Classical Resonances and an Arbitrary Trajectory Quantization Scheme for a Chaotic System

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The exponential decay of correlations in chaotic systems is often modulated and shows up as broad peaks in the power spectrum. For a particle sliding freely on a compact surface of constant negative curvature, we show that these classical resonances are directly related to the quantal eigenenergies of the system. We use this as the basis for proposing a quantization scheme which requires the knowledge of a single arbitrary ergodic classical trajectory. Our results are substantiated numerically.

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Chaotic dynamical systems have been the subject of considerable interest in recent years and a number of interesting results have emerged, both in the classical and quantum mechanical context. These systems possess unstable isolated periodic orbits and are characterized by an exponential decay of correlations. Interestingly, however, this decay is often strongly modulated and shows up as broad peaks in the power spectrum of autocorrelation functions [1]. The theory of such resonances in case of hyperbolic or axiom-A systems was developed by Ruelle [2] and Pollicott [3] and attempts at providing physical interpretations have been undertaken in some cases. Thus for intermittent systems, Baladi, Eckmann, and Ruelle [4] provide a dynamical mechanism for these modulations based on the probabilistic distribution of the duration of "laps" (time intervals with approximately independent behavior). Other observations [5] indicate that the peaks occur close to frequencies of dominant periodic motion. Physically this implies that the system preferentially exercises this motion but cannot do so forever due to its instability. The picture, however, is still not clear. We shall address this question here and show that for a particle sliding freely on a compact surface of constant negative curvature, the position of the peaks is directly related to quantal eigenenergies of the system.

Chaotic systems have also led to a significant development in the theory of semiclassical quantization. It is well known that the Einstein-Brillouin-Keller (EBK) quantization scheme works only in case of integrable systems where trajectories live on tori. For nonintegrable systems where direct methods are not applicable, Gutzwiller [6] and Balian and Bloch [7] developed a theory which requires as its input, the lengths, stability, and focusing properties of all periodic orbits of the system [8]. The density of states $\sum\delta(E-E_n)$ can then be expressed as a sum of a smooth average part and fluctuations that are comprised of oscillatory contributions from all the periodic orbits of the underlying classical system. For chaotic systems, it is expressed as

$$\sum_n \delta(E-E_n) = d_{\rm av}(E) + \frac{1}{\pi}\sum_p \sum_r \frac{T_p}{(|2-{\rm Tr}M_p^r|)^{1/2}} \times\cos[r(S_p-\phi_p)]\,, \quad (1)$$

where the sum runs over each primitive orbit (labeled by p) and its repetitions (labeled by r). In the above equation, $d_{\rm av}(E)$ refers to the average density of states (the Thomas-Fermi term), S_p and T_p the action and time period of a primitive orbit, ϕ_p an associated phase which depends on the focusing properties of the trajectory while M_p is a matrix which describes the stability of the orbit. In practice, there are two important factors which restrict a direct use of this result. First, it is generally difficult to enumerate the periodic orbits of a system or even to generate its symbolic dynamics. Second and more important, the resolution, Δ, of a quantal level is restricted by the length $l_{\max}$ of the longest orbit used ($\Delta\sim 1/l_{\max}$) and not the number N of periodic orbits used. Thus in chaotic systems where the proliferation of orbits is exponential [9], $\Delta\sim 1/\ln(N)$. The improvement in resolution is thus extremely slow. For few chaotic systems [10,11] it has been possible to generate a large number of periodic orbits due to the nature of their symbolic dynamics as well as pruning laws and for the low lying levels they give a fairly good approximation to the quantal energies. Improved techniques have, however, emerged involving cycle expansions of the dynamical zeta function [12] and these require fewer orbits to generate the desired resolution. In this Letter, we provide an alternate quantization scheme for a specific example of a chaotic system comprised of a particle sliding freely on a compact surface of constant negative curvature. The method follows from our result on classical resonances, and requires as its input a single arbitrary ergodic classical trajectory.

In what follows, we shall introduce the subject of classical resonances and subsequently establish the connection with quantal energies for the system under consideration. The quantization scheme follows as a natural extension and we demonstrate its viability numerically.

Classical resonances are a characteristic feature of axiom-A systems and these are reflected in peaks of the Fourier transform of correlation functions, $\rho_{BC}(t) = \langle B(\mathbf{f}^{t+\tau}(\mathbf{x}))C(\mathbf{f}^{\tau}(\mathbf{x}))\rangle_\tau - \langle B\rangle_\tau\langle C\rangle_\tau$ where B and C are differentiable functions and $\mathbf{f}^t$ is the flow of the system. Further, the position of the peaks is independent of the functions B and C [1]. A famous example is the geodesic flow on a manifold of negative curvature.

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According to the theorems of Ruelle [2] and Pollicott [3] and numerical evidence for other systems, the spectrum of a purely chaotic system consists of a series of resonances γ_n of multiplicity g_n:

$$\mathrm{tr}\mathcal{L}^t = \mathrm{tr}\,\delta(\mathbf{x}-\mathbf{f}^t(\mathbf{y})) = \int d\mathbf{x}\,\delta(\mathbf{x}-\mathbf{f}^t(\mathbf{x})) = \sum_n g_n e^{-\gamma_n t}\,, \tag{2}$$

where $\mathcal{L}^t$ is the evolution operator and $\gamma_n = \alpha_n + i\beta_n$. The connection between the eigenvalues of the evolution operator and the periodic orbits of the system is quite obvious. Cvitanovic and Eckhardt [13] have evaluated the contribution of each orbit to $\mathrm{tr}\mathcal{L}^t$ using a coordinate system with the longitudinal coordinate along the direction of the flow and the others transverse to it. The calculation is reminiscent of the quantum case [6] and they finally obtain a zeta function in terms of which $\mathrm{tr}\mathcal{L}^t$ is expressed. We state their result [13] in a form convenient for subsequent analysis:

$$\mathrm{tr}\mathcal{L}^t = \sum_p \sum_r \frac{T_p}{|2-\mathrm{Tr}M_p^r|}\,\delta(t-rT_p)\,. \tag{3}$$

The eigenvalues of the classical evolution operator are thus a result of the collective properties of periodic orbits. It should, however, be noted that Eq. (3) is exact while in the quantum case, the periodic orbit expansion involves the stationary phase approximation. (For spaces of constant negative curvature, however, the quantum relation is exact and is known as the Selberg trace formula [14].)

We shall now deal with the specific example of a particle sliding freely on a compact surface of constant negative curvature and establish a connection between classical resonances and quantal energies.

Compact surfaces of negative Gaussian curvature have been dealt with quite extensively since it was first suggested by Hadamard [15] that the classical free motion of a particle constrained to move on it is unstable. It is now known that the dynamics is in fact Bernoullian and hence extremely chaotic. In analogy with the sphere (which has constant positive curvature) such surfaces are also known as "pseudospheres" and can be globally embedded in a space endowed with a Minkowskian metric. We shall deal here with two-dimensional compact surfaces for which a flat representation (with a different metric) has considerable advantages especially in constructing the Hamiltonian dynamics. We give here some details of the Poincaré disk model which we use in our numerical calculations. An extensive survey of various models can be found in Balazs and Voros [16].

The Poincaré unit disk is a stereographic projection of the pseudosphere onto the (x_1,x_2) plane. The boundary of the disk corresponds to points at infinity and the pseudosphere itself is represented by the interior of the disk. The corresponding metric is $ds^2 = 4(dx_1^2+dx_2^2)/(1-x_1^2-x_2^2)^2$ and the geodesics are circular arcs (or diameters) orthogonal to the disk boundary. The metric being conformal, the angles are the same as in the Euclidean case though distances get distorted.

The Hamiltonian in this representation reads as

$$H = \mathcal{A}^{-1}(p_1^2+p_2^2)/2\mu\,, \tag{4}$$

where $\mathcal{A}$ is the conformal factor $4(1-x_1^2-x_2^2)^2$ and μ is the mass of the particle. Interestingly the free motion in the entire unit disk is integrable [16] since apart from the energy, E, there exists another constant of motion in $M = x_1p_2 - x_2p_1$. The orbits are thus diameters or circular arcs orthogonal to the disk boundary and the radius and center of the arcs can be conveniently expressed in terms of the constants of motion [16]. A particle moving freely thus escapes to infinity (boundary of the disk).

The dynamics, however, becomes extremely complex on a compact surface. This can be achieved [16] by tiling the unit disk with replicas of a fundamental domain and imposing periodic boundary conditions. Conversely one can choose a discrete subgroup G of the Lorentz group and identify the points that are connected by a Lorentz transformation within this subgroup. Thus the infinite free trajectory passes through different replicas each of which are superposed on the fundamental domain to generate a criss-cross pattern. It is simpler, however, to take the fundamental domain and glue together the sides according to the connection rules of the discrete group G. This space is now compact. Moreover the simplest of such surfaces with constant negative curvature has genus two (equivalent to a sphere with two handles).

We consider here this simplest case where the fundamental domain is a regular octagon with vertex angles $2\pi/8$ and area 4π, with opposite sides identified. The corresponding "octagon group" G is generated by four special boosts, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and their inverses that exchange opposite sides: $z' = gz = (az+b)/(cz+d)$. In this particular case they are

$$g_j = \begin{pmatrix} 1+\sqrt{2} & e^{ij\pi/4}(\sqrt{2}+2\sqrt{2}) \\ e^{-ij\pi/4}(\sqrt{2}+2\sqrt{2}) & 1+\sqrt{2} \end{pmatrix}, \quad j=0,1,2,3\,.$$

The classical dynamics even on this simplest compact surface is extremely chaotic. The periodic orbits can, however, be coded quite easily by first noting that each element of the group corresponds to a periodic geodesic and using an identity relation satisfied by the generators and their inverses [16,17]. The exponential proliferation in this case is a direct manifestation of the noncommutativity of the Lorentz group. Aurich and Steiner [17] have constructed the group elements and evaluated the lengths of more than 2×10^6 periodic orbits for this system.

The quantal energies of the octagon are given by the Schrodinger equation with periodic boundary conditions, $\psi(z) = \psi(g_jz)$. The levels are discrete and the ground state $E_0 = 0$. Interestingly, the relation between the quantal energies $\{E_n\}$ and length spectrum $\{l_p\}$ is exact in this case and is given by the Selberg trace formula [14]:

$$\sum_n h(p_n) = \frac{A}{4\pi}\int_{-\infty}^{\infty} dp\, p \tanh(\pi p) h(p) + \sum_p \sum_r \frac{l_p}{2\sinh(rl_p/2)} g(rl_p)\,, \qquad (5)$$

where A is the area of the fundamental domain, $p_n^2 = E_n - 1/4$, $h(p)$ is a (nearly) arbitrary even function that is holomorphic in the strip $|\mathrm{Im}p| \le 1/2+\varepsilon$, $\varepsilon > 0$ and vanishes asymptotically for $|p| \to \infty$ faster than $1/p^2$, and $g(x)$ is its Fourier transform. For convenience, we shall assume $\hbar = 2\mu = 1$ in the rest of this paper.

Starting with $h(p_n) = \cos(p_n l) e^{-\beta E_n}$, it is easy to show [18] that for $\beta \to 0^+$

$$\sum_p \sum_r \frac{l_n}{4\sinh^2(rl_p/2)} \delta(l - rl_p) = \coth(l/2) + \sinh^{-1}(l/2) \sum_{n=1}^{\infty} \cos(p_n l)\,, \qquad (6)$$

where we have neglected the term $\cosh(l/2)/[2\times\sinh^3(l/2)]$ on the right-hand side. Note that for this system $|2 - \mathrm{Tr}M_p^r| = 4\sinh^2(rl_p/2)$. Thus using Eq. (6) in Eq. (3), and noting that $t = l/2\sqrt{E}$, we have

$$\mathrm{tr}\mathcal{L}^t = \coth(l/2) + \sinh^{-1}(l/2) \sum_{n=1}^{\infty} \cos(p_n l)\,. \qquad (7)$$

Equation (7) forms the central result of this paper. For l sufficiently large, this reduces to

$$\mathrm{tr}\mathcal{L}^t = 1 + \sum_{n=1}^{\infty} e^{-(1/2 \pm ip_n)l}\,, \qquad (8)$$

which is identical in form to Eq. (2) with $\alpha_n = \alpha = \frac{1}{2}$ and $\beta_n = \pm p_n$. Thus the classical Ruelle-Pollicott resonances for a particle sliding on a compact surface of constant negative curvature are directly related to the quantal energies, $E_n = p_n^2 + \frac{1}{4}$.

We reserve a discussion on the general case (where the Gutzwiller trace formula is applicable) until later and in the following, proceed with the quantization scheme which follows quite logically.

In order to observe these resonances in classical correlation functions, it is necessary to consider an ergodic trajectory which induces an invariant measure μ, such that

$$\langle X \rangle = \lim_{T\to\infty} T^{-1} \int_0^T d\tau X(x(\tau)) = \int \mu(dx) X(x)\,.$$

Then the Fourier transform, $\hat{\rho}_B(f) = \int dt\, e^{2\pi i f t} \rho_B(t)$ of the autocorrelation function $\rho_B(t) = \langle B(\mathbf{f}^{t+\tau}(\mathbf{x})) B(\mathbf{f}^\tau \times(\mathbf{x}))\rangle_\tau - \langle B \rangle_\tau^2$ should display broad peaks at $\pm p_n$ [$\hat{\rho}_B(f)$ is the power spectrum of the signal, $B(x(\tau))$]. Note that the peaks are not strictly Lorentzian since Eq. (8) involves an approximation. However, the resolution of the peaks is determined by the full width 2α, also known as the "entropy barrier" [19]. Thus, the quantum levels can be determined using an arbitrary, ergodic trajectory up to an accuracy determined by the topological entropy.

We use the Poincaré disk representation to generate the Hamiltonian dynamics and choose the fundamental domain as a regular octagon. As mentioned earlier, each segment of a trajectory is a circular arc whose center (x_1^c, x_2^c) and radius R can be determined in terms of the constants of motion in the entire unit disk (the unbounded case). Thus we have [16] $R = \sqrt{E}/|M|$, $x_1^c = B_1/M$, $x_2^c = B_2/M$ where $B_1 = (1 - x_1^2 - x_2^2)p_2/2 + Mx_1$ and $B_2 = -(1 - x_1^2 - x_2^2)p_1/2 + Mx_2$. As the trajectory hits the boundary of the fundamental domain, the transformation $g_j z$ carries it to the opposite side from where the trajectory emerges once more as a circular arc but with a different radius and center coordinates. In order to use a fast Fourier transform (FFT) algorithm to evaluate the power spectrum, we have determined the position and momentum coordinates at equal time intervals, Δt.

For the plot displayed in Fig. 1, we consider a long trajectory with energy $E = 0.000001$ (this is chosen so that the time required in traversing the smallest arc is not too small; the dynamics at any other energy is equivalent since the system is homogeneous) and time interval $\Delta t = 7.2$, undergoing 4371 hits at the boundary. This is broken up into 30 segments with 16384 points in each. The power spectrum displayed is the average over these 30 segments as a function of $p = \pi f/\sqrt{E}$ for the function $B = \cos(x_1 x_2 p_1 p_2)$. The quantal energies for this system have been evaluated by Aurich and Steiner and are listed in [20]. We have marked the approximate positions of p_n with arrows in Fig. 1.

The lower part of the spectrum clearly displays prominent sidebands which make it difficult to distinguish the second excited state at $p_2 = 2.25$ [21]. This is primarily due to the pronounced peak at zero (not shown in the figure; it is several orders of magnitude higher than the peaks shown in Fig. 1) and its rapidly decaying sidebands

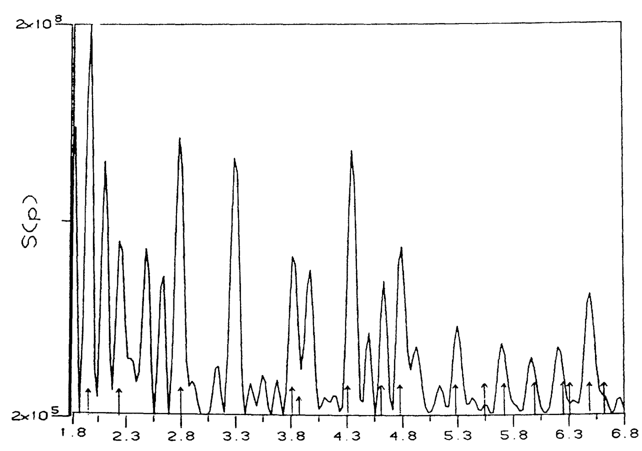

FIG. 1. Power spectrum, $S(p)$, of the function $B(x,p) = \cos(x_1 x_2 p_1 p_2)$. The arrows mark the position of the quantal energies [20].

which interfere with the lower part of the spectrum. The other levels (third excited state onwards) are quite prominent in general considering the limitations in resolution discussed earlier. Strangely, there is a pronounced peak at around 3.3 which persists for other initial conditions as well. Moreover, this and other peaks corresponding to quantal energies can be observed for various other functions, $B(x,p)$. The possibility of a missing level thus cannot be ruled out.

Our results therefore bear out the viability of this "arbitrary trajectory" quantization scheme for this system and can be compared profitably with the results obtained from a direct application of periodic orbits [10,18]. Further work along this direction is currently in progress and aimed at eliminating sidebands [21] and improving the resolution by "overcoming" the entropy barrier. A possible method is to evaluate first the autocorrelation function, multiply it by $e^{l/2}$, and then compute its Fourier transform so as to get sharp peaks with a resolution limited only by the length of the data set (of course at the cost of greater computational time). These and other related details will be communicated in due course.

A generalization of this scheme to other chaotic systems is not immediately obvious since the orbit dependent phases and stability indices that occur in the Gutzwiller trace formula prevent a direct inversion. A recent result [22] for the density of lengths, $\sum\delta(l-l_i)$ of chaotic billiards is, however, encouraging since it shows that leading order (in l) fluctuations in the density are indeed due to the quantal energies. The influence of the oscillatory corrections (which arise due to differences in quantal energies) on the power spectrum, however, needs to be investigated.

In conclusion, we have been able to show that for a particle sliding freely on a compact surface of constant negative curvature, the position of the peaks in the Fourier transform of classical autocorrelation functions are directly related to the quantal eigenenergies. A quantization scheme using a single arbitrary ergodic trajectory thus follows logically and we have demonstrated its viability in this Letter.

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