Theory of Fluctuations in Pseudointegrable Systems

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We provide a sum rule for the length spectrum of pseudointegrable systems in terms of the quantal energies. Further, we derive an expression for the form factor and infer the dependence of spectral fluctuations on the geometry of the system. Our analysis allows us to explain various numerical results obtained in recent years.

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An important aspect of dynamical studies in recent times has been to explore the manifestations of the underlying classical dynamics on quantal stationary state properties. Investigations carried out on certain statistical measures of the eigenvalue spectra reveal the existence of universality classes [1]. It is now known [2,3] that their origin lies in the nature of the corresponding classical counterpart. Most of the current understanding is based on the semiclassical periodic orbit theory [4], which allows one to express the density of states as a sum of a smooth average part and fluctuations arising from contributions of all the periodic orbits of the system. While the average part is system dependent, the fluctuations exhibit universalities governed by the nature of classical dynamics and thus form the object of statistical analysis.

The simplest statistical measure is the nearest-neighbor-spacing distribution (NNSD) $P(s)$. For generic integrable systems $P(s)$ is the Poisson distribution, $\exp(-s)$. The spectrum is thus characterized by near degeneracies. Chaotic systems, however, display level repulsion and the spacing distribution is well approximated by the Wigner distribution, $P(s)=(\pi s/2)\times\exp(-\frac{1}{4}\pi s^2)$. Of the higher-order correlations, the one most commonly encountered is the spectral rigidity $\Delta(L)$. Given a substretch of length L, $\Delta(L)$ measures the mean of the least-squares deviation of the spectral staircase, $\sum\Theta(E-E_i)$, from the best fitting straight line. For an uncorrelated (Poisson) spectrum, $\Delta(L)=L/15$, while for the chaotic systems with time reversal symmetry, $\Delta(L)=(1/\pi^2)\ln(L)-0.007$. Most of these results are now understood in the framework of the periodic orbit theory (POT) [2,3].

Pseudointegrable systems are the nearest step away from classical integrability since their invariant integral surface is N dimensional but has a genus, $g>1$. For instance, polygonal billiards with internal vertex angles $\alpha_i=m_i\pi/n_i$ $(m_1\neq 1)$ belong to this category. The periodic orbits here are similar to those of integrable billiards in that they occur in $N-1$ parameter families. Moreover, the asymptotic proliferation of these orbits is identical to that in the integrable case [5]. Interestingly enough though, numerical computations on the quantal spectra [6–10] show fluctuations closer to chaotic systems. This enigmatic result has generated considerable research interest in recent years and has so far defied explanation in terms of the POT.

Our purpose in this Letter is to resolve these seemingly contradictory observations. Our analysis is based on an expression for the length spectrum, $N(l)=\sum\Theta(l-l_i)$, that we derive by a suitable inversion of the trace formula [4,11]. We show that the proliferation rate is in fact different from integrable billiards for lengths that contribute significantly to the spectral rigidity, $\Delta(L)$, or a related measure, the number variance $\Sigma^2(L)$. Using this, we arrive at an expression for the form factor which gives a theoretical basis for understanding several past numerical results and explains the dependence of $\Delta(L)$ on the geometry of the system.

We shall first derive an expression for $N(l)$. The starting point of our analysis is the function

$$\sum_{n=1}^{\infty} J_0(\sqrt{E_n}x)\exp(-\beta E_n) = \int_\varepsilon^\infty dE\, J_0(\sqrt{E}x)\exp(-\beta E)\sum\delta(E-E_n)\,, \qquad (1)$$

where the summation is over the quantal energies E_n, J_0 is the Bessel function, and $\varepsilon<E_i$. Now the density of states can be expressed as ($\hbar=2m=1$, where m is the mass)

$$\sum_n \delta(E-E_n)=\frac{\mathcal{A}}{4\pi}+\frac{1}{4\pi}\sum_{j=1}^{\infty} a_j J_0(\sqrt{E}l_j)\,, \qquad (2)$$

where l_j refers to the lengths of the periodic orbits, a_j is the projected phase-space area of the jth family, and $\mathcal{A}$ is the area of the billiard. Substituting expression (2) in Eq. (1), and choosing $\varepsilon\to 0^+$ [12] we have in the limit $\beta\to 0^+$ [13]

$$\sum_{n=1}^{\infty} J_0(\sqrt{E_n}x)=\frac{1}{2\pi}\sum_j a_j\delta(x-l_j)/(xl_j)^{1/2}-b_0\,, \qquad (3)$$

where $x>0$ and

$$b_0=\int_0^\varepsilon dE\, J_0(\sqrt{E}x)\exp(-\beta E)\left[\frac{\mathcal{A}}{4\pi}+\frac{1}{4\pi}\sum_j a_j J_0(\sqrt{E}l_j)\right]. \qquad (4)$$

Multiplying by $2\pi x$ and integrating in the range $[l_0, l]$ where l_0 is smaller than the length of the shortest periodic orbit, we obtain

$$\sum_{l_j<l} a_j = b_0 \pi l^2 + 2\pi \int_{l_0}^{l} \sum_n x J_0(\sqrt{E_n} x) dx , \qquad (5)$$

where the summation is over those orbits for which $l_j < l$ and b_0 is a constant [14]. We shall later give a value for b_0 from other considerations.

Clearly, for integrable billiards a_j's are identical ($a_j = a_{\rm int}$) and hence

$$a_{\rm int} N(l) = b_0 \pi l^2 + 2\pi \int_{l_0}^{l} \sum_n x J_0(\sqrt{E_n} x) dx . \qquad (6)$$

For pseudointegrable systems we have a very different situation, since the a_j's are not identical. This in fact is a characteristic property of such billiards and has very significant consequences as we shall now show.

Note that for any given l, we can always write $\sum_{l_j<l} a_j = \bar{a}(l) N(l)$, where $\bar{a}(l)$ is the average projected area for orbits up to length l. It then follows that

$$N(l) = \frac{b_0 \pi l^2}{\bar{a}(l)} + \frac{2\pi}{\bar{a}(l)} \int_{l_0}^{l} \sum_n x J_0(\sqrt{E_n} x) dx$$
$$= N_{\rm av}(l) + N_{\rm osc}(l) . \qquad (7)$$

The behavior of $N(l)$ thus crucially depends on the form of $\bar{a}(l)$. The area a_j is the configuration space area of the jth band (or family) × number of independent directions, n_j, of the jth orbit. Clearly a_j [and hence the average $\bar{a}(l)$] is bounded between $a_{\min}$ and $a_{\max}$. Here $a_{\min}$ corresponds to a bouncing ball family covering the smallest configuration space area $\mathcal{A}_{\min}$ and is equal to $2\mathcal{A}_{\min}$. The quantity $a_{\max}$ is due to the generic (long) orbits which cover the entire billiard and possess the maximum number of directions $2\mathcal{N}$, allowed by the genus,

$$g = 1 + \frac{\mathcal{N}}{2} \sum \left[\frac{m_i - 1}{n_i} \right] ,$$

where $\mathcal{N}$ is the least common multiple of the integer set $\{n_i\}$ and the sum is over all polygonal vertices. Then we have $a_{\max} = \mathcal{A} \times (2\mathcal{N})$. Thus $\bar{a}(l)$ is an increasing function connecting these two limits and asymptotically reaches $a_{\max}$. Thus the l dependence of $N(l)$ in pseudointegrable systems is different from that in integrable systems. This has been corroborated for the case of the barrier billiard, using a different procedure [15].

For large lengths, however, $N(l) \sim l^2$ since $\bar{a}(l)$ saturates. This is in conformity with Gutkin's asymptotic result [5] for almost-integrable systems which are a subset of pseudointegrable systems.

We shall now use the above analysis to arrive at a form for the spectral rigidity $\Delta(L)$,

$$\Delta(L) = \left\langle \min_{a,b} \frac{1}{L} \int_{-L/2d_{\rm av}}^{L/2d_{\rm av}} dx [N(E_0 + x) - a - bx]^2 \right\rangle \qquad (8)$$

for pseudointegrable systems, where $d_{\rm av} = d_{\rm av}(E) = \mathcal{A}/4\pi$. We shall concern ourselves with the behavior of $\Delta(L)$ in the region $L \ll L_{\max}$, where $L_{\max}$ $(= h d_{\rm av}/T_{\min})$ is the outer scale of the spectrum.

The density of energy eigenstates in billiards, as given in Eq. (2), can be expressed in the semiclassical limit as

$$\sum_{n=1}^{\infty} \delta(E - E_n) = d_{\rm av}(E) + \sum_{j=1}^{\infty} A_j \exp(i S_j) , \qquad (9)$$

where the amplitude $A_j = (1/32\pi^3 \sqrt{E} l_j)^{1/2} a_j$, and the action $S_j = \sqrt{E} l_j$. Using Eq. (9) the spectral rigidity is given as [3]

$$\Delta(L) = 2 \int_0^{\infty} \frac{dT}{T^2} \phi(T) G(LT/2d_{\rm av}(E_0)) , \qquad (10)$$

where $T = l/(2\sqrt{E_0})$ and

$$\phi(T) = \left\langle \sum_i{}^{\dagger} \sum_j{}^{\dagger} A_i A_j \cos(S_i - S_j) \delta(T - (T_i + T_j)/2) \right\rangle . \qquad (11)$$

The superscript † denotes that we have taken into account the conjugate terms in Eq. (9). The selection function $G(y)$ picks orbits that contribute substantially and is given by $(1 - F^2(y) - 3[F'(y)]^2)$, where $F(y) = \sin(y)/y$. For $y < \pi/4$, $G(y)$ is nearly zero, while in the interval $\pi/4 < y < 3\pi/2$, it climbs steeply until it saturates to a value $\cong 1$ for $y > 3\pi/2$. [For a plot of $G(y)$, see Berry [3].]

Here, as in the case of integrable systems [3], the action differences $(S_i - S_j)$ in $\phi(T)$ are always large compared to unity as the proliferation rates are comparable and thus the off-diagonal terms in Eq. (11) can be neglected due to the averaging in Eq. (11). So $\phi(T) = \phi_D(T)$. (In contrast, for chaotic systems, the increase is exponential and hence for long orbits, the differences in actions are small compared to unity and the off-diagonal terms cannot be neglected.)

We shall thus evaluate the quantity

$$\phi_D(T) = \left\langle \sum_{i=1}^{\infty} A_i^2 \delta(T - T_i) \right\rangle . \qquad (12)$$

It is instructive to study the function $\mathcal{F}(T) = \int_0^T dT' \phi_D(T')$ in order to deduce the nature of $\phi_D(T)$. A reasonable estimate of $\mathcal{F}(T)$ (and the correct T dependence) can be obtained by substituting the relevant expressions for A_i and T in Eq. (12) and using Eq. (3). Thus we have

$$\mathcal{F}(T) = \frac{1}{16\pi^3} \sum_i \frac{a_i^2}{l_i} = \frac{\langle a(l) \rangle}{16\pi^3} \sum_i \frac{a_i}{l_i} , \qquad (13)$$

where we have introduced the quantity $\langle a(l) \rangle$ and the sum now goes over all orbits up to a period T. For sufficiently high energies [large E_0 in Eqs. (8) and (10)] the selection function, $G(y)$, does not pick up contributions from the short orbits. Thus, neglecting the fluctuating part of Eq. (3), we have $\mathcal{F}(T)=(b_0T/8\pi^2)\langle a(l)\rangle$.

Clearly the function $\langle a(l)\rangle$ lies between $a_{\rm min}$ and the largest area, a_l included in the sum. As l increases all subsequent entries in the sum approach $a_{\rm max}$ (i.e., $a_j = a_{\rm max}$, for $l>l_j>l_0$). Thus

$$\sum_{l_j<l}\frac{a_j}{l_j}=\sum_{l_j<l_0}\frac{a_j}{l_j}+a_{\rm max}\sum_{l_0<l_j<l}\frac{1}{l_j}\,.$$

Therefore, it is easy to see from above and Eq. (3) that

$$\langle a(l)\rangle = a_{\rm max}-(l_0/l)[a_{\rm max}-\langle a(l_0)\rangle]\,.$$

Hence $\langle a(l)\rangle$ smoothly increases with l and saturates asymptotically at $a_{\rm max}$. Further, note that the slope decreases as $1/l^2$. Thus on differentiating $\mathcal{F}(T)$ [and using the fact that $\langle a(l)\rangle$ is a slowly varying function of l] we have $\phi_D(T)=(b_0/8\pi^2)\langle a(l)\rangle$. Further, from the semiclassical sum rule of Berry [3] we have

$$\phi(T)=\frac{d_{\rm av}(E)}{2\pi}\quad \text{for}\ T\gg d_{\rm av}(E)\,. \qquad (14)$$

Comparing the value of $\phi_D(T)$ with Eq. (14) at large T, where $\langle a(l)\rangle$ saturates to $a_{\rm max}$, it follows that the constant $b_0=\mathcal{A}/a_{\rm max}$. Thus, for integrable billiards $b_0=\frac{1}{4}$, which can also be obtained from the Poisson summation formula. Now using the expression for b_0 and $\langle a(l_0)\rangle$ we have [16]

$$\phi(T)=\frac{\mathcal{A}}{8\pi^2 a_{\rm max}}\langle a(l)\rangle=\frac{\mathcal{A}}{8\pi^2}\left\{1-\frac{T_0}{T}\left[1-\frac{\langle a(l_0)\rangle}{a_{\rm max}}\right]\right\}. \qquad (15)$$

This forms the central result of this paper. Equation (15) yields the correct value, $d_{\rm av}(E)/2\pi$, [2,3], for integrable billiards since $\langle a(l)\rangle=a_{\rm max}=a_{\rm int}$. Note that $\phi(T)$ is a constant for integrable systems, while for the chaotic case it initially increases linearly [$\phi(T)=T/4\pi^2$] and then saturates to $d_{\rm av}(E)/2\pi$ at large T. For pseudointegrable systems $\phi(T)$ rises with a slope decreasing as $1/T^2$, and saturates at $\mathcal{A}/8\pi^2$.

It is evident from Eq. (10) that the behavior of the spectral rigidity is determined completely by the nature of the form factor $\phi(T)$. In our subsequent analysis we will thus focus on $\phi(T)$ and explain past numerical results on the basis of this.

The above form for $\phi(T)$ [Eq. (15)] suggests the following features: (i) When $\mathcal{A}_{\rm min}$ decreases (due to the shape of the billiard), with $\mathcal{A}$ and $\mathcal{N}$ remaining invariant, $\phi(T)$ drops further due to a decrease in the value of $\langle a(l_0)\rangle$ and thus $\Delta(L)$ will depart more from $L/15$. (ii) When the number of directions, $\mathcal{N}$, increases keeping $\mathcal{A}$ and $\mathcal{A}_{\rm min}$ constant, for instance, by changing the angles of the polygonal billiard slightly, the value of $\langle a(l_0)\rangle/a_{\rm max}$ must decrease due to an increase in $a_{\rm max}$. (Note that the short periodic orbits are not so sensitively dependent on $\mathcal{N}$ in this case, as $\mathcal{A}$ and $\mathcal{A}_{\rm min}$ are held constant, and thus $\langle a(l_0)\rangle$ is roughly independent of $\mathcal{N}$). Hence it is clear from Eq. (15) that $\phi(T)$, and thus $\Delta(L)$, will fall further from the Poisson result. (iii) With an increase in energy [E_0 in Eq. (8)] the selection function $G(y)$ picks longer orbits for which $\phi(T)$ is closer to $d_{\rm av}/2\pi$ compared to that at a lower energy where shorter orbits contribute. Thus $\Delta(L)$ shifts towards the Poisson result with an increase in E_0. (This is in contrast to the integrable or chaotic cases where the fluctuations are invariant with energy.) In the limiting case of very large energy, the selection function G will choose orbits of large length and so $\langle a(l)\rangle$ will saturate at $a_{\rm max}$. Thus $\phi_D(T)$ will be a constant $(d_{\rm av}/2\pi)$ from where it follows that $\Delta(L)$ will show a linear dependence.

Numerical studies on polygonal billiards corroborate the above observations. Cheon and Cohen [7] have studied the spectral fluctuations of a class of pseudointegrable systems which approximate the chaotic Sinai billiard by a series of steps replacing the circle. As the number of steps increase (the number of directions, $\mathcal{N}$, remains constant, but $\mathcal{A}_{\rm min}$ decreases) the approximation to the Sinai billiard gets better and they observe a shift towards the chaotic result. This follows quite simply from case (i) above.

Biswas [9] has carried out a similar study on a family of rational rhombus billiards with angles that approximate $2\pi/(\sqrt{5}+1)$ increasingly well. Here too there is shift towards the chaotic result. In this case the genus increases, resulting in an increase in $\mathcal{N}$, while $\mathcal{A}$ and $\mathcal{A}_{\rm min}$ remain constant. The nature of the fluctuations can again be understood in terms of case (ii) above. Further, it was observed [9] that $\Delta(L)$ evaluated at smaller energies was lower compared to that evaluated at higher energies. This is consistent with the energy dependence predicted by our theory [case (iii) above].

In summary, we provide a sum rule for the length spectrum of pseudointegrable systems in terms of the quantal energies, analogous to that for the eigenvalue spectrum, by a suitable inversion of the trace formula [17]. We find that the proliferation rate of the periodic orbits differs from the integrable case for lengths that are relevant for the spectral fluctuations. In the asymptotic case, however, we recover Gutkin's result for almost-integrable systems. Using this analysis, we derive an expression for the form factor and infer from it the dependence of spectral fluctuations on the geometry of the system. The theory allows us to explain various numerical results obtained in recent years [18].

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Appendix.— We can also use a local approximation for the function $\langle a(l)\rangle$ to obtain a form of the spectral rigidity via Eq. (10). This is valid since for values of L smaller than the outer scale $L_{\rm max}$ (defined earlier) the interval of lengths which gives the L dependence of spectral rigidity is restricted by the selection function $G(y)$. Hence only the local behavior of $\phi(T)$ [and hence $\langle a(l)\rangle$] is relevant.

[In the rest of the interval we do not need to assume any form for $\phi(T)$ as will be clear from the analysis below.] Thus using the *local* form $\langle a(l)\rangle = Cl^{\delta}$ where δ $(=\delta_{E_0})$ is a small constant whose value depends on E_0 [where E_0 determines the interval of lengths contributing to the spectral rigidity, see Eq. (8)], we can arrive at the following expressions:

$$\phi_D(T) = BE^{\delta/2}T^{\delta}\,, \tag{A1}$$

where $B = b_0 C 2^{\delta}/8\pi^2$. Equation (A1) yields the correct value [2,3] for integrable billiards since $C = 4\mathcal{A}$, $\delta = 0$, and $b_0 = \frac{1}{4}$. Defining $\tau = T/2\pi d_{\rm av}(E)$ and writing $\phi(T) = d_{\rm av}(E)K(\tau)/2\pi$, it follows that $K(\tau) = D\tau^{\delta}$ for $\tau_{\min} \ll \tau \ll 1$ and $K(\tau) = 1$ for $\tau \gg 1$, where $D = B(2\pi)^{\delta+1} \times E^{\delta/2} d_{\rm av}^{\delta-1}$. Substituting the relevant quantities in Eq. (10) we obtain [3]

$$\Delta(L) = \frac{L}{2\pi}\int_0^{\infty}\frac{dy}{y^2}K(y/\pi L)G(y)\,, \tag{A2}$$

where $y = \pi L \tau$. For $L \ll 1$, $K(\tau) = 1$, which gives $\Delta(L) = L/15$ [3]. When $1 \ll L \ll L_{\max}$, it is possible to divide the range of integration in Eq. (A2) into two parts by choosing a value of Y that satisfies $1 \ll Y \ll L$. Using the above expression for $K(\tau)$ we get

$$\Delta(L) = \mathcal{C}L^{1-\delta} + \mathcal{D}\,, \tag{A3}$$

where

$$\mathcal{C} = \frac{D}{2\pi^{1+\delta}}\left[-\int_0^{Y}\frac{dy}{y^{2-\delta}}F^2(y) - 3\int_0^{Y}\frac{dy}{y^{2-\delta}}[F'(y)]^2\right]$$

and

$$\mathcal{D} = \frac{1}{2\pi^2(1-\delta)}\int_0^{\infty}\frac{dx}{x^{1-\delta}}\frac{d}{dx}\left\{\frac{K(x)}{x^{\delta}}\right\}.$$

Since $\mathcal{D}$ is L independent we do not need to approximate $K(x)$ [and hence $\phi(T)$] in the interval (Y,∞) in order to infer the L dependence of the spectral rigidity. (This is in the same vein as in chaotic systems [3].) The form in Eq. (A3) has been observed in previous numerical studies on the rhombus billiards [19]. Arguments similar to those for $\phi(T)$ can be put forward for the variation of δ to explain the dependence of $\Delta(L)$ on the geometry of the system.

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[11] Note that one can obtain the length spectrum from purely classical considerations only for very special cases since the possibility of a simple tessallation of the billiard plane determines whether or not one can easily enumerate the periodic orbits. Moreover this classical approach is not particularly useful as the analysis has to be done case by case and a general form cannot be obtained.

[12] Note that ε cannot be zero as Eq. (2) is valid for only $E > 0$.

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[14] The following argument shows that b_0 is a constant: When ε is sufficiently small Eq. (4) becomes

$$b_0 = \int_0^{\varepsilon} dE\left[\frac{\mathcal{A}}{4\pi} + \frac{1}{4\pi}\sum_j a_j J_0(\sqrt{E}\,l_j)\right]$$

$$\cong \sum_j a_j \int_0^{\varepsilon} dE\, J_0(\sqrt{E}\,l_j)\,.$$

Now, for every ε there exists an $l_{\max} \sim 1/\sqrt{\varepsilon}$, such that for $l_j > l_{\max}$, $J_0(\sqrt{E}\,l_j)$ oscillates very rapidly and the integral vanishes. Therefore $b_0 \sim \varepsilon \sum a_j$, where the sum is restricted up to $l_j < l_{\max}$. Then it follows from Eq. (5) $(\sum a_j \sim l_{\max}^2)$ that b_0 is a constant. For any value of ε in the range $0 < \varepsilon < E_i$, the leading term for large l is still b_0.

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[18] Note that the dependence of spectral fluctuations on geometry is determined by the ratio $\langle a(l_0)\rangle/a_{\max}$ in Eq. (15). Thus there may be competing influences affecting $\phi(T)$ making it difficult to infer the effects conclusively.

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