Asynchronous updating of coupled maps leads to synchronization

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(Received 18 June 1999; accepted for publication 9 December 1999)

We investigate the spatiotemporal dynamics of two dimensional coupled map lattices, in the strong coupling phase, evolving under updating rules incorporating varying degrees of asynchronicity. Interestingly, we observe that parallel updates never allow synchronization among the sites, while asynchroncity has the effect of opening up windows in parameter space where the synchronized dynamics gains stability. As asynchronicity increases, the parameter range supporting synchronization gets rapidly wider. Detailed numerics, including bifurcation diagrams and patterns formed en route to synchronization, are reported. We also attempt a mean-field analysis of the system in order to try and account for the stability of the spatiotemporal fixed point under asynchronous updates. © *2000 American Institute of Physics.* [S1054-1500(00)01002-8]

The standard time evolution of coupled map lattices (CML) employ parallel (or synchronous) updates in which all individual maps of the lattice are iterated forward simultaneously. Here, on the other hand, we will study the CML under asynchronous evolution, that is, one in which the updates are not concurrent, but sequential instead. Such systems are relevant to situations where an extended system is comprised of a collection of elemental dynamical units which evolve asynchronously, for instance in neuromorphology, where the neurons, neuron groups, and functional layers of the brain are believed to be fully asynchronous. We consider a two-dimensional CML in the strong coupling regime, and investigate the pattern formation under varying degrees of asynchronicity, for different coupling strengths and coupling forms. We show that asynchronicity has a pronounced effect on the stability of the spatiotemporal fixed point phase. In fact, strictly parallel updating is singular, in that it supports no spatiotemporal fixed point phase, whereas any degree of asynchronicity (however small) opens a window of coupling parameter space in which the synchronized fixed point gains stability.

I. INTRODUCTION

The coupled map lattice (CML) was introduced as a simple model capturing the essential features of nonlinear dynamics of extended systems.[1] Over the past decade, research centered around CML has yielded suggestive conceptual models of spatiotemporal phenomena, in fields ranging from biology to engineering. In particular, this class of systems is of considerable interest in modeling phenomena as diverse as Josephson junction arrays, multimode lasers, vortex dynamics, and even evolutionary biology. The ubiquity of distributed complex systems has made the CML a focus of sustained research interest.

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A CML is a dynamical system with discrete time, discrete space and continuous state variables. It usually consists of dynamical elements on a lattice which interact with suitably chosen sets of other elements. The usual strategy of modeling dynamical phenomena in spatially extended systems by a CML is based on the following procedure: (a) choosing a set of variables (at a macroscopic level); for example temperature, pressure, fluid velocity or concentration, (b) decomposing the underlying phenomena into independent components, which are implemented as simple parallel dynamics on a lattice; for example, there may be

(1) A local dynamics consisting of the nonlinear transformation of the field variables **x**, given by a nonlinear map $f(x)$.
(2) A coupling term $g(x)$ among a set of suitably chosen neighbors.

Now, we describe the class of CML we study. The space on which the dynamics occurs is a *discrete two-dimensional square lattice*, with periodic boundary conditions, where the sites are denoted by two sets of integers i and j, $i,j = 1,\ldots,N$, where N is the linear size of the lattice. On each site is defined a continuous state variable denoted by $x_n(i,j)$, which corresponds to the physical variable of interest, with n denoting the discrete time. The local on-site map is chosen to be the logistic map as it has widespread relevance in the context of low-dimensional chaos

$$f(x) = rx(1-x).$$

The nonlinearity parameter r is chosen to be 4, i.e., the local dynamical map is completely chaotic.

The range of our coupling is nearest neighbor. We investigate two coupling forms: $g(x)=x$ and the future coupled (i.e., discretized Laplacian[1]) form $g(x)=f(x)$. The strength of coupling is given by ϵ (which is the most crucial parameter in the system and governs the nature of the emergent dynamics).

Now, the standard time evolution of the CML above employs parallel (or synchronous) updates in which all individual maps of the lattice are iterated forward simulta-

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neously. Here, on the other hand, we will study the same CML under *asynchronous evolution*; that is, one in which the *updates are not concurrent, but sequential instead.*

There exist specific physical situations where an extended system is comprised of a collection of elemental dynamical units which evolve asynchronously (for instance in neuromorphology, where the neurons, neuron groups, and functional layers of the brain are believed to be fully asynchronous). Lattice dynamical models of such cases must of course employ asynchronous updating schemes. For instance, for neural networks, an independent choice of update times for the various elements of the system is believed to provide a closer approximation to reality.[2] So, it is of importance to investigate the effects of asynchronicity (if any) on various prototype models.

So, though asynchronous updating is traditionally not used, there are strong reasons to revisit this fundamental issue in light of the fact that asynchronous updating can be closer to physical reality than synchronous updates in certain situations.[2–6]

It should also be noted that asynchronous update is in fact the natural thing in Monte Carlo simulations of equilibrium statistical mechanical systems. Equilibrium stochastic systems, such as Ising models, are simulated numerically by Monte Carlo algorithms where spins are updated asynchronously.[7] In fact, Ising models cannot be simulated by synchronous algorithms. Simulations of Langevin equations also respect asynchronous numerical schemes. This leads us naturally to the question of what happens to CMLs under asynchronous updating.

II. MODEL

Now, in order to effectively study the influence of varying degrees of asynchronicity, we investigate the following dynamics: We break the lattice into N_c disjoint subsets, and update the sites belonging to each subset simultaneously, while updating the different subsets sequentially.[8] To remove spurious correlations, the assignation of the subsets is done randomly, and is changed at each iteration. The union of the subsets covers the entire lattice. We index our time n in discrete units, where in the interval between n and $n+1$ the lattice is updated in N_c sequential steps, with one subset being updated at each step. At the end of each unit of time then, all the (randomly chosen) subsets have been updated once, one after another, and so all sites have been updated once.

For instance, in a lattice of 9 elements (lattice size $=3\times3$), say we wish to update in 3 unequal subsets ($N_c=3$), with the subsets being of size 3, 4, and 2, respectively. Now, we update the lattice between time n and $n+1$ in three sequential steps: first we choose at random three sites (i,j) and update them, then we update four sites randomly chosen from the remaining six sites, and finally we update the remaining two sites belonging to the last subset. In the subsequent unit of time, from $n+1$ to $n+2$ we again choose at random, first three sites and update them, then four sites from the remaining sites and update them, followed by the update of the last two; and so on.

The case of $N_c=1$ corresponds to the usual parallel updates, while $N_c=N^2$ corresponds to the completely asynchronous random update. So, the value of N_c indicates varying degrees of asynchronicity in the evolution.

The complete evolution equation for our dynamics is

$$x_{n+1}(i,j)=(1-\epsilon)f(x_n(i,j))+\frac{\epsilon}{4}\{g(x_{k_1}(i+1,j)) + g(x_{k_2}(i-1,j))+g(x_{k_3}(i,j+1)) + g(x_{k_4}(i,j-1))\}. \quad (1)$$

The indices $k_i, i=1,\ldots,4$ depend on the updating mode, and take values n or $n+1$. The usual simultaneous updating form is recovered when $k_i=n$ for all i. Different degrees of asynchronicity can be realized by suitably choosing k_i, and the probability of the indices being n or $n+1$ will be determined exactly by the sizes of the subsets.

For instance, when the updating is done sequentially for two equal randomly chosen subsets ($N_c=2$), half the sites get updated first, and then the other half. So, k_i has probability 1 of being n (and probability 0 of being $n+1$) for sites (i,j) belonging to the first subset. For sites belonging to the second subset, the probability is 1/2 of being n (and 1/2 of being $n+1$) as, on an average, half its neighbors would already have been updated. Similarly, when the updating is done sequentially for three equal randomly chosen subsets ($N_c=3$), one-third of the sites get updated first, and then another one-third, and after that the remaining one-third of sites. So k_i has probability 1 of being n (and 0 of being $n+1$) for the first subset, probability 1/3 of being $n+1$ (and 2/3 of being n) for the second subset, as a third of its neighbors would already have been updated, and finally probability 2/3 of being $n+1$ (and 1/3 of being n) for the last subset, as two-third of the lattice would already have been updated.

In a couple of very recent works, the significance of asynchronicity has been indicated. In Ref. 3 it was observed that the critical properties of a standard CML depend crucially on the updating rules: when the dynamics in a two-dimensional system was changed from simultaneous to asynchronous the corresponding critical properties also changed, and seemed to belong to the 2D Ising universality class. In Ref. 4 it was shown that asynchronous updating in a CML of spatiotemporal intermittency led to critical exponents falling in the universality class of directed percolation.

Further, in an important earlier study Lumer and Nicolis[9] had brought to light some remarkable effects of asynchronous evolution *vis-à-vis* synchronous evolution in a CML with coupling form $g(x)=f(x)$ in the coupling range $\epsilon \leqslant 0.5$. These effects were not subtle differences in critical exponents as in Refs. 3 and 4, but were apparent as completely different stable dynamical phases, manifest in large regions of parameter space.

In this paper we will attempt to find out if asynchronicity has similar far-reaching effects in the strong coupling phase and under different coupling forms. We will focus on two-dimensional lattices, and investigate the pattern formation under varying degrees of asynchronicity, for different coupling strengths and coupling forms. We will present details

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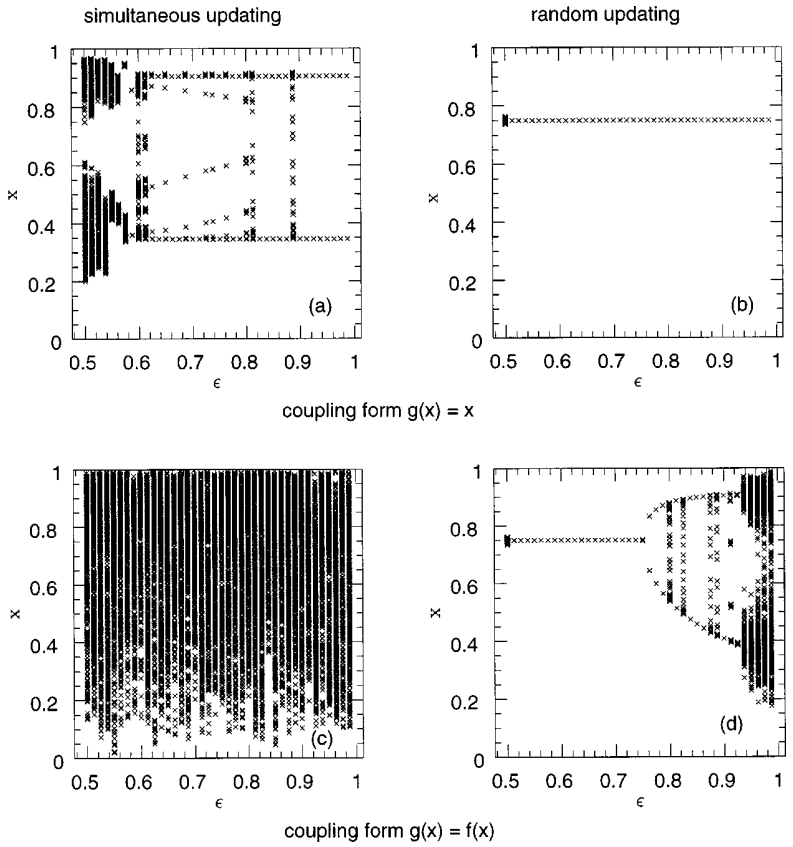

FIG. 1. Bifurcation diagrams, showing values of $x(i)$ $(i=1,\ldots,N^2)$ with respect to coupling strength ϵ, for a CML with coupling form $g(x)=x$, evolving under: (a) simultaneous updates (i.e., $N_c=1$) and (b) asynchronous random updates (i.e., $N_c=N$); and for a CML with coupling form $g(x)=f(x)$, evolving under: (c) simultaneous updates (i.e., $N_c=1$) and (d) asynchronous random updates (i.e., $N_c=N$). Here, the linear size of the lattice is $N=20$ and we plot one random initial configuration after a transience time of 2000.

of the phenomenology obtained from extensive numerical simulations. Further, we will analyze the stability of the system under varying degrees of asynchronicity, and show that there is reasonable agreement between numerics and analytical results.

III. PHENOMENOLOGY

Here, we describe the dynamical phases arising from updating rules with varying degrees of asynchronicity, with respect to coupling strength ϵ, in the strong coupling phase (i.e., $0.5<\epsilon<1.0$). We use the two different coupling forms in the two-dimensional CML mentioned above [namely: $g(x)=x$ and $g(x)=f(x)$ in Eq. (1)]. Our results will be obtained by sampling a large set of random initial conditions. The simulations were done with lattices whose sizes ranged from 10×10 to 20×20.

A. Coupling form $g(x)=x$

In the limiting case of simultaneous updating (i.e., $N_c=1$) one obtains two cycles in space, i.e., a kind of ''antiferromagnetic'' order, characterized by alternating sites. For most initial conditions these two-cyclic configurations are exact, i.e., without any kinks or defects. Temporally, the individual elements evolve as approximate two cycles in this phase. [See Fig. 1(a) for the bifurcation diagram.]

Now, the effect of increasing the number of subsets N_c is to *open up windows in parameter space where a stable synchronized state gains stability*, i.e., where one finds all lattice sites synchronized at $x(i)=x^*=0.75$, for all i.

We then have for all $N_c\geqslant2$, a stable region of synchronized fixed points in the parameter interval: $\epsilon_{\text{bifur}}<\epsilon<1.0$, with ϵ_{bifur} being dependent on N_c. For example, for the case of N_c *equal* subsets (i.e., each subset having N^2/N_c number of sites) we obtain the following scenario:

(1) $N_c=2$ has $\epsilon_{\text{bifur}}\sim0.628$;
(2) $N_c=3$ has $\epsilon_{\text{bifur}}\sim0.564$;
(3) $N_c=4$ has $\epsilon_{\text{bifur}}\sim0.537$;
(4) $N_c=5$ has $\epsilon_{\text{bifur}}\sim0.528$;
(5) $N_c=8$ has $\epsilon_{\text{bifur}}\sim0.514$;
(6) $N_c=10$ has $\epsilon_{\text{bifur}}\sim0.510$;
(7) Limiting case of $N_c=N$ has $\epsilon_{\text{bifur}}\sim0.5$.

The bifurcation diagram of the limiting case of $N_c=N$, namely the completely asynchronous updating is shown in Fig. 1(b).

Interestingly here, dimensionality does not affect the bifurcation points. The values of the lower end of the stable range ϵ_{bifur} obtained for two dimensional lattices are almost equal to that obtained for one-dimensional chains.

In order to closely examine the transition from two cycles to spatial homogeneity, we study the following illustrative situation. We track the spatial patterns in a lattice where the updates occur in two *unequal* subsets. We denote the first set as having N_1 sites and the second N_2 sites ($N_2=N-N_1$, and $N_1\neq N_2$). So, the probability of a site being in the first subset is $p_1=N_1/(N\times N)$, and the probability of it being in the second subset is $p_2=1-p_1$. Now, as p_1 is varied one obtains a *continuous transition* from simultaneous

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updating (when $p_1=0$ and $p_2=N$, i.e., $N_c=1$) to sequential equal subset updating (when $p_1=p_2=0.5$, i.e., the case of $N_c=2$ above).

In Fig. 2 we have followed this system, at constant ϵ, through varying p_1. The increasing asynchronicity in updating brings about a clearly defined transition from two cycles in space to a spatiotemporal fixed point. Note how the zigzag two-cyclic spatial pattern smooths out to yield a flat profile. It does this via increasing areas of progressively shallow two cycles, i.e., two-cycles where the peaks and troughs approach each other, yielding less marked undulations, which in the limit of $p_1 \to 0.5$ yields complete synchronization.

B. Coupling form $g(x)=f(x)$

Here, $N_c=1$, namely simultaneous updating, leads to a spatiotemporally irregular and noisy lattice [see Fig. 1(c)]. Again, the effect of increasing N_c is to open up windows in parameter space where synchronized dynamics gains stability.

So, for equal subsets, we have for all $N_c \geqslant 2$ a stable synchronized fixed point phase in the range $\epsilon_{\text{bifur}} < \epsilon < 0.75$. Again, the lower end of the range gets shifted down with increasing N_c. For example

(1) $N_c=2$ has $\epsilon_{\text{bifur}} \sim 0.705$;
(2) $N_c=3$ has $\epsilon_{\text{bifur}} \sim 0.592$;
(3) $N_c=4$ has $\epsilon_{\text{bifur}} \sim 0.560$;
(4) $N_c=5$ has $\epsilon_{\text{bifur}} \sim 0.540$;
(5) $N_c=8$ has $\epsilon_{\text{bifur}} \sim 0.523$;
(6) $N_c=10$ has $\epsilon_{\text{bifur}} \sim 0.516$;
(7) Limiting case of $N_c=N$ has $\epsilon_{\text{bifur}} \sim 0.5$;

Note however, unlike the case of coupling through $g(x)=x$, here, significant differences arise with changing dimensionality. For instance, *in two dimensions a stable range is obtained even for* $N_c=2$, *whereas in one-dimension it occurs only for* $N_c \geqslant 3$.[10] See Fig. 3 for comparative ϵ_{bifur} in one and two dimensions. Notice that the ϵ_{bifur} for two dimensions is consistently lower than that obtained for one dimension. This indicates the stabilizing role of increasing dimensionality (i.e., increasing connectivity in the lattice) which leads to larger parameter ranges for the stable spatiotemporal fixed point state.

Now, we will again examine the transition from randomness to spatial homogeneity by following the system with two unequal subsets, with p_1 going from 0 to 0.5. Here, one obtains a transition from random spatial profiles (at $p_1=0$) to synchronization as $p_1 \to 0.5$.

Figure 4 tracks this system at constant ϵ through varying p_1 and finds the details of the clearly defined transition from disorder to the synchronization. Note how the random spatial pattern smooths out to yield a flat profile. It does this by first becoming more jagged, i.e., approaching approximate two cycles in space. These spatial zigzag patterns then become increasingly shallow, yielding in the limit of $p_1 \to 0.5$, complete synchronization.

Now, in the case of coupling through $g(x)=f(x)$, thespatiotemporal fixed point again loses stability beyond the above-mentioned parameter regime. At the upper end, when

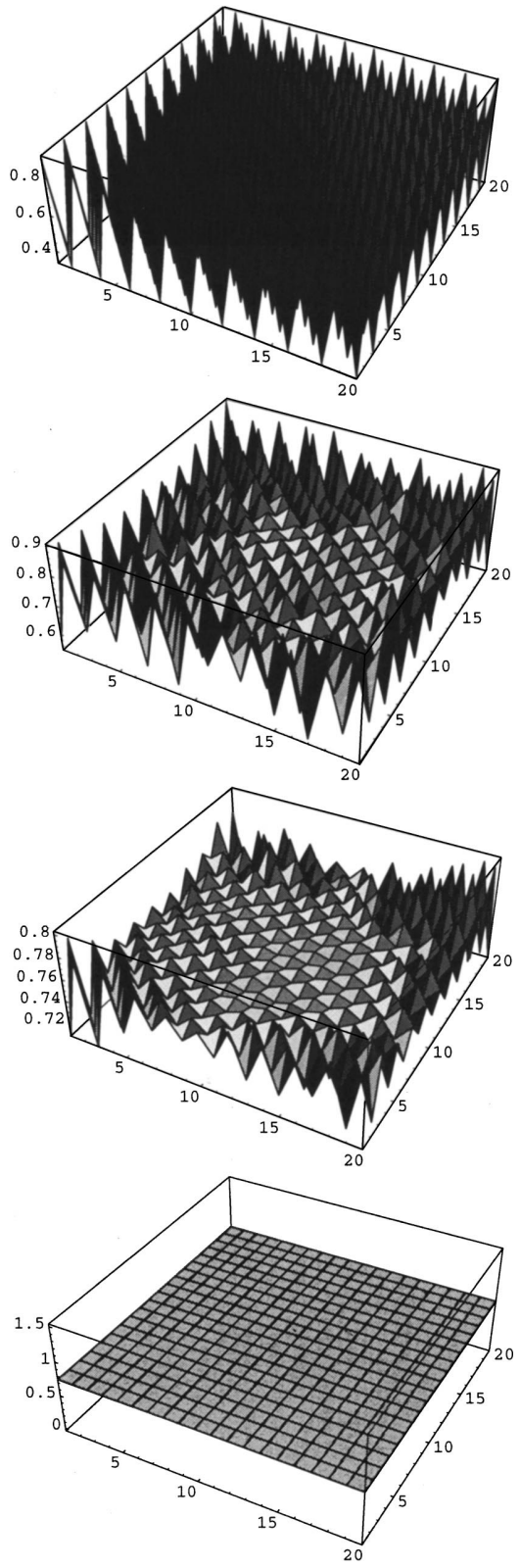

FIG. 2. Surface plots showing the transition from spatial two cycles to synchronization in a 20×20 lattice, for the case of two unequal subsets updating sequentially, with respect to p_1, where p_1 is the fraction of sites in the first subset. Here, $\epsilon=0.8$, $g(x)=x$ and (top to bottom) $p_1=$ 0, 0.12, 0.14, and 0.16.

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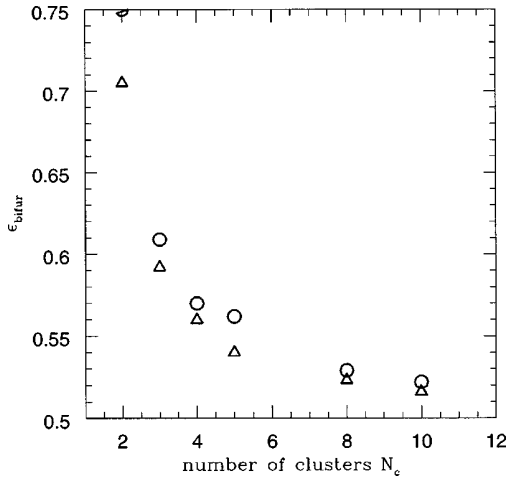

FIG. 3. The ϵ_{bifur} (i.e., the lower end of the stable range supporting the synchronized fixed point state) for the case of N_c equal subsets updating sequentially, with respect to N_c, for a one-dimensional chain of length 100 (O), and a 10×10 two dimensional lattice (△). Here, the coupling is through $g(x)=f(x)$.

$\epsilon>0.75$ we observe exact anti-ferro-type (frozen) ordering in space. Above $\epsilon\sim0.95$ one observes noisy anti-ferro-type domains. [See Fig. 1(d), for the bifurcation diagram of the limiting case of $N_c=N$, namely the completely asynchronous updating].[11]

Again, we examine the transition from spatial randomness to spatial cycles, at $\epsilon>0.75$, by following the system with two unequal subsets, with p_1 going from 0 to 0.5. Here one obtains a transition from a random profile (at $p_1=0$) to two cycles in space as $p_1\to0.5$.

Figure 5 tracks this system at a constant ϵ with varying p_1 and finds the details of the clearly defined transition from disorder to antiferro order. Note how the random spatial pattern becomes increasingly checkered, finally yielding exact spatial two cycles in the limit of $p_1\to0.5$.

C. Discussion

So, the phenomenology very clearly shows that, unlike simultaneous updates, asynchronous updating leads to large parameter regimes of regular homogeneous behavior, with all lattice sites synchronized exactly at $x(i,j)=x^*=0.75$.

Further, the synchronized spatiotemporal fixed point gains stability for very small asynchronicities in updating rules. The parameter range where one finds synchronization gets wider very rapidly with increasing N_c (i.e., with increasing degree of asynchronicity) and then saturates.

The robust spatiotemporal fixed point supported by asynchronous updating may have significant real-life ramifications. It has immediate relevance to the important and open problem of controlling/synchronizing extended chaotic systems—which is essentially a search for algorithms which yield stable synchronized states.[12,13] So, the emergence of stable spatiotemporal fixed points from asynchronous evolution of the CML may have practical utility in the control oflarge interactive systems. This key observation that *asyn-*

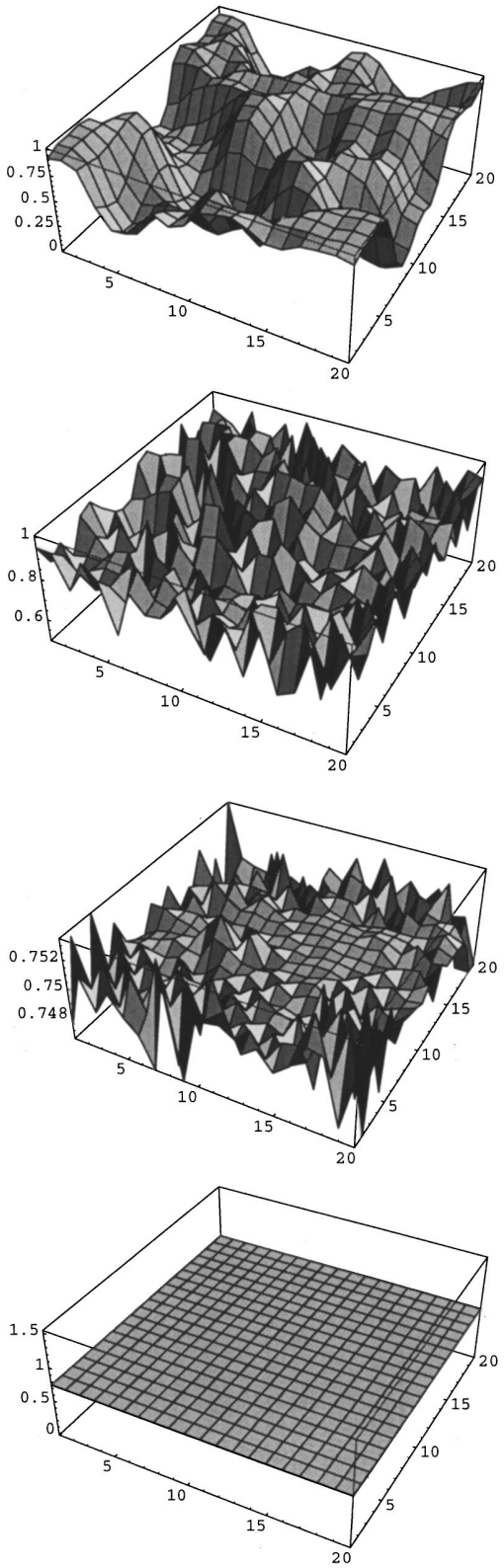

FIG. 4. Surface plots showing the transition from random spatial profile to synchronization in a 20×20 lattice, for the case of two unequal subsets updating sequentially, with respect to p_1, where p_1 is the fraction of sites in the first subset. Here, $\epsilon=0.74$, $g(x)=f(x)$ and (top to bottom) $p_1=$ 0, 0.25, 0.40, and 0.45.

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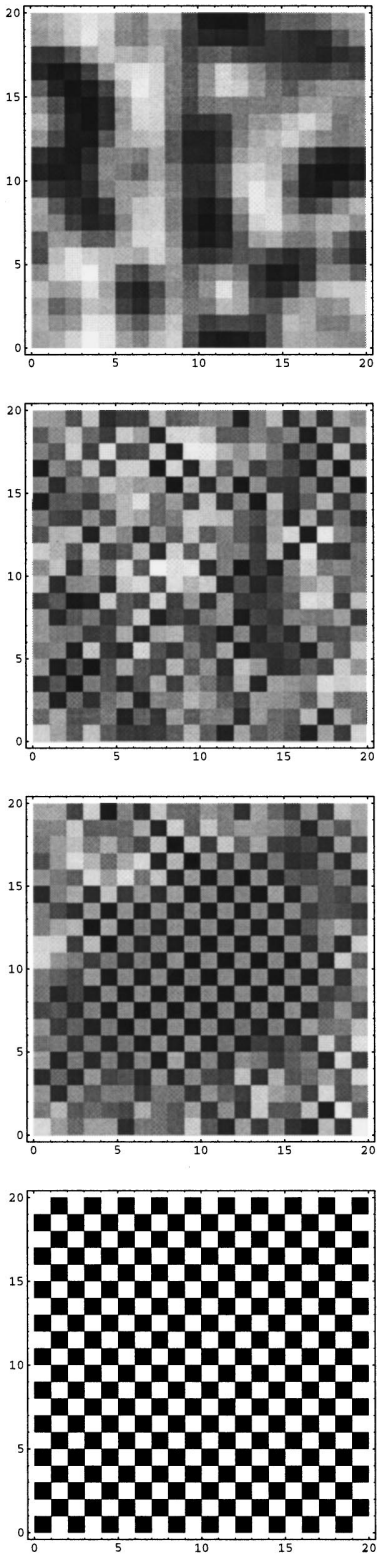

FIG. 5. Density plots showing the transition from random spatial profile to spatial two cycles in a 20×20 lattice, for the case of two unequal subsets updating sequentially, with respect to p_1, where p_1 is the fraction of sites in the first subset. Here, $\epsilon=0.8$, $g(x)=f(x)$ and (top to bottom) $p_1=$ 0, 0.19, 0.24, and 0.25.

chronicity yields synchronization may then help to suggest methods of controlling spatially extended interactive systems.

We shall now attempt to do a mean-field analysis of this system, and try to see if we can roughly account for the much enhanced stability of the homogeneous phase under asynchronous evolution.

IV. STABILITY ANALYSIS

In order to discuss synchronization in this model, the first issue we need to solve is the nature of the synchronized state. With the exception of the $N_c=1$ case (parallel updating), the updating operation ''travels'' randomly through the lattice, and therefore the only possibility for synchronization that we have is that of a fixed point, which in this case is $x^*=3/4$.

To calculate the stability of this state we need to understand first how the updating works, in terms of the number of subsets. Since the subset assignment of a given site is randomly decided, we will construct an *average probabilistic evolution rule* for the sites, which becomes a sort of *mean-field version of the dynamics*. Some effects due to fluctuations are lost, but as a first approximation we have found this approach qualitatively right, and quantitatively close to the numerical results.

Now, here we will focus again on the illustrative and interesting situation of two unequal subsets updating sequentially.

We take into account the following:

(a) A site (i,j) has probability p_1 of being in the first subset, where all neighbors contribute $g_n(x)$ to the coupling term during updating. (Further, note that there is only one combination of sites possible, as all the neighbors contribute equivalently.)

(b) A site (i,j) has probability p_2 of being in the second subset. Now the updating can have five possible situations, as p_1 fraction of the lattice has already been updated. These situations are:

(1) All neighbors contribute $g_n(x)$ to the coupling term during updating. (Here again there is only one combination of sites possible: i.e., the one involving all the neighbors.)

(2) Three neighbors contribute $g_n(x)$ and one neighbor contributes $g_{n+1}(x)$ to the coupling term, during updating. [Here, there are four combinations of sites possible: with either site $(i+1,j)$ or $(i-1,j)$ or $(i,j+1)$ or $(i,j-1)$ contributing $g_{n+1}(x)$, and the rest of the sites contributing $g_n(x)$].

(3) Two neighbors contribute $g_n(x)$ and two neighbors contribute $g_{n+1}(x)$ to the coupling term, during updating. [Here, there are six combinations of sites possible: any two sites contributing $g_{n+1}(x)$ combine with the remaining two sites contributing $g_n(x)$, and one can have six possible combinations of two sites from four sites.]

(4) Three neighbors contribute $g_{n+1}(x)$ and one neighbor contributes $g_n(x)$ to the coupling term, during updating. [Here, there are four combinations of sites possible: with

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either site $(i+1,j)$ or $(i-1,j)$ or $(i,j+1)$ or $(i,j-1)$ contributing $g_n(x)$, and the rest of the sites contributing $g_{n+1}(x)$].

(5) All neighbors contribute $g_{n+1}(x)$ to the coupling term during updating. (Here again there is only one combination of sites possible: i.e., the one involving all the neighbors.)

Further, note that a neighbor has probability p_1 of having updated already and therefore contributing $g_{n+1}(x)$, and a probability p_2 of not having updated as yet and therefore contributing $g_n(x)$ to the coupling term.

Using the simplifying notations: $A(i,j)$ where all four nearest neighbors contribute $g(x_n)$, i.e.,

$$A(i,j) \equiv g(x_n(i-1,j)) + g(x_n(i+1,j)) + g(x_n(i,j-1)) + g(x_n(i,j+1)), \tag{2}$$

and $B(i,j)$ where all four nearest neighbors contribute $g(x_{n+1})$, i.e.,

$$B(i,j) \equiv g(x_{n+1}(i-1,j)) + g(x_{n+1}(i+1,j)) + g(x_{n+1}(i,j-1)) + g(x_{n+1}(i,j+1)), \tag{3}$$

we then have the following scenario: The averaged evolution equation of a site (i,j) is

$$x_{n+1}(i,j) = (1-\epsilon) f(x_n(j)) + \frac{\epsilon}{4}\{F(p_1,p_2)A(i,j) + G(p_1,p_2)B(i,j)\}, \tag{4}$$

with

$$F(p_1,p_2) = p_1 + p_2(p_2^4 + 3p_1p_2^3 + 3p_1^2p_2^2 + p_2p_1^3) = p_1 + p_2^2,$$

and

$$G(p_1,p_2) = p_2(p_1^4 + 3p_2p_1^3 + 3p_1^2p_2^2 + p_1p_2^3) = p_1p_2 = 1 - F(P_1,p_2).$$

Now, we will consider the specific case of $g(x)=x$. To calculate the stability of the coherent state, we perform the usual linearization, replacing $x_n(i,j) = x^* + h_n(i,j)$, and expanding to first order to find

$$h_{n+1}(i,j) = (1-\epsilon) f'(x^*) h_n(i,j) + \frac{\epsilon}{4} \times \{F[h_n(i-1,j) + h_n(i+1,j) + h_n(i,j-1) + h_n(i,j+1)] + G[h_{n+1}(i-1,j) + h_{n+1} \times (i+1,j) + h_{n+1}(i,j-1) + h_{n+1}(i,j+1)]\}.$$

For stability considerations one can diagonalize the above expression using a Fourier transform $[h_n(\mathbf{k}) = \Sigma_{\mathbf{q}} \phi_n(\mathbf{q}) \exp(i\mathbf{k}.\mathbf{q})$, where $\mathbf{q}$ is the wave number and $\mathbf{k}$ contains the site indices], which finally leads us to the following growth equation:

$$\phi_{n+1} = \frac{2f'(x^*)(1-\epsilon) + \epsilon F(p_1,p_2)(\cos q_1 + \cos q_2)}{2 - \epsilon G(p_1,p_2)(\cos q_1 + \cos q_2)} \phi_n, \tag{5}$$

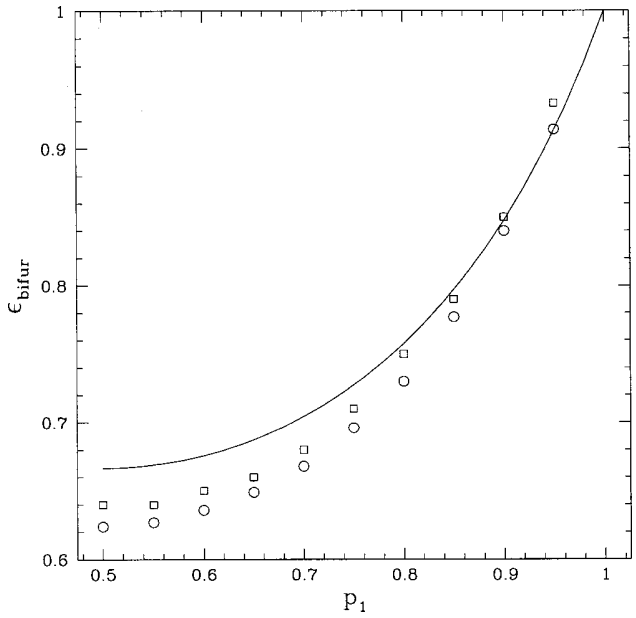

FIG. 6. The ϵ_{bifur} (i.e., the lower end of the stable range supporting the synchronized fixed point state) for the case of two unequal subsets updating sequentially, with respect to p_1, where p_1 is the fraction of sites in the first subset. The connected line is the result of the mean-field analysis, and the points are obtained from numerical simulation for two different lattice sizes: a 10×10 lattice (○), and a 20×20 lattice (□). Here, the coupling is through $g(x)=x$.

with q_1, q_2 going from 0 to π. Replacing $f'(x^*) = -2$, one finds that the growth coefficient that appears in this formula is smaller than 1 in magnitude if

$$\frac{1}{2 - F(p_1,p_2) + G(p_1,p_2)} = \frac{1}{1 + 2p_1p_2} < \epsilon < 1. \tag{6}$$

This gives the largest stable range for $p_1 = p_2 = 0.5$ (i.e., when the subsets are equipartitioned) for which the lower end of the stable window ϵ_{bifur} is minimum, with $\epsilon_{bifur} = 2/3$. (This is easily seen by noting that the minimum of the expression $2p_1^2 - 2p_1 + 1$ is obtained for $p_1 = 0.5$).

The range of stability narrows as p_1 and p_2 become increasingly unequal. The limit of vanishing synchronous fixed points (i.e., $\epsilon_{bifur} \to 1$) is reached when $p_1 \to 1$ and $p_2 \to 0$ (see Fig. 6). Also note that the stability range is independent of whether the larger subset or the smaller one updates first (hence, the bifurcation is symmetric around $p_1 = p_2 = 0.5$).

The same procedure can be employed for the other coupling form $g(x) = f(x)$ as well. To calculate the stability of the coherent state, we again perform the usual linearization. Replacing $x_n(i,j) = x^* + h_n(i,j)$, and expanding to first order, we obtain

$$h_{n+1}(i,j) = (1-\epsilon) f'(x^*) h_n(i,j) + \frac{\epsilon f'(x^*)}{4} \times \{F[h_n(i-1,j) + h_n(i+1,j) + h_n(i,j-1) + h_n(i,j+1)] + G[h_{n+1}(i-1,j) + h_{n+1} \times (i+1,j) + h_{n+1}(i,j-1) + h_{n+1}(i,j+1)]\}.$$

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This expression is identical in form to that obtained for $g(x)=x$, with the only difference being the rescaling of F and G by a multiplicative factor of $f'(x^*)$. So, doing exactly the same steps as before, we obtain $\epsilon_{\text{bifur}} \to 1$, i.e., a vanishingly small stable range. This is consistent with the fact that for coupling form $g(x)=f(x)$, for $N_c=2$, no synchronization is observed in one-dimension. But, in two-dimensions a small stable range in ϵ is opened up by asynchronicity, which this current analysis does not capture. One needs to take care of higher order effects and go beyond first-order treatments and mean field approximations in order to obtain this subtle effect of dimensionality.

Note that the above machinery goes through for higher N_c as well, but with greatly enhanced algebraic manipulations. To give a flavor of the extension to higher N_c, we indicate here the approach for the case next in complexity, namely that of three equal subsets: $N_c=3$. For that we will have to use the following:

(a) A site (i,j) will have probability 1/3 of being in the first subset, where all neighbors contribute $g_n(x)$ to the coupling term during updating.
(b) A site (i,j) has probability 1/3 of being in the second subset, where the updating can occur under the five possible situations enumerated earlier in detail for $N_c=2$. Now, a neighbor has probability 1/3 of having updated already and therefore contributing $g_{n+1}(x)$, and a probability 2/3 of not having updated as yet and therefore contributing $g_n(x)$ to the coupling term.
(c) A site (i,j) has probability 1/3 of being in the third subset, where the updating can again have five possible situations, with the difference from the second subset being that two-thirds of the lattice has already been updated now, and so a neighbor has probability 2/3 of having updated already and therefore contributing $g_{n+1}(x)$, and a probability 1/3 of not having updated as yet and therefore contributing $g_n(x)$ to the coupling term.

So, again using the simplifying notations: $A(i,j)$ where all the four nearest neighbors contribute $g(x_n)$, and $B(i,j)$ where all the four nearest neighbors contribute $g(x_{n+1})$, we then have the following scenario: The averaged evolution equation of a site (i,j) again is

$$x_{n+1}(i,j)=(1-\epsilon)f(x_n(j))+\frac{\epsilon}{4}\{FA(i,j)+GB(i,j)\}, \qquad (7)$$

now with

$$F=p_1(1+p_1^4+6p_1^2p_2^2+4p_1p_2(p_1^2+p_2^2)+p_2^4),$$

where $p_1=1/3$ and $p_2=2/3$, and $G=1-F$.

Note that for all N_c the averaged evolution equation has the same form, namely that given in Eq. (7, *but with different algebraic expressions for the factors F and G*. In order to obtain the expression for coefficients F and G, we have to consider the contributions from N_c subsets separately. These contributions will be determined by the differing probabilities of having updated neighbors for different subsets. From this probabilistic evolution equation one can try to obtain the approximate stable ranges by linear stability analysis around the coherent state. The analysis above gives an idea of the general strategy employed to handle cases of greater asynchronicity, i.e., higher N_c.

V. CONCLUSION

We have investigated the spatiotemporal dynamics of coupled map lattices evolving under updating rules incorporating varying degrees of asynchronicity. Interestingly, we observe that parallel updates never allow synchronization among the sites,[14] while asynchroncity has the effect of opening up windows in parameter space where the synchronized dynamics gains stability. As asynchronicity is increased, the parameter range supporting synchronization, rapidly gets wider first and then saturates. Further, we have tried to analytically investigate the stability of the spatiotemporal fixed point under asynchronous updates, via a mean-field approach. Our treatment was successful for the case of $g(x)=x$, but did not agree well with the numerics for the case of $g(x)=f(x)$ for two dimensions.

In summary then, we have shown that asynchronicity has a pronounced effect on the stability of the spatiotemporal fixed point phase. In fact strictly parallel updating is singular, in that it supports *no* spatiotemporal fixed point phase, whereas any degree of asynchronicity (however small) opens a window of coupling parameter space in which the synchronized fixed point is stable.

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[5] In Ref. 4 the significance of asynchronous updating was argued on the basis of the connection between CML and physical spatially extended systems (which is not straightforward at all). Heuristically, the closest physical parallel is probably a collection of coupled elements, each evolving chaotically. In order to derive a discrete local map for the sites, one may apply a Poincare section. In the absence of an external periodic forcing, the time interval between consecutive crossings of the section will generically not be a constant and in fact vary from unit to unit in space. The state of the extended system at fixed time intervals can then be roughly mimicked by using asynchronous updates, in which the elements experience slightly different update times.
[6] There are cases where synchronous updates can lead to complicated spatial structures in cellular automata [Ref: M. Nowak and R. M. May, Nature (London) **359**, 826 (1992)] which disappear under asynchronous updating [B. A. Huberman and N. Glance, Proc. Natl. Acad. Sci. USA **90**, 7716 (1993)] and are therefore considered unphysical in the absence of an external clock.
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[11] Note, for coupling form $g(x)=f(x)$, in the range $\epsilon\sim 0.8$–0.85, we observed very long quasistationary transient configurations, that have antiferro spatial order broken by thin lines of kinks or defects, for certain initial conditions. We are further investigating the nature of this transient phenomenon.
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[14] Note that very small lattices (of 2–4 maps) can synchronize spatially under simultaneous updates (though usually not to a temporal fixed point), in certain cases, as reported in some references in Ref. 12. Of course, here we have been interested in synchronization of *very large systems* and typically we deal with very large numbers of maps ($N>100$). Such large system sizes also validate our mean-field approach in the analytical treatment of the problem.

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