Using thresholding at varying intervals to obtain different temporal patterns

Sudeshna Sinha*
The Institute of Mathematical Sciences, Taramani, Chennai 600 113, India
(Received 16 August 2000; published 26 February 2001)

We show how stroboscopic threshold mechanisms can be effectively employed to obtain a wide range of stable cyclic behavior from chaotic systems, by simply varying the frequency of control. We demonstrate the success of the scheme in a prototypical one-dimensional map, as well as in a three-dimensional system modeling lasers where the threshold action is implemented on any one of the variables. It is evident that thresholding is capable of yielding exact limit cycles of varying periods and geometries when implemented at different intervals (even when very infrequent). This suggests a simple and potent mechanism for selecting different regular temporal patterns from chaotic dynamics.

DOI: 10.1103/PhysRevE.63.036212 PACS number(s): 05.45.−a

I. INTRODUCTION

Mechanisms that enable a system to maintain a fixed activity (the ''goal'' or ''target'') even when intrinsically chaotic have many applications [1,2] in situations ranging from biology (as in the control of cardiac rhythms [3]) to engineering. It is thus of considerable interest and potential utility to devise algorithms capable of achieving the desired type of behavior in strongly nonlinear systems.

In recent years, there has been intense research activity devoted to the design of effective control techniques [1,2]. A large body of work derives from the Ott-Grebogi-Yorke (OGY) idea [1], which seeks to use small perturbations to place chaotic orbits onto desired (unstable) periodic orbits. Since chaotic orbits are ergodic on the attractor they eventually wander close to the desired periodic orbit and because of this proximity can be ''captured'' by a small control.

Here we describe an alternate control strategy: the simple and easily implementable *threshold* mechanism. We will demonstrate the scope of the threshold action implemented *at varying intervals* to yield a wide range of regular orbits, whose period and geometry depend on the frequency of control, i.e., we achieve control to different temporal patterns simply by varying the frequency of control.

First we will introduce the general formalism below, and then we will investigate two representative examples: a one-dimensional map, and a multidimensional system, namely, a set of three coupled ordinary differential equations (ODEs) modeling a laser system.

II. THRESHOLD MECHANISM

Consider a general N-dimensional dynamical system, described by the evolution equation

$$\frac{d\mathbf{x}}{dt}=F(\mathbf{x};t),$$

where $\mathbf{x}\equiv(x_1,x_2,\ldots,x_N)$ are the state variables, and variable x_i is chosen to be monitored and threshold controlled. The prescription for threshold action in this system is as follows: control will be triggered whenever the value of the monitored variable exceeds a critical threshold x^* and the variable x_i will then be reset to x^*, i.e.,

$$\begin{aligned}&\text{if } x_i\leqslant x^* \quad \text{then no action,}\\ &\text{if } x_i>x^* \quad \text{then } x_i\rightarrow x^*.\end{aligned} \tag{1}$$

The dynamics continues until the next occurrence of x_i exceeding the threshold, when control resets its value to x^* again. No knowledge of $F(\mathbf{x})$ is involved, and no computation is needed to implement the threshold action.

All the system parameters are left invariant by this method as it acts only on a state variable. In fact the method requires no knowledge of the parameters, which is advantageous. The moment the thresholding is removed the system is back to its original dynamics.

The threshold action is necessarily *stroboscopic*, as the threshold condition can be checked only at finite intervals. Here we will study the interesting and often unexpected effects of implementing the threshold action at varying intervals. We will show that changing the frequency of thresholding leads to many different regular temporal patterns. In fact even very infrequent thresholding is capable of yielding amazingly simple and regular orbits.

First we will study the one-dimensional example both numerically and analytically, and then we will investigate a multidimensional laser system through extensive numerical simulations.

III. ONE-DIMENSIONAL MAP

In the case of one-dimensional maps [4,3], where the evolution of the uncontrolled system is given by

$$x_{n+1}=f(x_n) \tag{2}$$

with f being a nonlinear function, the threshold mechanism is simply implemented as the following condition: if variable $x_{n+1}>x^*$ then the variable is adjusted back to x^*. The threshold x^* is the critical value the state variable is not allowed to exceed, and controlling action is triggered whenever the state variable grows larger than the prescribed

*Electronic address: sudeshna@imsc.ernet.in

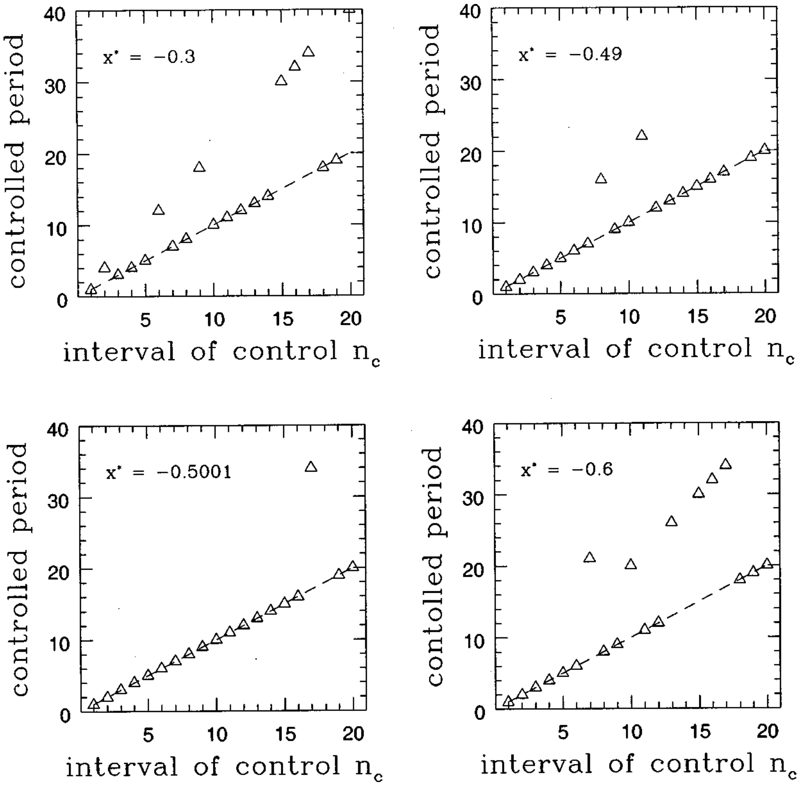

FIG. 1. The chaotic logistic map $[x_{n+1} = f(x_n) = 1 - 2x_n^2]$, under threshold control with threshold value equal to -0.5001, 0.1, 0.4, 0.6. The control acts in intervals of n_c, with $n_c = 1,...,20$. The periods of the resultant cycles p are displayed vs the interval of control. Note the marked concentration along the $p = n_c$ line (shown by a dashed line in the figures), i.e., the controlled period is often equal to the interval of control. Such figures can serve as easy look-up diagrams from which one can read off the thresholding frequency necessary for obtaining a desired period.

threshold [3,5]. It was shown in [4,3] that this simple threshold action, implemented after every iterate n, controlled the fully chaotic map onto orbits of all orders. In this scheme the trajectory does not have to be close to any particular unstable fixed point before control is implemented. Once the system exceeds the threshold, it is caught immediately in a stable orbit. So control transience is very short.

Now we will implement the threshold mechanism at varying intervals n_c, with $1 < n_c \leqslant 20$, i.e., the thresholding frequency ranges from once every two iterates of the chaotic map to once every 20 iterates. We find that for all n_c in this range the chaotic map gets controlled onto an exact and stable orbit of periodicity $p \geqslant n_c$.

Figure 1 shows the periods p of the different orbits resulting from threshold action on the chaotic map, with the thresholding implemented at different intervals. Notice that the periods are concentrated on the $p = n_c$ line, i.e., the period of the resultant cycle is often equal to the interval of thresholding, especially for thresholds $x^* \sim -0.5$ [6].

It is thus evident that one can fix a threshold and obtain different temporal behavior by simply varying the interval of thresholding. It is very interesting to see how very effective infrequent thresholding can be in regulating systems. In fact it can serve as an extra tool for selecting different cyclic behaviors from the chaotic dynamics.

This has particular utility in obtaining higher order periods, which are difficult to obtain by adjusting the threshold levels alone, as one has to make finer and finer threshold settings. Here, on the other hand, the threshold level can be kept invariant, and only the interval of thresholding adjusted, in order to obtain the desired orbit.

This scheme can thus prove useful in applications where one does not want to invest effort in changing the threshold but wishes to obtain different periodic behaviors. Further, the infrequent threshold action, involving infrequent monitoring and resetting to obtain the result, reduces the cost of control. We will now further analyze the stroboscopic threshold scheme below.

A. Analysis

For the one-dimensional map the analysis can be done exactly. That is, one can directly calculate the period corresponding to a particular threshold x^* and interval of control n_c.

The starting point of the analysis is the fact that the ergodicity of chaos guarantees that the system will exceed threshold at some point in time. At that point its state is reset to x^*. One then studies the forward iterates of the map, starting from this state $x = f_0(x^*) = x^*$, i.e., $f_1(x^*), f_2(x^*), \ldots$, where $f_k(x^*)$ is the kth iterate of the map. Specifically, for $f(x) = 1 - 2x^2$, $x \in [-1,1]$, this is

(1) $k=0$, $f_0(x^*) = x^*$,

(2) $k=1$, $f_1(x^*) = 1 - 2(x^*)^2$,

(3) $k=2$, $f_2(x^*) = 1 - 2[1 - 2(x^*)^2]^2 = 8(x^*)^2 - 8(x^*)^4 - 1$,

and so on. In general

$$f_k(x^*) = f \circ f_{k-1}(x^*) = f \circ f \circ \cdots f \circ (x^*).$$

Now let us denote the n_cth iterate of the chaotic map, $f_{n_c}(x^*)$, by $F(x^*)$. So $F_k(x^*) = f_{n_c \times k}(x^*)$.

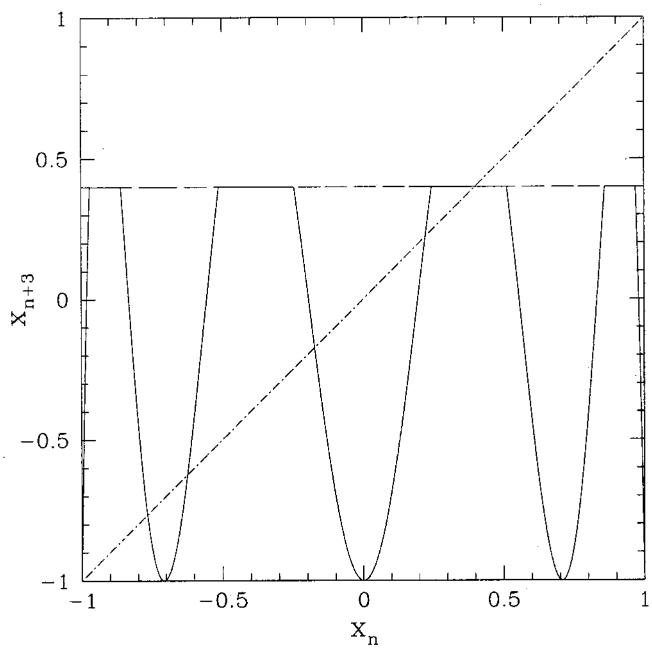

FIG. 2. The map $(x_{n+3}=f\circ f\circ f(x_n)=1-2\{1-2[1-2x_n^2]^2\}^2)$, under threshold control with threshold value equal to 0.4 (indicated by a dot-dashed line in the figure). This is the effective map for the situation where the control acts at intervals of $n_c=3$, i.e., after every third iterate. The thresholded chaotic map is controlled to period 3, as the $x_{n+3}=x_n$ line intersects the flat-top region. So the fixed point $x_{n+3}=x_n$ is superstable, as the slope is zero at that point.

The effective action of thresholding at intervals of n_c is to yield a beheaded $F\equiv f_{n_c}$ map. This ''flat-top'' map of x^* to $F(x^*)$ can yield stable periodic orbits of various orders for (1) different threshold values, which determine the level at which the map is chopped off; and (2) different n_c values, which determine the form of the map being beheaded, namely, how many crests and troughs the map has. For $n_c=1$, it is the usual unimodal map, with one hump in the interval $[-1,\ 1]$ as $F\equiv f$. In general $F=f_{n_c}$ has 2^{n_c-1} maxima in the interval.

The controlling action of the threshold mechanism is best rationalized through the fixed points of the map x_{n+n_c} vs x_n, where $x_{n+n_c}=f_{n_c}(x_n)=F(x_n)$, under varying heights of truncation determined by different thresholds x^*. When the flat portion of the kth iterate of F, $F_k\equiv f_{n_c\times k}$, intersects the 45° line we have a superstable period $n_c\times k$, with the points on the orbit being $x^*=f_0(x^*)$, $f_1(x^*)$, $f_2(x^*)$, ..., $f_{n_c\cdot k-1}(x^*)$. In terms of probability densities, the chaotic map under threshold mechanism, i.e., with the flat top, will map large intervals onto a severely contracting region. This is why the control transience is so short, and the method is so powerful.

For example, Fig. 2 shows the case of $n_c=3$, i.e., where the threshold action is implemented every third iterate, for threshold value 0.4. Here $F(x)=f_3(x)$. Clearly the 45° line intersects the flat portion (slope 0) of the beheaded map, and this yields the fixed point $x_{n+3}=x_n$, i.e., a period 3 cycle.

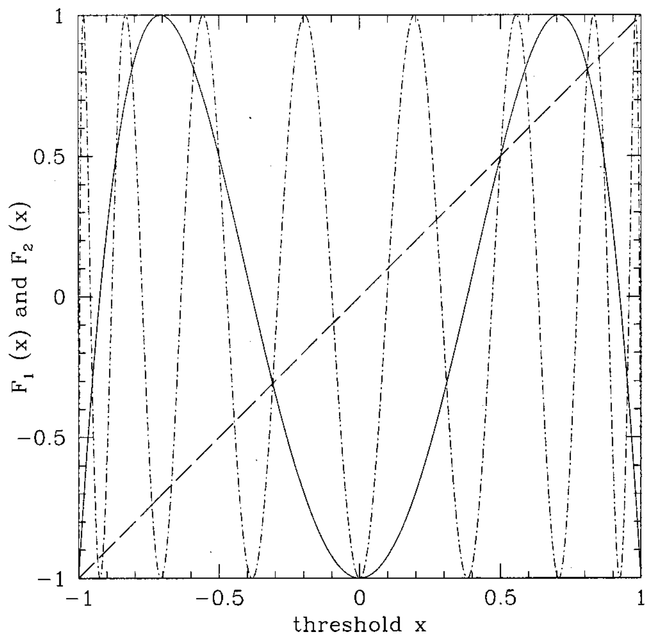

FIG. 3. Curves of $F_k(x)=f_{n_c\cdot k}(x)$ vs threshold x, with $n_c=2$. Here $k=1$ [i.e., $F_1(x)=f_2(x)$] is shown by the solid line, and $k=2$ [i.e., $F_2(x)=f_4(x)$] is shown by the dot-dashed line. The 45° line (dashed) is also displayed. Whenever the F_k curve crosses above this line $F_k(x)>x$, i.e., the kth iterate of F exceeds the threshold value x.

This cycle is *superstable* as $F'(x)=0$ at the point $x_{n+3}=x_n$. So here we obtain an orbit with period equal to the interval of thresholding, namely $p=n_c=3$, with the points on the cycle being $x^*=0.4$, $f_1(x^*)=0.68$, $f_2(x^*)=0.08$ [the next iterate $f_3(x^*)=0.98>x^*$ and so is reset to x^*, i.e., back to the first point of the cycle].

Alternately one can analyze the situation as follows. Whenever the $F_k(x^*)$ vs x^* curve crosses above the $F_0(x^*)=x^*$ line (i.e., the 45° line) we have an $n_c\times k$ cycle, as this implies that the $(n_c\times k)$th iterate exceeds the critical value x^* and thus is adapted back to x^* (which is the first point in the cycle). For instance, for $n_c=2$, we have $F_1(x)=f_2(x)=8(x^*)^2-8(x^*)^4-1$. The $F_1(x)$ and $F_2(x)=F_1\circ F_1(x)\equiv f_4(x)$ curves are displayed in Fig. 3. It is clear that in the range $-1\leqslant x^*<-0.3$ and in the range $0.5<x^*<0.8$ the $F_1(x^*)$ curve lies above the F_0 curve [i.e., $F_1(x^*)>x^*$]. So the chaotic element is adapted back to x^* after every n_c iterates, yielding a period n_c cycle, with $n_c=2$ here.

In the ranges $-0.3<x^*<0.5$ and $-0.8<x^*<1$ the $F_1(x^*)$ curve dips below the 45° line, but the $F_2(x^*)$ curve lies above the 45° line in the ranges $-0.3<x^*<-0.1$, $0.1<x^*<0.28$, $0.81<x^*<0.85$, and $x^*\sim 0.98$. So there are four windows of period $2n_c=4$, as the $(2\times n_c)$th iterate of the map (starting from $x=x^*$) exceeds threshold and is adapted back to x^*. So the threshold mechanism always leads to cycles, as the system is guaranteed to exceed the threshold during the course of its evolution, for all threshold values x^* smaller than the bounds of the attractor. When the iterate exceeds the threshold it is trapped in a cycle whose period is determined by the considerations outlined above-namely, the cycle at each value of threshold is the smallest k

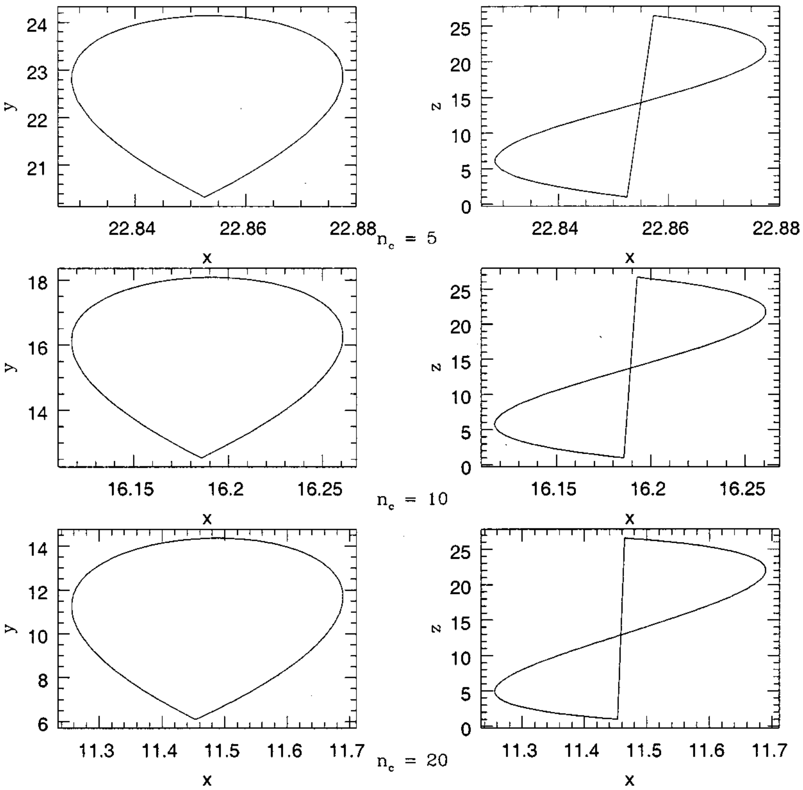

FIG. 4. The chaotic (Lorenz-like) IR NH_3 laser system under threshold control of variable z, with threshold value $z^*=1$. The control acts at intervals of $n_c \times \delta t$, with $\delta t=0.01$. Here $n_c=5,10,20$, i.e., the frequency of control ranges from once in an interval of 5×0.01 to once in an interval of 20×0.01. The controlled cycles in x-y space and x-z space are displayed.

such that the kth iterate of the map F (starting from $x=x^*$) is greater than x^*, i.e., $F_k(x^*)>x^*$.

It is thus evident through both numerical simulation and analytical treatment that a chaotic system can yield *a wide variety of dynamical behaviors under fixed threshold, by simple variation of the interval of control*. Changing the interval of thresholding then acts as an effective mechanism for selecting different temporal patterns, thus suggesting a tool for control.

Once one obtains the ''bifurcation diagram'' of the controlled cycles with respect to threshold value and interval of thresholding (for instance, as in Fig. 1), one can use this knowledge as a look-up table for very swift control, requiring no further run-time knowledge of the system. Calibrating the system characteristics at the outset with respect to threshold and interval of thresholding gives one all the information one needs to directly and simply effect control at all consequent times, at no additional cost of studying the system.

IV. APPLICATION TO LASER SYSTEMS

Now we demonstrate the action of infrequent threshold control on a system of three coupled ODEs: a chaotic Lorenz-like attractor known to be relevant to lasers [7]. It is given by

$$\dot{x}=\sigma(y-x),$$

$$\dot{y}=rx-y-xz,$$

$$\dot{z}=xy-bz. \tag{3}$$

The correspondence between the laser and the system above is as follows: the z variable corresponds to the normalized inversion and the x and y variables to normalized amplitudes of the electric field and atomic polarizations, respectively. The three parameters corresponding to the coherently pumped far-infrared ammonia laser system, obtained by detailed comparison with experiments [7] are $\sigma=2$, $r=15$, and $b=0.25$.

A crucial issue in multidimensional systems is whether or not the thresholded state variable can enslave the rest of the variables to some regular dynamical behavior, especially so when the interval of control is large. It is interesting to determine how infrequent one can make the the threshold action and still effectively manage to control chaotic systems onto regular temporal behavior. Here we investigate this through extensive numerical simulations.

To check the efficacy of the threshold mechanism in this multidimensional system, we impose the threshold condition on any one of the three variables of the system, i.e., one demands that variable x, y, or z must not exceed the prescribed threshold values x^*, y^*, and z^*, respectively. Figures 4–11 show some representative results of this threshold action for a range of control intervals $n_c\times\delta t$, with n_c ranging from 1 to 1500 ($\delta t=0.01$) for different state variables. It is clear that the mechanism (at fixed threshold value) successfully controls to limit cycles of varying sizes and geometries by simply varying the interval of control. Interestingly, very infrequent control, for instance $n_c=1500$, alsomanages to yield a clean, exact, and simple limit cycle (see Fig. 8).

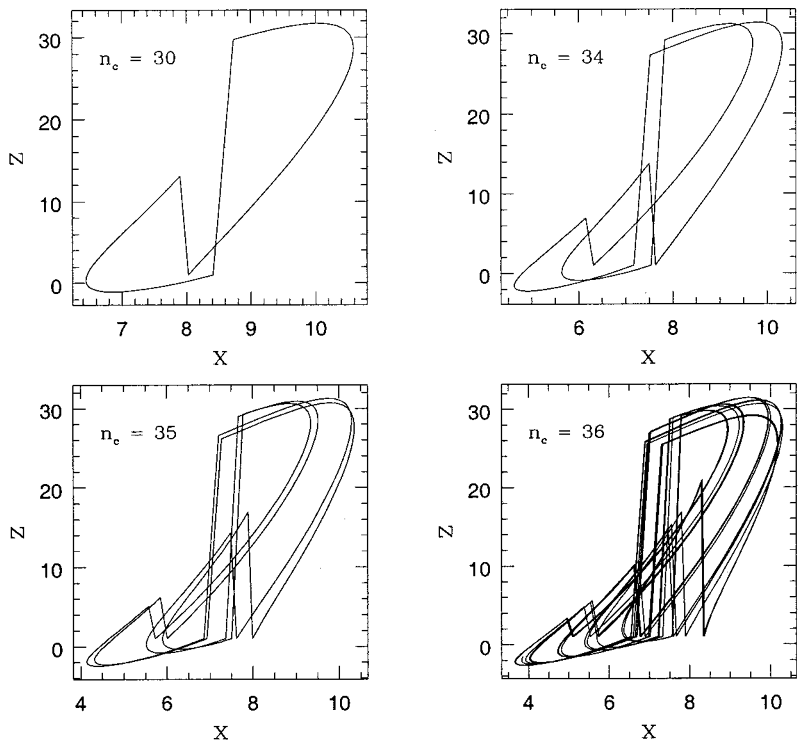

FIG. 5. The chaotic laser-Lorenz system under threshold control of variable z, with threshold value $z^*=1$. The control acts at intervals of $n_c \times \delta t$, with $\delta t=0.01$. Here $n_c=30$, 34, 35, 36. The controlled cycles in x-z space are displayed. Notice the period doubling of the cycles as the control interval n_c is increased.

Consider the particular case of the threshold mechanism imposed on the z variable. The stroboscopic threshold action occurs at an interval of $n_c \delta t$. Figures 4–8 show the different temporal patterns obtained when the threshold is fixed at $z^*=1$ and the n_c is increased from 1 to 1500. Thus the frequency of control is decreased from once every 0.01 unit of time to once every 15 units of time, i.e., spanning three orders of magnitude.

When the frequency of control is high, i.e., when the interval of threshold action n_c is low, one obtains fixed points

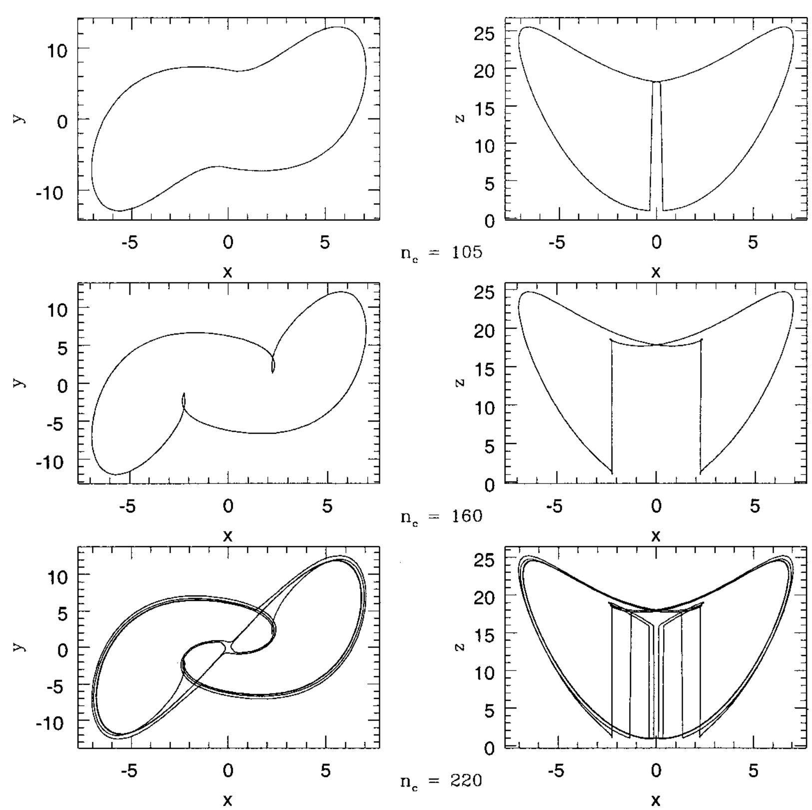

FIG. 6. The chaotic laser-Lorenz system under threshold control of variable z, with threshold value $z^*=1$. The control acts at intervals of $n_c \times \delta t$, with $\delta t=0.01$. Here $n_c=105,160,220$. The controlled cycles in x-y space and x-z space are displayed. Notice the period doubling of the cycles as the control interval n_c is increased.

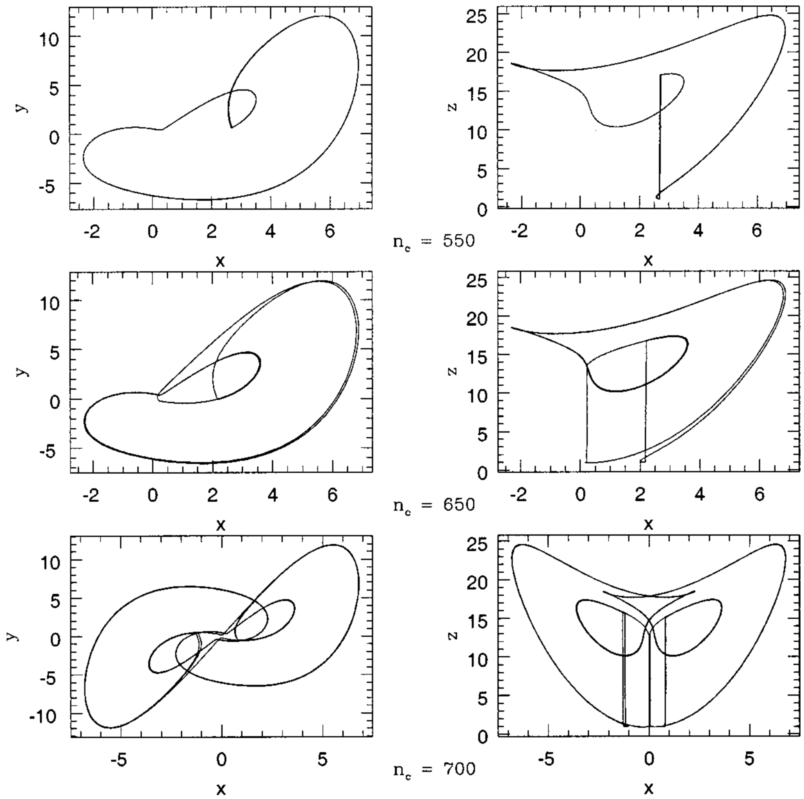

FIG. 7. The chaotic laser-Lorenz system under threshold control of variable z, with threshold value $z^* = 1$. The control acts at intervals of $n_c \times \delta t$, with $\delta t = 0.01$. Here $n_c = 550,650,700$. The controlled cycles in x-y and x-z space are displayed.

(in the stroboscopic sense). That is, right after every threshold control event, the state variables are always exactly at the same set of values (Fig. 4).

After stroboscopic fixed points, limit cycles are obtained. On increasing n_c, typically, doubled limit cycles are obtained (i.e., the limit cycle develops strands). Then on further increase of the interval of control fuzzier cycles arise. And then often, interestingly, further decreasing the frequency of control yields exact simple limit cycles of a different geometric family. Figures 5–8 show several specific examples

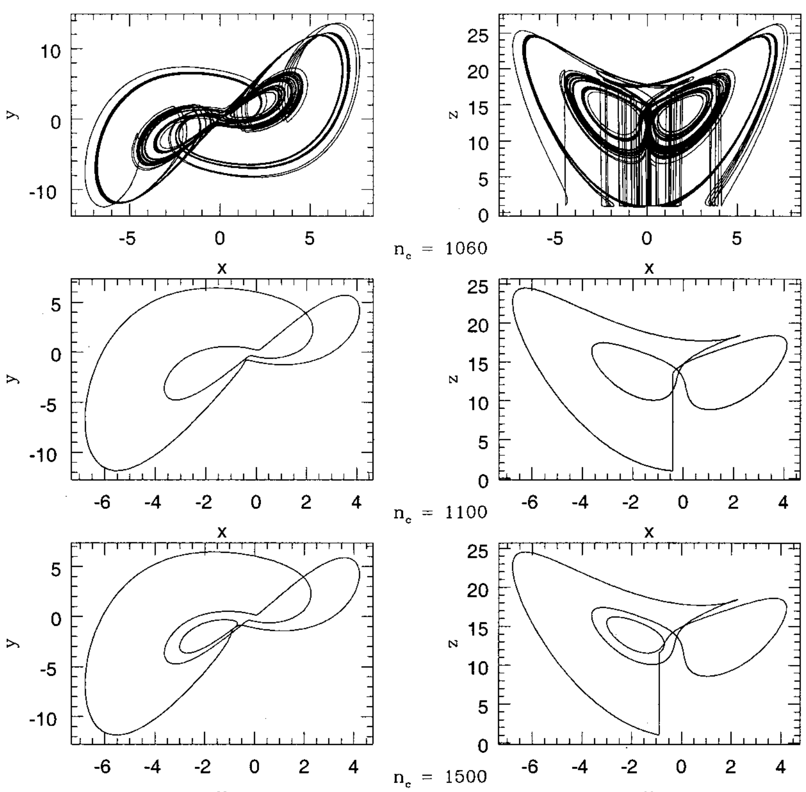

FIG. 8. The chaotic laser-Lorenz system under threshold control of variable z, with threshold value $z^* = 1$. The control acts at intervals of $n_c \times \delta t$, with $\delta t = 0.01$. Here $n_c = 1060,1100,1500$. The controlled cycles in x-y and x-z space are displayed.

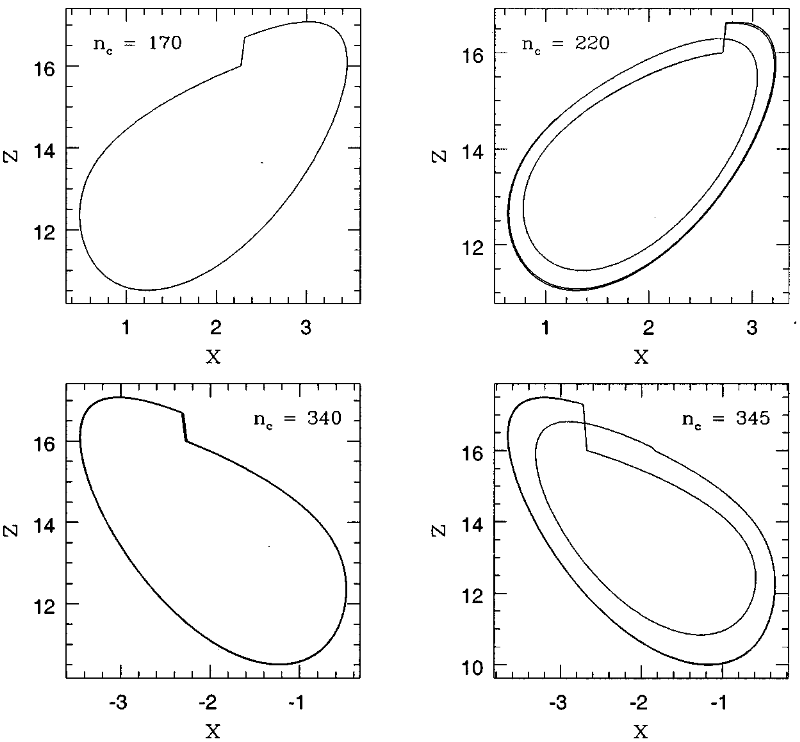

FIG. 9. The chaotic laser-Lorenz system under threshold control of variable z, with threshold value $z^* = 16$. The control acts at intervals of $n_c \times \delta t$, with $\delta t = 0.01$. Here $n_c = 170,220,340,345$. The controlled cycles in x-z space are displayed. Note that the limit cycles obtained at $n_c = 170$ and $n_c = 340$ have the same period $340 \times \delta t = 3.4$.

of this wide range of temporal patterns obtained by variation in the interval of control.

Note that one actually has situations where more regular temporal patterns are obtained from more infrequent control. For instance, compare the orbits obtained with thresholding at intervals of $n_c = 1060$ and 1500, shown in Fig. 8. Clearly, when $n_c = 1060$, the resultant orbit is noisy while when $n_c = 1500$, i.e., with significantly more infrequent control, the orbit is exact and geometrically simple.

Figure 9 shows the resultant orbits obtained with threshold fixed at the large value $z^* = 16$, with the interval of thresholding varying from ~ 150 to ~ 350. Figure 10 shows

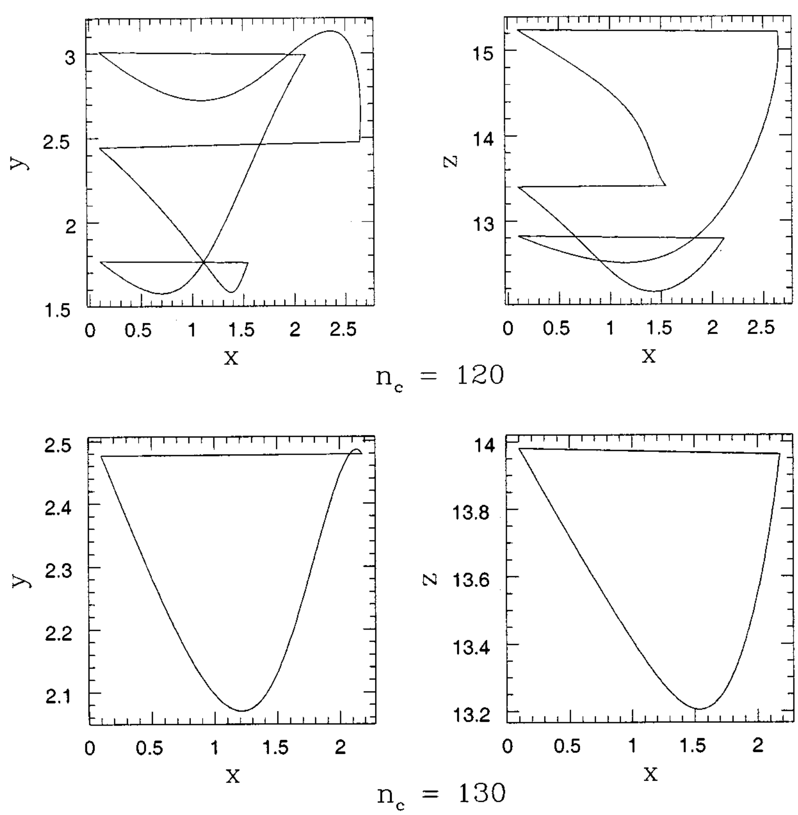

FIG. 10. The chaotic laser-Lorenz system under threshold control of variable x, with threshold value $x^* = 0.1$. The control acts at intervals of $n_c \times \delta t$, with $\delta t = 0.01$. Here $n_c = 120,130$. The controlled cycles in x-y and x-z space are displayed. Note that the orbit has looped around three times as the control interval is decreased from $n_c = 130$ to $n_c = 120$.

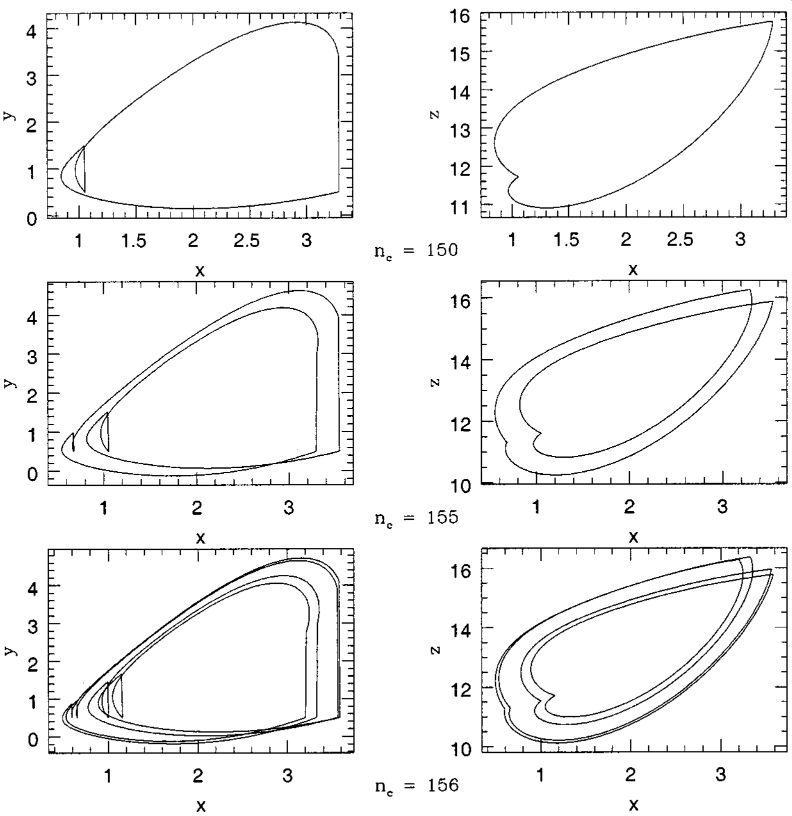

FIG. 11. The chaotic laser-Lorenz system under threshold control of variable y, with threshold value $y^* = 0.5$. The control acts at intervals of $n_c \times \delta t$, with $\delta t = 0.01$. Here $n_c = 150,155,156$. The controlled cycles in x-y and x-z space are displayed. Notice the period doubling of the cycles as the control interval n_c is increased.

the regular temporal patterns obtained from threshold action on the x variable, with threshold value $x^* = 0.1$, and Fig. 11 displays the control achieved by thresholding the y variable, with threshold fixed at $y^* = 0.5$, for a range of control frequencies.

When the interval of control is too large, the threshold mechanism is unable to effect control to exact limit cycles. This failure to control at very infrequent thresholding occurs earlier for higher thresholds. For instance, thresholding is no longer capable of yielding temporal regularity for n_c beyond ~ 3000 for $z^* = 1$ while it fails beyond $n_c \sim 350$ for threshold values close to the bounds of the attractor, e.g., $z^* = 16$.

In conclusion, here we have obtained a large range of numerical evidence to show that stroboscopic threshold action of any variable in this multidimensional chaotic system successfully yields regular temporal patterns, displaying a wide variety of periods and geometries. In fact, the interval of control may be very large in many cases and still lead to very effective control onto simple limit cycles. So varying the interval of control offers flexibility and cost effectiveness in regulating chaotic systems onto different cyclic patterns.

V. DISCUSSION

Varying the interval of thresholding thus acts as an effective mechanism for selecting different temporal patterns, suggesting a tool for control. One now has the possibility of obtaining sustained temporal regularity from chaos by making very infrequent changes to a state variable. Further, in multidimensional systems thresholding is implemented on a *single* variable alone in order to control the entire system.

This mechanism works in marked contrast to the OGY method. In the OGY method the chaotic trajectories in the vicinity of unstable fixed points are controlled onto these points. In threshold control, on the other hand, the system does not have to be close to any particular fixed point before implementing the control. Here the trajectory merely has to exceed the prescribed threshold. So the control transience is typically very short. Also unlike OGY (or related) control thresholding does not entail any computation during the run time of the implementation.

This technique has a certain similarity with periodic impulse methods [8], in that they are both stroboscopic in operation and act only on state variables. They share the advantage that they do not require knowledge of the system's dynamics or parameters, and both yield stable orbits after control. The difference lies primarily in that our method acts only when the system is above a threshold and is thus very infrequent, while the periodic pulse method acts at fixed intervals. Further, the control action here is a resetting of one variable, while the periodic pulse method involves an additive (negative/positive) or multiplicative pulse to one or more state variables.

In summary, stroboscopic threshold mechanisms can be effectively employed to control chaotic systems onto different stable limit cycles by simply varying the frequency of control. The success of the mechanism is demonstrated in a prototypical one-dimensional chaotic map (both analytically and numerically), as well as in a three-dimensional system modeling lasers (through extensive simulations). In multidi mensional systems, the threshold condition is imposed on only one variable, and this manages to regulate the entire system onto various exact limit cycles (even when the thresholding is very infrequent). A wide range of cyclic behavior is obtained by varying the frequency of thresholding. This suggests that thresholding at varying intervals can serve as a simple and potent mechanism for selecting different regular temporal patterns in chaotic systems.

[1] E. Ott, C. Grebogi, and J. Yorke, Phys. Rev. Lett. **66**, 1196 (1990); T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke, Nature (London) **363**, 411 (1993); V. Petrov, B. Peng, and K. Showalter, J. Chem. Phys. **96**, 7506 (1992).

[2] B. A. Huberman and E. Lumer, IEEE Trans. Circuits Syst. **37**, 547 (1990); S. Sinha, R. Ramaswamy, and J. Subba Rao, Physica D **43**, 118 (1990); S. Sinha, Phys. Lett. A **156**, 475 (1991); A. Hubler, Helv. Phys. Acta **62**, 343 (1989); E. A. Jackson and A. Hubler, Physica D **44**, 407 (1990); E. A. Jackson, Phys. Lett. A **151**, 478 (1990); Phys. Rev. A **44**, 4839 (1991); S. Rajasekhar and M. Lakshmanan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 201 (1992); A. Garfinkel, M. Spano, W. Ditto, and J. Weiss, Science **257**, 1230 (1992); K. Hall *et al.*, Phys. Rev. Lett. **78**, 4518 (1997).

[3] L. Glass and W. Zheng, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 1061 (1994).

[4] S. Sinha and D. Biswas, Phys. Rev. Lett. **71**, 2010 (1993); S. Sinha, Phys. Rev. E **49**, 4832 (1994); Phys. Lett. A **199**, 365 (1995); Int. J. Mod. Phys. B **9**, 875 (1995).

[5] The threshold control action is quite analogous to the current algorithm employed in commercial cardiac pacemakers for the regularization of cardiac rhythm, where an electrical stimulus will be delivered when the interbeat interval is longer than some predetermined value. See S. Ursell and N. El-Sherif, in *Electrical Therapy for Cardiac Arrhythmias*, edited by S. Saksena and N. Goldschlager (Saunders, Philadelphia, 1990), p. 205.

[6] If $x^* = -0.5$ exactly, then the next iterate is the fixed point at $x = 0.5$. All subsequent iterates remain at 0.5, which is greater than the threshold value $x^* = -0.5$. So after every n_c iterates, when the threshold condition is checked, the system will be found to be higher than threshold and reset to -0.5. At the next iterate it goes back to 0.5 and stays there for $n_c - 1$ more iterates till the threshold condition is checked again. So its periodicity is always $p = n_c$, and the orbit is -0.5, 0.5, 0.5,...,0.5, i.e., $x = x^* = -0.5$ followed by $n_c - 1$ iterates at $x = 0.5$.

[7] U. Hubner, N. B. Abraham, and C. O. Weiss, Phys. Rev. A **40**, 6354 (1989), and references therein.

[8] M. A. Matias and J. Guemez, Phys. Rev. Lett. **72**, 1455 (1994); Z. Liu and S. Chen, Phys. Lett. A **232**, 55 (1997); Phys. Rev. E **56**, 168 (1997).