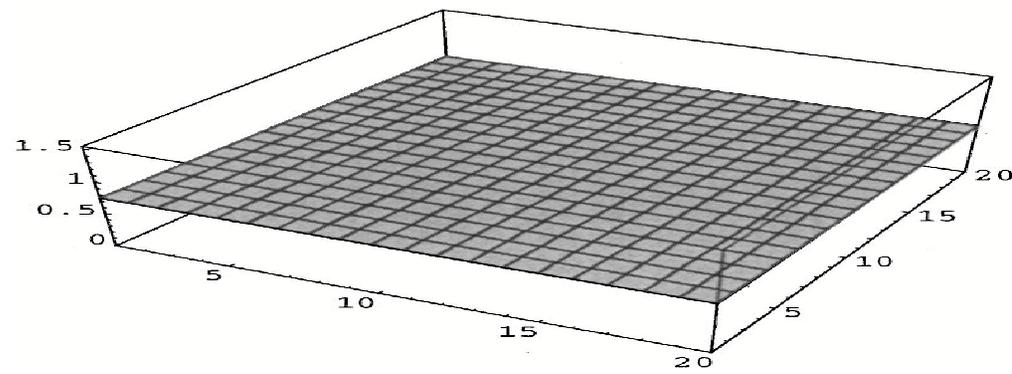
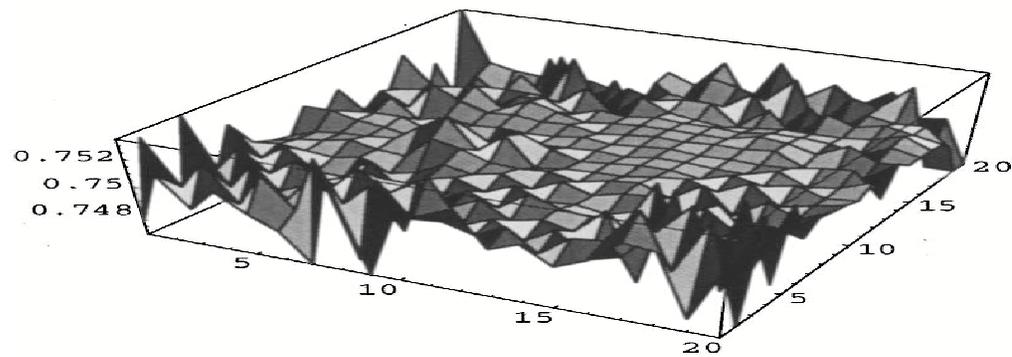
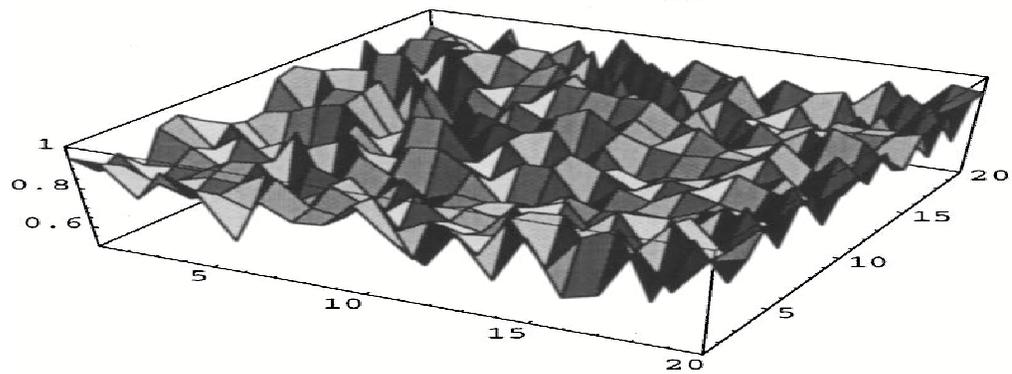
Chaos, Vol. 10, No. 2, 2000



coupling form $g(x) = x$



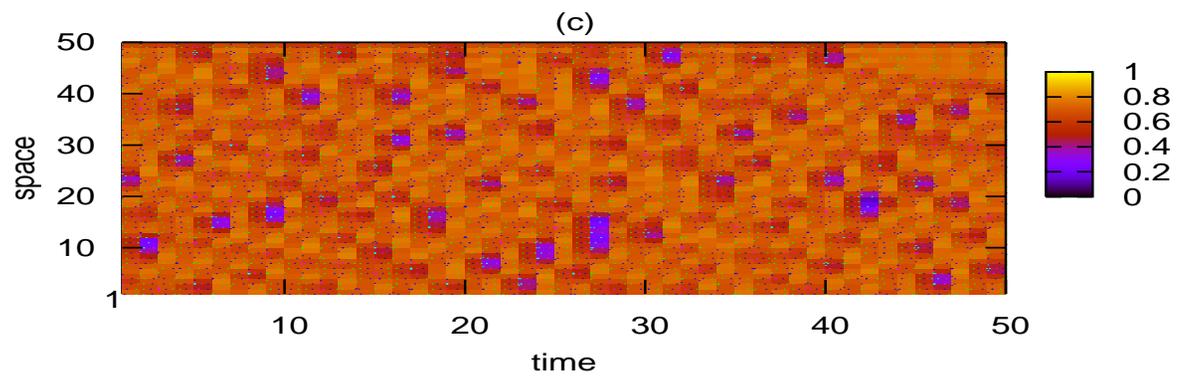
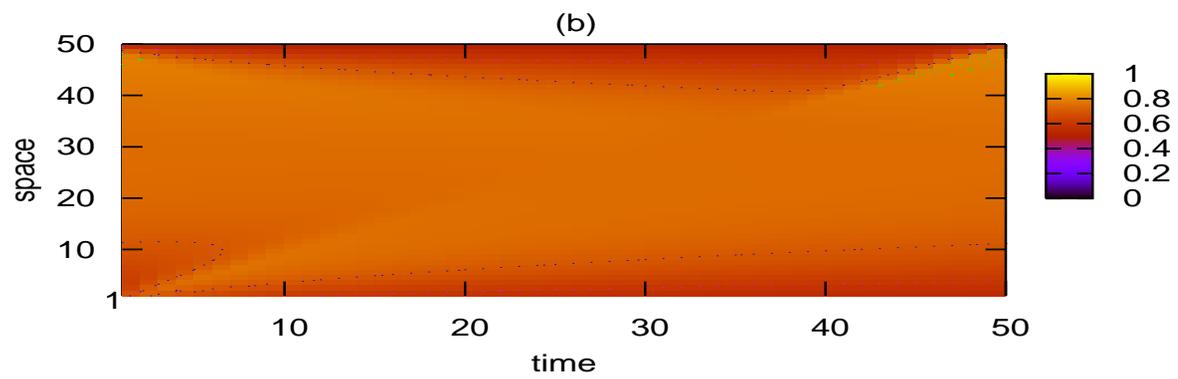
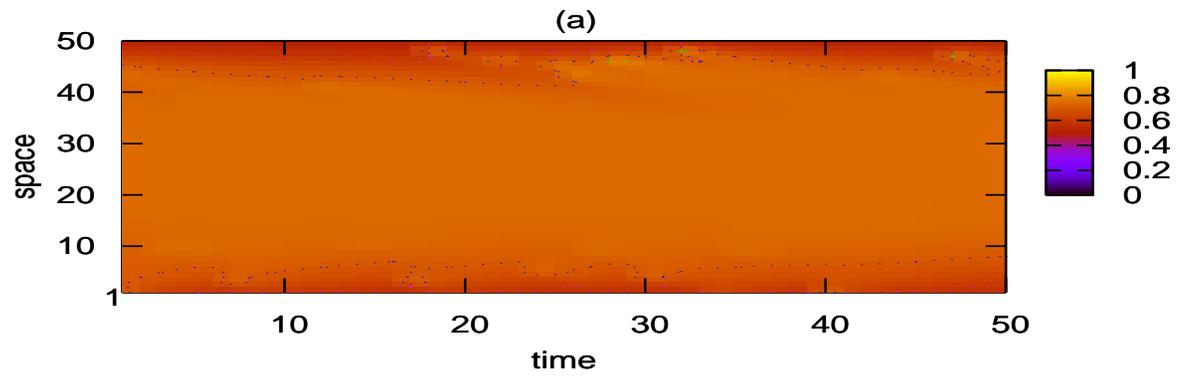
coupling form $g(x) = f(x)$



Stability Analysis with **Probabilistic Dynamical Evolution Equation** yields good match with numerics

Threshold Coupled Systems

Local neuronal dynamics



Outlook

- The regularising effect of random non-local coupling suggests a mechanism for regulation in natural physical and biological systems
- It also can help in designing efficient control methods for spatially extended interactive systems
- Significance of the Interplay between nodal dynamics and connectivity
- Significance of dynamic rewiring vis-a-vis static rewiring

- S. Sinha, *Phys. Rev. E*, v. 66 (2002) 016209
- S. Sinha, *Phys. Rev. E*, v. 69 (2004) 066209
- P.M. Gade & S. Sinha, *Phys. Rev. E*, v. 72 (2005)
- P. M. Gade & S. Sinha, *Int. J. of Bif. and Chaos* (2006)
- S. Rajesh, S. Sinha & S. Sinha, *Phys. Rev. E*, v. 75 (2007)
- Maruthi, Sonawane, Gade & S. Sinha, *Phys. Rev. E*, v. 75 (2007)
- M. Mehta and S. Sinha, *Chaos* v. 10 (2000)
- M. Shrimali, S. Sinha & K. Aihara, *Phys. Rev. E* v. 76 (2007)