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ADAPTIVE CONTROL IN NONLINEAR DYNAMICS

Sudeshna SINHA[1], Ramakrishna RAMASWAMY[2]
School of Physical Sciences[3], Jawaharlal Nehru University, Vedanta Desika Marg, New Delhi 110 067, India

and

J. SUBBA RAO
School of Environmental Sciences, Jawaharlal Nehru University, Vedanta Desika Marg, New Delhi 110 067, India

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We extend an adaptive control algorithm recently suggested by Huberman and Lumer to multi-parameter and higher-dimensional nonlinear systems. This control mechanism is remarkably effective in returning a system to its original dynamics after a sudden perturbation in the system parameters changes the dynamical behaviour. We find that in all cases, the recovery time is linearly proportional to the inverse of control stiffness (for small stiffness). In higher dimensions there is an additional optimization problem since increasing stiffness beyond a certain value can retard recovery. The control of *fixed point* dynamics in systems capable of a wide variety of dynamical behaviour is demonstrated. We further suggest methods by which periodic motion such as *limit cycles* can be adaptively controlled, and demonstrate the robustness of the procedure in the presence of (additive) background noise.

1. Introduction

A variety of physical and biological systems are well modelled by coupled nonlinear equations [1–3]. In most cases such systems are capable of displaying several types of dynamical behaviour – limit cycles, bistability, birhythmicity, hard excitation or chaos, for example [3]. Typically, the nature of the motion depends on the value of one or more variable parameters (in real systems these may be quantities such as electric fields, temperature or pressure gradients, or kinetic rates) in the system. In spite of the inherent capability of such systems to operate in a variety of behavioural regimes through the intrinsic nonlinearity, it is nevertheless observed that many real systems behave in a smooth fashion where predominantly a single type of behaviour obtains. This happens even when the parameters of the system can change owing to fluctuations driven by the environment. The scope of control or self-regulation in systems capable of complex dynamics is thus of considerable interest [4–6].

Recently Huberman and Lumer (HL) [7] have suggested a simple and effective adaptive control algorithm (for one-dimensional systems) which utilizes an error signal proportional to the difference between the goal output and the actual output of the system. This error signal governs the change of the parameters of the system, which readjust so as to reduce the error to zero. For a

[1]Also at: Chemical Physics Group, TIFR, Homi Bhabha Road, Bombay 400 005, India.
[2]Present address: Institute for Molecular Science, Myodaiji, Okazaki 444, Japan.
[3]Contribution No. 41.

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general N-dimensional dynamical system

$$\dot{X} \equiv \frac{dX}{dt} = F(X; \mu; t), \tag{1a}$$

where $X \equiv (X_1, X_2, \ldots, X_N)$ are variables and $\mu \equiv (\mu_1, \mu_2, \ldots, \mu_M)$ are parameters whose values determine the nature of the dynamics, the HL prescription for effecting adaptive control is through the additional dynamics,

$$\dot{\mu} = \epsilon(X - X_s), \tag{1b}$$

where X_s is the desired steady state value and ϵ indicates the "stiffness" of control. (For a single degree of freedom such as the examples HL treated, there is no ambiguity in eq. (1b), but in higher-dimensional cases X can, in principle, be *any* one of the dynamical variables. This aspect is dealt with in section 2.3 below.)

This algorithm is remarkably effective and rapid, and is of utility in a large variety of systems, ranging from biological units to control engineering. The efficacy of this idea was demonstrated by HL [7] in application to discrete maps with a single control parameter. When this parameter was perturbed so as to drive the system from a fixed point into the chaotic regime, say by changing the parameter instantaneously by an amount δ (a "shock"), the control mechanism was capable of pulling iterates back to the initial state. The recovery time, defined as the time taken to reach the desired state within finite precision after a shock was found to be inversely proportional to the stiffness of control. Furthermore, at each ϵ there was a maximum strength of shock, δ_{max}, beyond which the system failed to recover. HL also (empirically) observed that the δ_{max} was a linearly increasing function of ϵ, and so the product of the speed of relaxation and δ_{max} was constant [7].

The need for control mechanisms in order that a system is guaranteed to maintain a fixed activity even when subject to environmental fluctuations has long been discussed in system theory [8], as well as in biology [9]. Situations in the latter context, wherein these are thought to play a role, include the pupillary servomechanism [10], biological thermostats and the regulation of cell reactions [9]. Although the details of the control mechanism operating in a given situation may be system specific, from a theoretical stand point it is important to study the general principle by which systems can be brought back to a desired state by self-regulation. The concepts developed through the study of model systems may prove useful in understanding the more complex mechanisms by which biophysical processes maintain a steady state.

In the context of system theory, much effort has been directed toward the development of robust adaptive algorithms which remain effective in generic nonlinear systems [4, 5]. Indeed, it is widely recognized [8] that models of realistic systems are seldom completely known, and if known, are seldom linear. This points to the importance of understanding the stability characteristics of nonlinear control systems [11], and of developing control methodology for such situations [8].

In this paper we study the algorithm [7] outlined above in eqs. (1). One concern here is in exploring the control behaviour in higher dimensions, when the nature of the dynamics can become considerably more complicated than in the one-dimensional case. We find that the linear form specified in eq. (1b) appears to be adequate although an obvious generalization to

$$\dot{\mu} = \epsilon g(X - X_s), \tag{1c}$$

where $g(X - X_s)$ is some suitable function with $g(0) = 0$, may be necessary in specific contexts. We take as examples both continuous and discrete dynamical systems with several degrees of freedom and more than one controlling parameter. The study reveals a number of interesting features which are quite distinct from those in discrete 1D maps. Some properties of adaptive control, however, seem to be independent of the details of the system.

The organization of this paper is as follows. In section 2 we study the control of fixed points. We deal first with continuous time systems which undergo Hopf bifurcations. (Such systems are believed [2] to underlie the oscillations observed, for instance, in Cheyne–Stokes respiration, some types of muscle tremor and hematological disorders.) Examples of varying complexity are analyzed numerically and, wherever possible, analytically. For small control stiffness we observe a trend similar to that in discrete 1D maps, namely that recovery time is inversely proportional to ϵ. However, for larger ϵ some novel features emerge. We find that there is an optimum control stiffness, beyond which increasing ϵ can actually decrease the speed of recovery. In multiparameter systems, a simple extension of the adaptive mechanism works well in bringing back the system to steady state after two (or more) parameters have been perturbed. Application is made to examples in one and two dimensions. As for single parameter control, we observe that for small control stiffness the recovery time varies inversely proportional to ϵ, while for higher ϵ, there is saturation followed by an increase in recovery time.

Another motivation for the present study derives from the practical importance in controlling complex periodic behaviour, for example *limit cycles*. Many biological processes depend on the stabilization of such cycles (for example in glycolytic oscillations, peristaltic waves, electrical activity of the cortex, circadian rhythms, population dynamics, etc. [1, 2, 12]) and so it is important to find a feedback mechanism that can return a nonlinear system to a specific cycle after the parameters have been perturbed into other dynamical regimes. Accordingly, in section 3 we extend the control algorithm so as to regulate periodic behaviour. We do this by suitably modifying the error signal in the feedback loop, and demonstrate the efficacy of this method both in continuous and discrete systems.

It is also important to determine how robust this present adaptive control mechanism is. In section 4 the effect of external background noise on the control procedure is discussed, as also is the effect of changing the control function $g(X)$. Our results, which are summarized in section 5, are mainly empirical, and indicate that the technique is extremely well behaved under a variety of conditions.

2. Control of fixed points

We apply the adaptive control algorithm (1) to nonlinear systems in $d = 2$ and 3 dimensions with a single parameter which is varied. All systems have a stable regime where asymptotically the motion goes to a fixed point attractor, as well as regimes with a limit cycle (following a Hopf bifurcation) or more complicated periodic and aperiodic behaviour.

In studying control we have the following situations [5, 7]. Given an initial value of the parameter (denoted by subscript in), the system evolves – following eq. (1a) – to its stable steady state. At time $t = t_0$, say, the system is perturbed: here this implies an instantaneous change in the parameter value. Subsequently, the system evolves under control dynamics, i.e. eqs. (1a) and (1b), and returns to the original steady state (see fig. 1). Clearly, a convenient quantifier of this process is the time of recovery τ, which depends both on the perturbation as well as on the stiffness of control.

2.1. Application to single parameter perturbation

2.1.1. The Poincaré oscillator

The first system we study is a textbook example with one nontrivial degree of freedom, described by the following equations [2, 3]:

$$\dot{r} = \alpha r - r^3, \tag{2a}$$

$$\dot{\theta} = \omega. \tag{2b}$$

The sign of α determines the dynamics: when $\alpha < 0$ the system evolves to a fixed point ($r = 0$), and when $\alpha > 0$ there is a supercritical Hopf

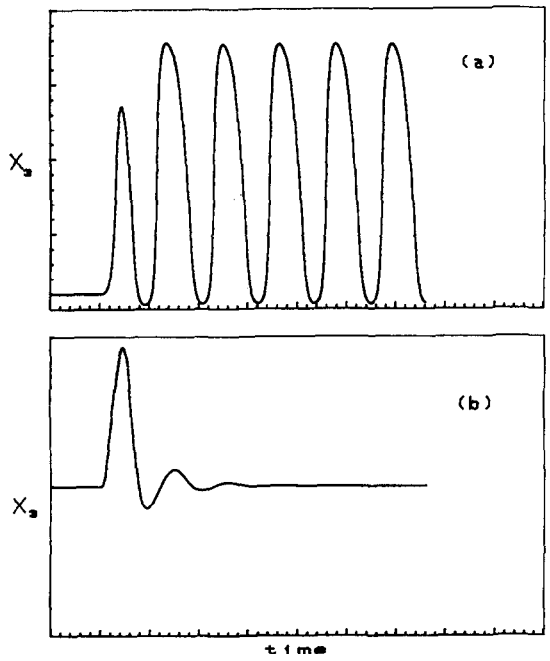

Fig. 1. Dynamics of a nonlinear system (described by eqs. (18)) after a sudden perturbation which changes the parameter value by a factor of 20, (a) without control and (b) with adaptive control.

bifurcation (or soft excitation) and the system evolves to a (circular) limit cycle of radius $r_c = \alpha^{1/2}$. The control dynamics is determined by the error signal, i.e. the difference between the goal output and the actual output:

$$\dot{\alpha} = \epsilon(r - \langle r \rangle), \tag{2c}$$

where $\langle r \rangle$ is the steady state value of r. To pull the system back to the fixed point, $\langle r \rangle$ in eq. (2c) must be 0.

The above system allows analytic treatment. Initially, $\alpha = \alpha_{in} < 0$ and the dynamics is attracted to $r = 0$. When α is suddenly perturbed to a positive value (at $t = 0$), we study how the system relaxes back to the fixed point. It is convenient to rewrite eq. (2) as a Riccati equation,

$$\frac{dr}{d\alpha} = \frac{1}{\epsilon}(\alpha - r^2). \tag{3}$$

Further transformation to an equation of second order,

$$\frac{d^2u(\alpha)}{d\alpha^2} + \frac{\alpha u}{\epsilon^2} = 0, \tag{4}$$

allows $r(\alpha)$ to be written in terms of the two linearly independent solutions, u_1 and u_2, of eq. (4),

$$\frac{r}{\alpha}(Cu_1 + u_2) + C\frac{du_1}{d\alpha} + \frac{du_2}{d\alpha} = 0. \tag{5}$$

For positive α, and using the variable $z = \alpha/\epsilon^{2/3}$, eq. (4) can be expressed as

$$\frac{d^2u(z)}{dz^2} + zu = 0, \tag{6}$$

which has the Airy solutions $u_1 = \mathrm{Ai}(z)$, $u_2 = \mathrm{Bi}(z)$. Substituting this in eq. (5) we get

$$r = -\epsilon^{1/3}\frac{C\mathrm{Ai}'(z) + \mathrm{Bi}'(z)}{C\mathrm{Ai}(z) + \mathrm{Bi}(z)}. \tag{7}$$

For small ϵ (i.e. large z), we can approximate [13] the Airy functions to obtain (in terms of $\zeta = \frac{2}{3}z^{3/2}$)

$$\begin{aligned} r(\alpha) &= -\epsilon^{1/3}\frac{C\left(-\frac{1}{2}\pi^{-1/2}z^{1/4}e^{-\zeta}\right) + \pi^{-1/2}z^{1/4}e^{\zeta}}{C\left(\frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\zeta}\right) + \pi^{-1/2}z^{-1/4}e^{\zeta}} \\ &\approx -\alpha^{1/2}. \end{aligned} \tag{8}$$

Then solving

$$\dot{\alpha} = \epsilon r = -\epsilon\alpha^{1/2}, \tag{9}$$

we get

$$\alpha^{1/2} = -\tfrac{1}{2}\epsilon t + \alpha_0^{1/2}, \tag{10}$$

where α_0 is the value of α at $t = 0$. Thus

$$r(t) = -\tfrac{1}{2}\epsilon t + \alpha_0^{1/2}, \tag{11}$$

and the recovery time τ is then given approximately by

$$\tau = 2\alpha_0^{1/2}/\epsilon. \tag{12}$$

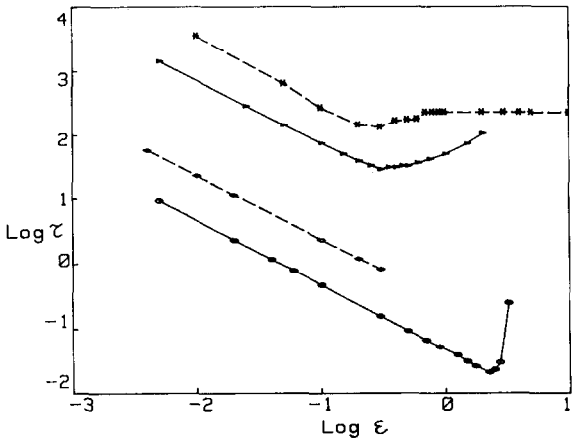

Fig. 2. Recovery time (in arbitrary units) versus stiffness of control for nonlinear systems of varying complexity given by eqs. (2) (◇), eqs. (13) (▷), eqs. (18) (*), and eqs. (19) (o). In these examples only one parameter was regulated.

Although the above analysis is valid for small ϵ, in practice we find that eq. (12) describes the recovery behaviour over a wider range (see fig. 2). This system (unlike the logistic or other unimodal maps examined by HL) has globally attracting steady states and so there is no perturbation, however large, from which the system does not recover in finite time. The recovery time τ, is always close to the estimate provided by eq. (12).

2.1.2. *The Brusselator*

We next analyze the Brusselator [14], for which the evolution equations are

$$\dot{x} = \alpha + x^2 y - \beta x - x, \tag{13a}$$

$$\dot{y} = \beta x - x^2 y. \tag{13b}$$

The nature of the dynamics is determined by the parameter β. When the asymptotic motion is attracted to a fixed point, the steady state values of the Brusselator can easily be seen to be $y_s = \beta_{in}/\alpha_{in}$ and $x_s = \alpha_{in}$. When β is perturbed, the equation for control dynamics becomes

$$\dot{\beta} = -\epsilon(y - y_s). \tag{13c}$$

Linear stability analysis of eqs. (13) yields the following conditions for equilibrium,

$$\beta < \alpha_{in}^2 + 1 \tag{14a}$$

and

$$\alpha_{in}^4 + \alpha_{in}^3(\epsilon - \beta_{in}) + \alpha_{in}^2(1 - \epsilon\beta_{in}) > 0. \tag{14b}$$

With no loss of generality we set $\alpha_{in} = 1$ and eq. (14a) then gives the condition for attraction to the fixed point to be $\beta_{in} < 2$; for $\beta_{in} > 2$ the system is attracted to a limit cycle. Further, eq. (14b) gives a stability window determined by

$$1 + (1 - \beta_{in})(\epsilon + 1) > 0. \tag{15}$$

Note that eq. (15) sets a restriction on the value of ϵ so here the range of control stiffness is limited by stability considerations. Fig. 2 shows τ as a function of ϵ within the stability window.

Examination of the dependence of τ on ϵ reveals a novel feature which is not observed in the 1D case [7]. For small ϵ, $\tau \sim 1/\epsilon$. But τ does not decrease monotonically with ϵ and beyond an optimum value of ϵ, τ actually starts increasing. A rough argument accounting for the linear relation between τ and $1/\epsilon$ in the small ϵ regime makes use of the observation that $y \approx \beta_{in}/x$ (for small ϵ). Substitution in eq. (13a) gives $x(t) = 1 - \text{const.}\, e^{-t}$, and eq. (13c) becomes

$$\dot{\beta} = -\epsilon(\beta/x - \beta_{in}). \tag{16}$$

Assuming small ϵ and a small perturbation, we then have

$$\beta \approx \beta_{in} - C e^{-\epsilon t} \approx \beta_{in} - \text{const.}(1 - \epsilon t), \tag{17}$$

suggesting that recovery time $\tau \propto 1/\epsilon$.

2.1.3. *A biochemical network*

The third example we study is a complex dynamical system [15] which describes various biochemical processes responsible for the coherent behaviour observed in spatio-temporal organiza-

tion. The equations contain positive and negative feedback loops such as are typically thought to occur in a variety of processes within living cells. This system, which gives rise to a variety of behavioural patterns, is

$$\dot{X}_1 = \frac{a_1}{a_2 + X_3^n} - KX_1, \tag{18a}$$

$$\dot{X}_2 = a_3 X_1 - \phi(X_2, X_3), \tag{18b}$$

$$\dot{X}_3 = a_4 \phi(X_2, X_3) - qX_3, \tag{18c}$$

where

$$\phi(X_2, X_3) = \frac{TX_2(1+X_2)(1+X_3)^2}{L + (1+X_2)^2(1+X_3)^2}. \tag{18d}$$

Choosing parameters (a_1, a_2, a_3, a_4, L, T and n) suitably, we have a system whose dynamics can be varied by tuning the parameters K and q. For instance, for $q = 0.5$, we get a limit cycle when $K = 0.001$, complex oscillations when $K = 0.003$, chaos when $K = 0.004$, complex oscillations and period doublings when $0.005 < K < 0.02$, a limit cycle again when $0.03 < K < 0.5$ and a steady state when $K = 1$. For control we let K evolve as

$$\dot{K} = -\epsilon(X_1 - \langle X_1 \rangle). \tag{18e}$$

Eq. (18e) is very effective in returning the system back to the original steady state ($K = 1$) when K is perturbed into any of the other above mentioned regimes, including the chaotic regime. The behaviour of τ with respect to ϵ is also shown in fig. 2. For small ϵ, τ remains proportional to ϵ^{-1}.

2.1.4. *A discrete dissipative system*

We finally apply the control to a two-dimensional discrete system given by [16]

$$X_{n+1} = 1 - \frac{\alpha X_n^2}{1 + X_n^4} - \beta Y_n,$$

$$Y_{n+1} = X_n, \tag{19}$$

which is similar to the Hénon map [17], with the additional feature that the global asymptotic dynamics is on an attractor for $\beta < 1$. When α is varied, this map gives rise to the entire repertoire of behaviour seen in unimodal chaotic maps. For regulation of the steady state the control dynamics given by the equation

$$\alpha_{n+1} = \alpha_n - \epsilon(X_n - X_s) \tag{20}$$

is very effective. For small ϵ the recovery time τ is proportional to $1/\epsilon$ (see fig. 2), but beyond $\epsilon = \epsilon_{opt}$, τ increases with ϵ, similar to what is observed for the Brusselator.

The above dissipative systems with more than one degree of freedom can show novel behaviour quite distinct from the 1D case. For example HL observed that $\delta_{max} \propto \epsilon$. Here, however, there is a maximum strength of shock for every value of ϵ beyond which the system does *not* recover. Below $\epsilon = \epsilon_c$ recovery is possible for shocks of all strengths (provided that the shock does not throw the perturbed value of α outside the allowed range). For $\epsilon > \epsilon_c$ the system fails to recover from all shocks however small in magnitude, so that the δ_{max} versus ϵ characteristic is a step function (fig. 3) rather than the linearly decreasing function obtained by HL.

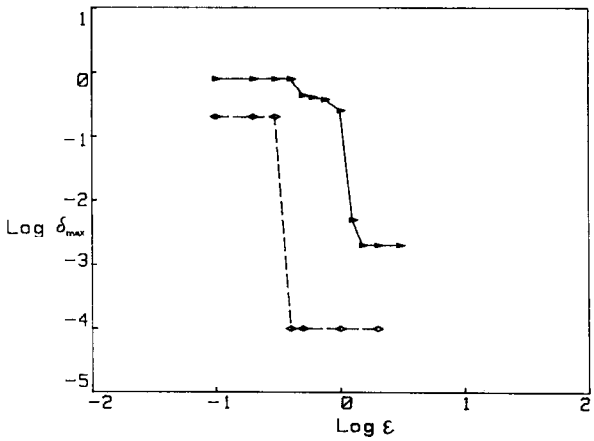

Fig. 3. Maximum strength of perturbation versus stiffness of control for discrete systems described by eqs. (19) (◇) and eqs. (26) (▷).

2.2. Applications to multi-parameter systems

Typically, a dynamical system has several parameters which govern the overall behaviour. In order to regulate such systems, the control has to be applied to each relevant parameter.

A representative one-dimensional map with two parameters (which is of interest in population dynamics [12]) is given by

$$X_{n+1} = \alpha X_n (1 + X_n)^{-\beta}. \tag{21}$$

For particular α and β the map has a globally stable equilibrium state; when the parameters α and β are varied the map yields a rich variety of dynamical behaviour [12]. To regulate the steady state of the system we control both parameters, in an obvious manner, through

$$\begin{aligned} \alpha_{n+1} &= \alpha_n + \epsilon (X_n - X_s), \\ \beta_{n+1} &= \beta_n + \epsilon (X_n - X_s), \end{aligned} \tag{22}$$

where ϵ is the control stiffness.

Similarly, for a two-dimensional discrete map of two driven coupled oscillators, which has been used [18] to model SQUIDS and convection in conducting fluids, given by

$$\begin{aligned} x_{n+1} &= \alpha x_n (1 - x_n) + \beta (y_n - x_n), \\ y_{n+1} &= \alpha y_n (1 - y_n) + \beta (x_n - y_n), \end{aligned} \tag{23}$$

the same control dynamics is effective. In both cases the recovery time remains linearly dependent on ϵ, and the results for the dependence of τ on ϵ are summarized in fig. 4. Note the similarity to fig. 2 in that there is an optimal ϵ beyond which τ increases with ϵ.

2.3. Discussion

In the control dynamics implemented here, we have always chosen an error indicator which utilizes a single variable, i.e. $X_i - X_s$. In higher-dimensional systems there is an ambiguity regarding the choice of the state variable X_i to be used in eq.

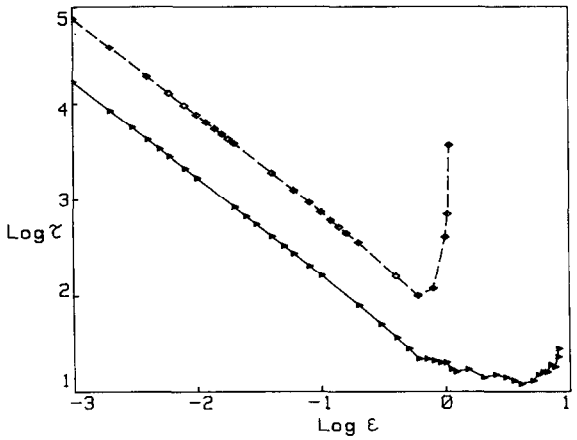

Fig. 4. Recovery time (in arbitrary units) versus stiffness of control for discrete multiparameter nonlinear systems described by eqs. (21) (◇), and eqs. (23) (▷). In these examples two parameters were regulated.

(1b) (or eq. (1c)). Empirically, we observe that in most systems *any* of the variables can effect control, since the equilibrium condition will lead all other variables to their steady state values when any one of them is forced to reach steady state. However there are exceptions: for instance, in the Brusselator, using coordinate y in eq. (13c) results in control whereas using x does not. This is because the steady state value of x ($x_s = \alpha$) does not constrain β to be the desired value, whereas the steady state of y ($y_s = \alpha\beta$) does. On the other hand, in the three-dimensional system given by eqs. (18), *all* three variables can effect control. X_1 works most efficiently, however, as the magnitude of X_1 (and hence the error signal) is small, leading to a more stable control dynamics.

One method of removing the above mentioned ambiguity is by employing AND logic in the control, i.e. by requiring that all variables reach their steady state values, X_i^s, $i = 1, 2, \ldots, N$. The equation for control dynamics then becomes (cf. eq. (1b))

$$\dot{\mu} = \epsilon \sum_{i=1}^{N} (X_i - X_i^s). \tag{1b'}$$

In the examples we have studied, either eq. (1b)

with proper choice of variable, or eq. (1b') works equally efficiently.

3. Control of limit cycles

For the simple oscillator [2, 3] described in eq. (2), limit cycles can be adaptively controlled. In this case defining the error signal is quite unambiguous as every limit cycle is uniquely characterized by its radius r_c. The difference between the actual radius and the radius of the limit cycle to be controlled can be used effectively for regulatory feedback. This is done, for example, by setting $\langle r \rangle$ in eq. (2c) equal to r_c. The dynamics of the controlled system is shown graphically in fig. 5a. When perturbed onto a different limit cycle (radius not equal to r_c) or into the fixed point regime ($\alpha > 0$), the system rapidly relaxes to the original limit cycle. For small ϵ, τ is inversely proportional to ϵ, but for large ϵ we observe a different phenomenon: τ oscillates about a saturation value (see fig. 6) which is roughly constant for all values of perturbation.

When r_c is small and the system is perturbed to a much larger radius the control dynamics for small ϵ is determined by a set of equations similar to eq. (2c) (as $r > r_c$); we can therefore expect the same linear trend. When the system is kicked to the fixed point regime ($r \to 0$) the control dynamics is approximated by

$$\dot{\alpha} = -\epsilon r_c, \tag{24}$$

so that (for small ϵ)

$$\alpha(t) = -\epsilon r_c t + \alpha_0, \tag{25}$$

from which the inverse dependence of τ on ϵ follows. For large ϵ this is not valid (see fig. 6).

More generally, we can extend the above procedure to control cycles in discrete systems. What is required is an error indicator that encodes as much information about the cycle as is necessary for its unique characterization. An error signal

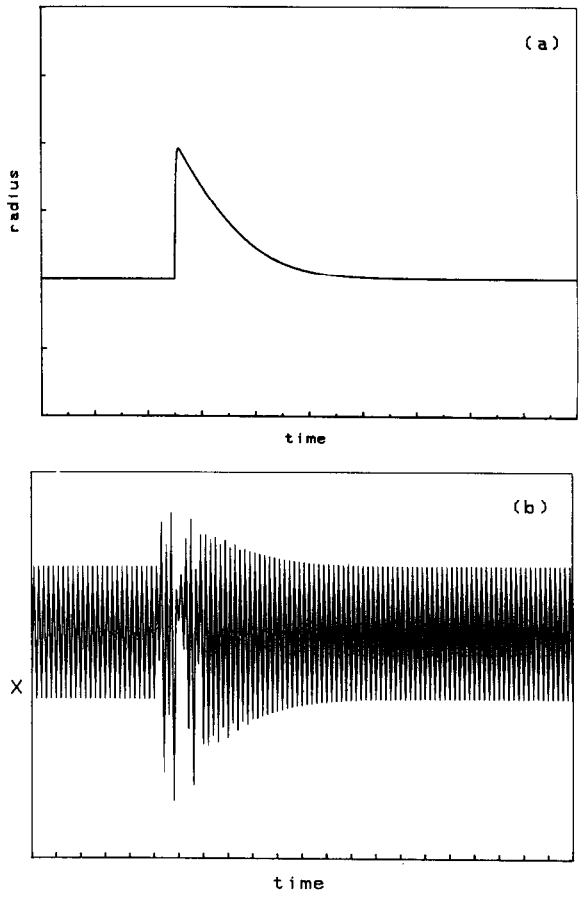

Fig. 5. Dynamics of controlling a limit cycle in two systems: (a) a continuous time system given by eqs. (2), and (b) a discrete time system given by eqs. (26).

depending on $X_{n+2} - X_n$ suffices in bringing the system back to some period 2-cycle, but not to a *specific* 2-cycle: $X_{n+2} - X_n = 0$ for *all* period 2-cycles, and so cannot guide the control dynamics onto any particular cycle. To regulate specific cycles we require an error that is unique for every 2-cycle: one possibility is an error proportional to $|X_{n+1} - X_n| - |X_1^c - X_2^c|$, where X_1^c and X_2^c are the values of the iterates of X in the 2-cycle we want to control.

We implement this for the logistic map

$$X_{n+1} = \alpha X_n(1 - X_n). \tag{26}$$

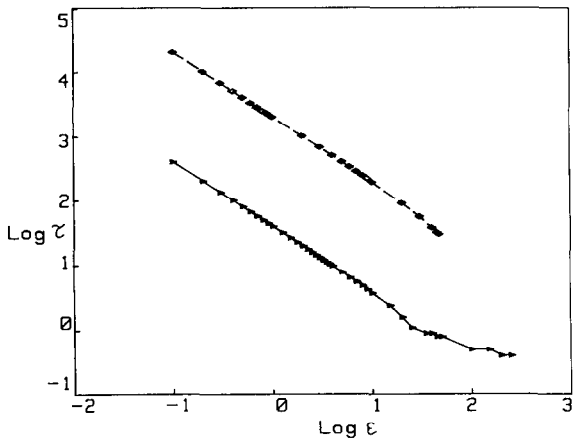

Fig. 6. Recovery time (in arbitrary units) versus stiffness of control for the regulation of cycles in nonlinear systems described by eqs. (2) (▷) and eqs. (26) (◇).

The control dynamics follows from

$$\alpha_{n+2} = \alpha_n - \epsilon(|X_{n+1} - X_n| - |X_1^c - X_2^c|). \quad (27)$$

Results are shown in fig. 5b, from which it is clear that this control mechanism very effectively returns the system to the *desired* 2-cycle. The recovery time varies inversely as ϵ (fig. 6). There is also a maximum strength of shock, δ_{max} (depending on ϵ) beyond which the system fails to recover; this δ_{max} versus ϵ curve shows a step function pattern.

Similar error indicators can, in principle, be constructed for higher-order periodic orbits although the technique diminishes in utility with increase in period. This is a problem of practicality, since higher-order cycles typically have narrow windows of stability; as a consequence the control dynamics becomes very unstable.

For discrete dynamical systems, however, more effective algorithms can be devised. One, which employs a logical OR structure in the error indicator, is

$$\dot{\alpha} = \epsilon \prod_{i=1}^{k} (X - X_i^c), \quad (28)$$

where X_i^c, $i = 1, 2 \ldots, k$ is the stable period k-orbit to be controlled. Since it implies that the desired state is either $X = X_1^c$ or $X = X_2^c$ or ... $X = X_k^c$ this adaptive algorithm works at *every* iteration step. For higher-order periodic orbits, this latter method is far superior to that embodied in eq. (27). For controlling the 2-cycle, for instance, the control equation analogous to eq. (27) is

$$\alpha_{n+1} = \alpha_n - \epsilon(X_n - X_1^c)(X_n - X_2^c). \quad (29)$$

Even with this latter form, the inverse dependence of τ on ϵ is unchanged.

4. Variations in the control dynamics

Apart from sudden perturbations in the system environment that lead to parameters changing value drastically – the primary case studied above – there are additional noise effects that can occur. In particular, it is interesting to consider the effect of random background noise on the control algorithm. In an effort to explore this question, we study the discrete maps eqs. (26) and (19), with additional Gaussian noise. The control dynamics remains unchanged. The variance σ of this noise clearly determines the control behaviour: for small σ, recovery times with and without noise are virtually identical, and beyond a value $\sigma = \sigma_{max}$ recovery is not possible. However, most importantly, this control procedure is remarkably *robust* (see fig. 7) for $\sigma < \sigma_{max}$, and recovery time continues to remain inversely proportional to stiffness.

A question of some importance is whether the control algorithm is sensitive to the specific form of the control dynamics, namely the choice for $g(X - X_s)$ in eq. (1c). In realistic systems, the control dynamics that can be incorporated may be of arbitrary functional form, arising, for example, from physico-chemical or engineering design considerations specific to the system. It is thus neces-

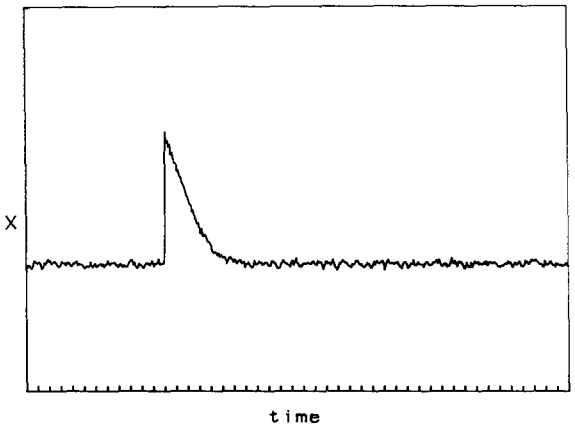

Fig. 7. Control dynamics of a nonlinear system in the presence of background noise ($\sigma = 0.005$).

sary to determine whether the linear recovery we observe in the examples above is an artefact of using a linear control function, and also whether such adaptive control is more generally applicable with different functions $g(y)$. In order to explore the features of control with different (nonlinear) functions, we have varied the control function, using $g = y^2$, $y^{1/2}$, $\sin y$, $1 - e^y$ and $y(1-y)$. Results are shown in fig. 8: for all functional forms recovery times remain inversely proportional to

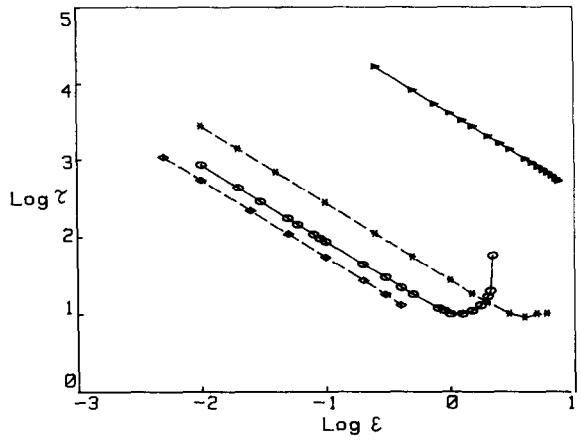

Fig. 8. Recovery time (in arbitrary units) versus stiffness of control for different control functions $g(X - X_s)$ in eq. (1c). The various functional forms used are $g(y) = y$ (o), $\sin y$ (o), $1 - e^y$ (o), y^2 (▷), $y^{1/2}$ (◇) and $4y(1-y)$ (*).

control stiffness for small ϵ. This strongly suggests that linear recovery may be a universal feature of the HL adaptive control algorithm.

5. Discussion

From our study of higher-dimensional systems of varying complexity, it appears that it is possible to provide efficient regulation of the steady state of nonlinear systems through adaptive control mechanisms. The procedure studied herein utilizes an error signal proportional to the difference between the goal output and the actual output of the *state variables* and should be contrasted with mechanisms [9] using an error signal proportional to a similar difference in the *parameter* value. In the latter case the control will, of course, bring the parameter back to its original value, but this does not ensure that the system will regain its specific original dynamical state. An instance where this distinction is important is in systems undergoing subcritical Hopf bifurcation or exhibiting bistability, when at a given parameter value, different initial conditions lead to different dynamics. In such a case, the present adaptive control ensures that both the original parameter value *and* the original dynamics will be recovered.

From numerical experiments studying the dependence of controllability on the stiffness and on the strength of perturbation, we find a number of interesting phenomena quite distinct from that seen in HL, but typical of most real systems. For multi-parameter systems, a simple extension of the adaptive mechanism suffices in regulating the system; furthermore it can also be adapted to regulate periodic behaviour such as limit cycles.

The HL procedure [7] is robust both to the existence of a background noise, and to the variation of the form of the control function. Most interestingly, recovery times are always inversely proportional to stiffness of control, for small stiffness (see figs. 2, 4, 6 and 8), which may be a universal feature of such adaptive control.

Biological situations where control is believed to play a crucial role include, for instance, the maintainance of homeostasis [2] (the relative constancy of the internal environment with respect to variables such as blood pressure, pH, blood sugar, electrolytes and osmolarity). Clinical experiments on animals show, for example, that following a quick mild hemorrhage (a sudden perturbation in arterial pressure) the blood pressure is restored to equilibrium values within a few seconds [19]. The control of fixed points, explored in detail in section 2, thus has potential utility in such physicochemical contexts. Cycles are also central to a variety of biophysical and biochemical processes. Variations in these – for example by the replacement of periodic by aperiodic behaviour, or the emergence of new periodic cycles – is often associated with disease [2]. The control of cycles suggested in section 3 has applicability in the regulation of biologically significant oscillatory phenomena.

In summary, our study confirms that adaptive control provides a simple, powerful and robust tool for regulating multidimensional systems capable of complicated behaviour. The concepts developed through the study of model systems can serve as a paradigm for understanding more complex regulatory mechanisms widespread in nature. These may also be of utility in helping formulate efficient and robust design principles.

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