THE TEMPERLEY-LIEB ALGEBRA

A Thesis submitted in partial fulfilment of the requirements for the award of the degree of

Master of Science

by

S. Sundar

The Institute of Mathematical Sciences
Chennai 600113

HOMI BHABHA NATIONAL INSTITUTE

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BONAFIDE CERTIFICATE

Certified that this dissertation titled **The Temperley-Lieb Algebra** is a bonafide record of work of **Mr. S. Sundar** who carried out the project under my supervision.

Dr. V. S. Sunder
Mathematics
Institute of Mathematical Sciences
Chennai

Preface

The main aim of this thesis is to determine the maximal $C^\star$ quotient of the Temperley-Lieb algebra $T_n(\tau)$.

In chapter 1, we define $T_n(\tau)$ for every $n \in \mathrm{N}$ and for every non zero complex number τ. The algebra $T_n(\tau)$ is defined as the universal C algebra generated by $1, e_1, e_2, \cdots e_{n-1}$ satisfying the following relation:

$$\begin{aligned} e_i^2 &= e_i \quad \text{ for } i \in \{1, 2, \cdots, n-1\} \\ e_i e_j &= e_j e_i \ \text{ if } \ |i-j| \geqq 2 \\ e_i e_j e_i &= \tau e_i \ \text{ if } \ |i-j| = 1 \end{aligned}$$

We prove that $T_n(\tau)$ is a $\star$ algebra by identifying $T_n(\tau)$ with the diagram algebra $D_n(\beta)$ when $\tau = \frac{1}{\beta^2}$.

In chapter 2, Jones- Wenzl idempotents are defined. Wenzl's theorem, which states that if $TL(\tau) = \cup_{k=1}^{\infty} T_k(\tau)$ admits a non-trivial $C^\star$ representation then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{n+1}) : n \geq 2\}$, is proved.

In chapter 3, we obtain $C^\star$ representations of $TL(\tau)$ when the parameter $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{n+1}) : n \geq 2\}$. Jones' basic construction for inclusion $N \subset M$ of finite dimensional $C^\star$ algebras together with a faithful trace is explained. When the trace is Markov of modulus τ, we can repeat the Jones' basic construction and obtain a tower of finite dimensional $\mathrm{C}^\star$ algebras called the Jones tower and a sequence of projections e_n^J called the Jones projections and consequently a sequence of quotients $J_n(\tau)$ for $T_n(\tau)$.

In chapter 4, we obtain the maximal $C^\star$ quotient of $T_k(\tau)$. If $\tau \leq \frac{1}{4}$, the quotient map $\phi : T_k(\tau) \to J_k(\tau)$ is $\star$ algebra isomorphism. When the parameter $\tau = \frac{1}{4} sec^2(\frac{\pi}{n+1})$, the map $\phi : T_k(\tau) \to J_k(\tau)$ is an isomorphism for $1 \leq k \leq n-1$. For $k \geq n$, Let $\tilde{1} : T_k(\tau) \to \mathrm{C}$ be the trivial map for which $\tilde{1}(e_i) = 0$. Then we prove that $(J_k(\tau) \oplus \mathrm{C}, \phi \oplus \tilde{1})$ is the maximal C^* quotient of $T_k(\tau)$ when $k \geq n$. Much of the material in this thesis can be found in [Jon].

Acknowledements

I thank V.S.Sunder for suggesting me this topic. The discussions that I had with him helped me in understanding the topic better. I also thank him for making me think and write mathematics better.

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SUNDAR

Contents

Chapter 1

The Temperley-Lieb Algebra

1.1 The Temperley-Lieb algebra $T_n(\tau)$

We consider only C algebras. Let τ be a nonzero complex number.

Definition 1. *For $n \geq 2$, let $T_n(\tau)$ be the* C *algebra generated by $1, e_1, e_2 \cdots e_{n-1}$ subject to the following relations :*

$$\begin{aligned} e_i^2 &= e_i \quad \textit{for } i \in \{1, 2, \cdots, n-1\} \\ e_i e_j &= e_j e_i \quad \textit{if} \quad |i-j| \geq 2 \\ e_i e_j e_i &= \tau e_i \quad \textit{if} \quad |i-j| = 1 \end{aligned}$$

$T_n(\tau)$ has the following universal property. Let A be a unital C algebra. Let $f_1, f_2, \cdots, f_{n-1} \in A$ be such that

$$\begin{aligned} f_i^2 &= f_i \quad \text{for } i \in \{1, 2, \cdots, n-1\} \\ f_i f_j &= f_j f_i \quad \text{if} \quad |i-j| \geq 2 \\ f_i f_j f_i &= \tau f_i \quad \text{if} \quad |i-j| = 1 \end{aligned}$$

Then there exists a unique algebra homomorphism $\phi : T_n(\tau) \to A$ such that $\phi(e_i) = f_i$ and $\phi(1) = 1_A$ where 1_A denotes the multiplicative identity of A.

We now proceed to prove that $T_n(\tau)$ is finite dimensional. By a word on $1, e_1, e_2, \cdots, e_{n-1}$ we mean a product $e_{i_1} e_{i_2} \cdots e_{i_p}$. By convention empty product denotes 1. Note that words on $1, e_1, e_2, \cdots, e_{n-1}$ span $T_n(\tau)$.

Lemma 1. *Let w be a word on $1, e_1, e_2 \cdots, e_{n-1}$. Then*

$$w = \tau^k (e_{i_1} e_{i_1 - 1} \cdots e_{j_1})(e_{i_2} e_{i_2 - 1} \cdots e_{j_2}) \cdots (e_{i_p} e_{i_p - 1} \cdots e_{j_p})$$

where $k \in \mathrm{N} \cup \{0\}$ *and*

$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1$$
$$1 \leq j_1 < j_2 < \cdots j_p \leq n-1$$
$$i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p$$

*Proof.*The proof can be found in [Jon]. We prove this by induction on n. Clearly the result is true for $n = 2$. Now assume that any word in $1, e_1, e_2, \cdots, e_{n-1}$ is of the required form. Let w be a word in $1, e_1, e_2, \cdots, e_n$. If w does not contain e_n then we are done. So suppose that w contains e_n.

Assertion. $w = \tau^k w_1 e_n w_2$ where w_1, w_2 are words in $1, e_1, e_2, \cdots, e_{n-1}$.

w has the form $v_1 e_n v e_n v_2$ where v_1, v_2 are words in $1, e_1, e_2, \cdots, e_n$ and v is a word in $1, e_1, e_2, \cdots, e_{n-1}$.
If v does not contain e_{n-1} then e_n commutes with v and hence $w = v_1 v e_n v_2$.
If v contains e_{n-1} then by induction hypothesis $v = \tau^r u_1 e_{n-1} u_2$ where u_1, u_2 are words in $1, e_1, e_2, \cdots, e_{n-2}$. Now

$$w = \tau^r v_1 u_1 e_n e_{n-1} e_n u_2 v_2$$
$$w = \tau^{r+1} v_1 u_1 e_n u_2 v_2$$

In any case w is τ^l multiple of a word which has one e_n less. Repeating this process proves the assertion.

Hence $w = \tau^k w_1 e_n w_2$ where w_1, w_2 are words in $1, e_1, e_2, \cdots, e_{n-1}$. By induction hypothesis

$$w_2 = \tau^l v_2 (e_{n-1} e_{n-2} \cdots, e_{j_p})$$

where v_2 is a word in $1, e_1, e_2, \cdots, e_{n-2}$. (The product $(e_{n-1} e_{n-2} \cdots e_{j_p})$ could be empty). Hence

$$w = \tau^s w_1 v_2 (e_n e_{n-1} \cdots e_{j_p})$$

where $w_1 v_2$ is a word in $1, e_1, e_2, \cdots, e_{n-1}$

Hence by induction hypothesis,

$$w = \tau^k (e_{i_1} e_{i_1 - 1} \cdots e_{j_1})(e_{i_2} e_{i_2 - 1} \cdots e_{j_2}) \cdots (e_{i_p} e_{i_p - 1} \cdots e_{j_p})$$

where $k \in \mathrm{N} \cup \{0\}$ and

$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1$$
$$i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p$$

Hence we have written w in the form needed with $i's$ increasing. Now consider such an expression which has the least length. Then we claim that $j's$ are also increasing. Let

$$w = \tau^k(e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p})$$

be such an expression. Suppose $j_1 \geq j_2$. Then

$$\begin{aligned}
w &= \tau^k(e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p}) \\
w &= \tau^k(e_{i_1}e_{i_1-1}\cdots e_{j_1+1})(e_{i_2}\cdots e_{j_1}e_{j_1+1}e_{j_1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p}) \\
w &= \tau^{k+1}(e_{i_1}e_{i_1-1}\cdots e_{j_2})(e_{i_2}e_{i_2-1}\cdots e_{j_1+2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p})
\end{aligned}$$

which has length decreased by one which is a contradiction. Hence $j_1 < j_2$. Similarly $j_r < j_{r+1}$. This completes the proof. □

Now we consider the following combinatorial problem. Consider $\mathrm{Z}^2 \subset \mathrm{R}^2$. Consider paths on Z^2. The only allowed moves are either up or right i.e. from (a, b) one can go to either $(a + 1, b)$ or $(a, b + 1)$.

Proposition 1. *The number of paths from $(0, 0)$ to (n, n) where $n \in \mathrm{N}$ which lie in the region $y \leq x$ is $\frac{1}{n+1}\binom{2n}{n}$. Let $p_n = \frac{1}{n+1}\binom{2n}{n}$. Then p_n satisfy the following recurrence*

$$\begin{aligned}
p_1 &= 1 \\
p_n &= \sum_{i=1}^{n} p_{i-1}p_{n-i}, \textit{for } n \geq 2.
\end{aligned}$$

For a proof,we refer to [GHJ]. □

The relevance of proposition 1 in our context is as follows:
Given $(i_1, i_2, \cdots, i_p)$ and $(j_1, j_2, \cdots, j_p)$ such that

$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1,\ 1 \leq j_1 < j_2 < \cdots j_p \leq n-1,\ i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p$$

one can associate the path from $(0, 0)$ to (n, n) given by

$$(0, 0) \to (i_1, 0) \to (i_1, j_1) \to (i_2, j_1) \to \cdots (i_p, j_p) \to (n, j_p) \to (n, n)$$

This is clearly a bijection from the set of paths from $(0, 0)$ to (n, n) to the set of ordered pairs $((i_1, i_2, \cdots, i_p), (j_1, j_2, \cdots, j_p))$ which satisfies the following condition.

$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1,\ 1 \leq j_1 < j_2 < \cdots j_p \leq n-1,\ i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p$$

Hence we get an onto map from the set of paths from $(0,0)$ to (n,n) to

$$\{(e_{i_1}e_{i_1-1}\cdots e_{j_1})(e_{i_2}e_{i_2-1}\cdots e_{j_2})\cdots(e_{i_p}e_{i_p-1}\cdots e_{j_p}) :$$
$$1 \leq i_1 < i_2 < \cdots i_p \leq n-1; 1 \leq j_1 < j_2 < \cdots j_p \leq n-1; i_1 \geq j_1, i_2 \geq j_2, \cdots, i_p \geq j_p\}$$

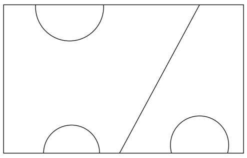
which spans $T_n(\tau)$ by Lemma 1. Hence we have proved the following result.

Proposition 2. *The algebra $T_n(\tau)$ is finite dimensional and it's dimension is atmost $\frac{1}{n+1}\binom{2n}{n}$.*

1.2 Diagram algebra $D_n(\beta)$

Fix a non-zero complex number β. Let m, n be nonegative integers such that $m-n$ is even. By an (m,n) **Kauffman** diagram we mean a rectangle in the plane with m points on the top and n points on the bottom and $\frac{n+m}{2}$ curves which connect pairs of points such that the curves do not intersect.

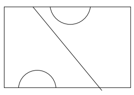
A $(3,5)$ diagram is shown below

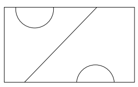


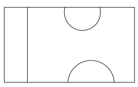
Let a be an (m,n) diagram and b be an (n,p) diagram. Let $b \odot a$ denote the (m,p) diagram obtained by placing a on the top and b on the bottom and removing the loops. Define

$$ba = \beta^r b \odot a$$

where r denotes the number of loops removed.
For example,

$a =$ 

$b =$ 

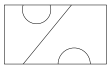
$ba = \beta$ 

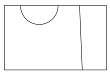
Let $Hom(m,n)$ denote the C vector space with (m,n) Kauffman diagrams as basis. The 'multiplication' that we have defined on diagrams extends to a bilinear map

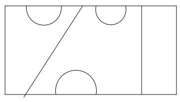
$$Hom(m,n) \times Hom(n,p) \to Hom(m,p)$$

which is associative.

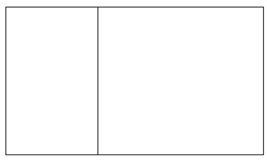
For a an (m,n) diagram and b a (p,q) diagram, $a \otimes b$ denote the $(m+p, n+q)$ diagram obtained by horizontal juxtaposition.
For example,

$a =$ 

$b =$ 

$a \otimes b =$ 

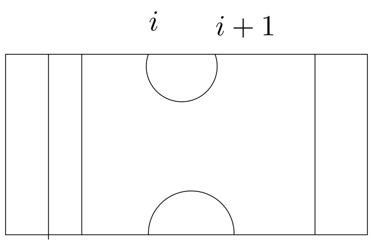
Let $1 \in Hom(1,1)$ denote the $(1,1)$ diagram shown below:

$1 =$ 

Let $1_n = 1 \otimes 1 \otimes 1 \cdots \otimes 1$, the (n,n) diagram with all strands coming vertically down.

Define $D_n(\beta) = Hom(n,n)$. Then $D_n(\beta)$ is a unital C algebra with 1_n as the multiplicative identity. The map $a \to a \otimes 1$ is an embedding of $D_n(\beta)$ into $D_{n+1}(\beta)$. With this embedding in mind, we write $D_n(\beta) \subset D_{n+1}(\beta)$.

Let E_i denote the following diagram in $D_n(\beta)$



Then we have the following relations:

$$E_i^2 = \beta E_i \quad \text{for } i \in 1, 2, \cdots, n-1$$
$$E_i E_j = E_j E_i \quad if\ |i-j| \geqq 2$$
$$E_i E_j E_i = E_i \quad if\ |i-j| = 1$$

Let $e_i^D = \frac{1}{\beta} E_i$.
Then we have the following relations:

$$\begin{aligned} (e_i^D)^2 &= (e_i^D) \quad \text{for } i \in 1, 2, \cdots, n-1 \\ e_i^D e_j^D &= e_j^D e_i^D \quad if \;\; |i-j| \geqq 2 \\ e_i^D e_j^D e_i^D &= \frac{1}{\beta^2} e_i^D \quad if \;\; |i-j| = 1 \end{aligned}$$

For $0 \neq \tau \in$ C, a nonzero complex number, let β be such that $\beta^2 = \frac{1}{\tau}$. Then by the universal property of $T_n(\tau)$, there exists a unique unital homomorphism $\phi : T_n(\tau) \to D_n(\beta)$ such that $\phi(e_i) = e_i^D$. We now proceed to prove that ϕ is an isomorphism.

Lemma 2. *The dimension of $D_n(\beta)$ is $\frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let p_n denote the number of (n, n) Kauffman diagrams. Think of an (n, n) Kauffman diagram as a disk with $2n$ points on the boundary with n curves connecting pairs of points without any intersection. Then we have the following recurrence relation

$$\begin{aligned} p_0 &= p_1 = 1 \\ p_n &= \sum_{i=1}^{n} p_{i-1} p_{n-i}, \text{for } n \geq 2. \end{aligned}$$

Hence, by proposition 1, $p_n = \frac{1}{n+1}\binom{2n}{n}$. □

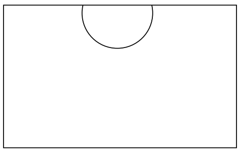
Lemma 3. $\{1, E_i : i = 1, 2, \cdots, n-1\}$ *generate the algebra $D_n(\beta)$*

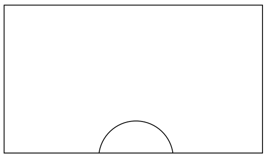
Proof. We prove this result by induction on n. If $n = 2$ the result is clear. Let a be an (n, n) Kauffman diagram. If that a has a strand that comes straight down then $a = b \otimes 1 \otimes c$ with $b \in D_r(\beta)$ and $c \in D_s(\beta)$ with $r, s < n$. Hence by induction hypothesis a can be written as a scalar multiple of E_i' s and we are done. Now we consider two cases.

Case 1. a has a through string i.e a string which joins a top point with a bottom point. Let us call a strand that comes vertically down a vertical string. Pick the rightmost through string. Let $\nu(a)$ be the number of vertices to the right of the rightmost through string of a(inclusive of the vertices that the rightmost through string joins).
We prove that a can be written as a scalar multiple of a product of $E_i's$ by induction on $\nu(a)$. If $\nu(a) = 2$ then the rightmost through string is vertical and we are through. Assume that it slants from right to left. Then

$a = b \otimes 1 \otimes c \otimes d$ with $b \in Hom(l,k)$, $c \in Hom(0,2r)$, $d \in Hom(t,s)$ for some non negative integers l,k,r,s,t with $r > 0$.

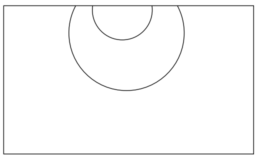
Let $\cup \in Hom(2,0)$ and $\cap \in Hom(0,2)$ be the following diagrams.





Let $\cup^r = \cup \otimes \cup \otimes \cdots \otimes \cup$ (r *times*). Similarly $\cap^r$ is defined. Note that $1 \otimes c = (1 \otimes \cup^r \otimes c)(\cap^r \otimes 1)$. Let $\bar{b} = 1_k \otimes 1 \otimes \cup^r \otimes c \otimes 1_s$ and $\bar{c} = b \otimes \cap^r \otimes 1 \otimes d$. Then $a = \bar{b}\bar{c}$ where $\bar{b}$ has a vertical string and $\nu(\bar{c}) < \nu(a)$. Hence by induction a can be written as a scalar multiple of a product of $E_i' \, s$. The proof is similar when the rightmost through string slants from left to right.

Case 2. a has no through strings. By a concentric loop we mean a Kauffman diagram which is either $\cup^r \circ (1 \otimes \alpha \otimes \cap^{r-1} \otimes 1)$ where α is a $(2r-2,0)$ Kauffman diagram ($r \geq 2$) or $(1 \otimes \gamma \otimes \cup^{2s-2} \otimes 1) \circ \cap^s$ where γ is a $(0,2s-2)$ Kauffman diagram ($s \geq 2$). An example of a concetric loop is given below:



If a does not have a concentric loop, then $a = E_1 E_3 \cdots$. Hence assume that a has concentric loops. Then $a = b \otimes c \otimes d$ where c is a concetric loop in $Hom(2k+2,0)$ (assuming c is on top) and where $b \in Hom(r,s)$ and $d \in Hom(p,q)$ for some nonegative integers p,q,r,s,k with $k > 0$. Then $c = \cup^{k+1}(1 \otimes a \otimes \cap^k) \otimes 1)$. Let $\bar{c} = 1_r \otimes 1 \otimes a \otimes \cap^k \otimes 1 \otimes 1_p$. Let $\bar{b} = b \otimes \cup^{k+1} \otimes d$. Then $a = \bar{b}\bar{c}$ where both $\bar{b}$, $\bar{c}$ has one concentric loop less than that of a. Therefore, by induction on the number of concetric loops that a has, it follows that a can be written as a product of diagrams which have no concentric loop. Hence a is a product of $E_i's$. This completes the proof. $\square$

Theorem 1. *Let β be a nonzero complex number. Let $\tau = \frac{1}{\beta^2}$. Then the unique unital algebra homomorphism $\phi : T_n(\tau) \to D_n(\beta)$ such that $\phi(e_i) = e_i^D$ is an isomorphism.*

Proof. By Lemma 3, ϕ is onto. By rank-nullity theorem,

$$rank(\phi) + nullity(\phi) = dim\ T_n(\tau) \leq \frac{1}{n+1}\binom{2n}{n}$$
$$\frac{1}{n+1}\binom{2n}{n} + nullity(\phi) \leq \frac{1}{n+1}\binom{2n}{n}$$

Hence nullity(ϕ) = 0. Thus ϕ is one-one. Therefore ϕ is an isomorphism. □

From now on we will identify $T_n(\tau)$ with $D_n(\beta)$ when $\tau = \frac{1}{\beta^2}$ and e_i with e_i^D. Note that the natural map $i : T_n(\tau) \to T_{n+1}(\tau)$ is injective since $\phi(ia) = \phi(a) \otimes 1$ for $a \in T_n(\tau)$.

1.3 Trace and Conditional expectation on $D_n(\beta)$

Definition 2. *Let $N \subset M$ be unital* C *algebras such that $1_N = 1_M$. A linear map $E : M \to N$ is said to be a conditional expectation if*

1. *$E(nm) = nE(m)$ and $E(mn) = E(m)n$ $\forall n \in N, m \in M$*
2. *$E(n) = n$ $\forall n \in N$*

Now we describe a conditional expectation $\epsilon_n : D_{n+1}(\beta) \to D_n(\beta)$ as follows: Let $\tilde{\epsilon_n} : D_{n+1}(\beta) \to D_n(\beta)$ be defined by $\tilde{\epsilon_n}(a) = (1_n \otimes \cup)(a \otimes 1)(1_n \otimes \cap)$. If a is an $(n+1, n+1)$ diagram, then $\tilde{\epsilon_n}(a)$ is obtained by just closing up the last strand. Hence if $a \in D_n(\beta)$ then $\tilde{\epsilon_n}(a) = \beta a$. Let $\epsilon_n(a) = \frac{1}{\beta}\tilde{\epsilon_n}(a)$ for $a \in D_n(\beta)$. Then ϵ_n is a conditional expectation.

Definition 3. *Let M be a unital* C *algebra. Let $\rho : M \to$* C *be linear. Then ρ is said to be a trace if $\rho(ab) = \rho(ba) \forall a, b \in M$. The functional ρ is said to be unital if $\rho(1) = 1$.*

Let $tr_n : D_n(\beta) \to$ C be defined by $tr_n(a) = (\epsilon_1\epsilon_2 \cdots \epsilon_{n-1})(a)$. Note that $tr_n(a) = tr_{n+1}(a)$ if $a \in D_n(\beta)$. Hence we can and will denote tr_n by tr. If a is a diagram, let $c(a)$ be the number of loops one gets when one closes all the strands. Then $tr(a) = \beta^{c(a)-n}$

$tr : D_n(\beta) \to$ C is a unital trace and satisfy the following properties:

1. $tr(x) = tr(\epsilon_n(x))$ $\forall$ $x \in D_{n+1}(\beta)$.
2. $e_n x e_n = \epsilon_{n-1}(x)e_n$ $\forall$ $x \in D_n(\beta)$.
3. $tr(e_i) = \tau$ where $\tau = \frac{1}{\beta^2}$.

1.4 $\star$ structure on $D_n(\beta)$

Definition 4. *Let M be a C algebra. A $\star$ structure on M is a function $\star : M \to M$ (We write $\star(a) = a^\star$) such that the following holds*

1. $(a+b)^\star = a^\star + b^\star \ \forall \ a, b \in M$
2. $(\alpha a)^\star = \bar{\alpha} a^\star \ \forall \ a \in M, \alpha \in \mathrm{C}$
3. $(ab)^\star = b^\star a^\star \ \forall \ a, b \in M$
4. $(a^\star)^\star = a \ \forall \ a \in M$

A $\star$ algebra is a C algebra together with a $\star$ structure.

Now we make $D_n(\beta)$ a $\star$ algebra. The $\star$ structure is defined on the level of diagrams (and then extends conjugate linearly) as follows:
For a diagram a, $a^\star$ denotes the diagram obtained by reflecting along the horizontal middle line. Then $E_i^\star = E_i$. If β is real, then $(e_i^D)^\star = e_i^D$. Thus for $\tau > 0$, $T_n(\tau)$ is a $\star$ algebra with e_i selfadjoint.

Chapter 2

C* representations of $TL(\tau)$

In this chapter we will prove Wenzl's result. It characterises the values of τ for which $TL(\tau)$ admits a nontrivial $C^\star$ represntation.

Definition 5. *Let M be a $\star$ algebra. By a $C^\star$ representation of M we mean an algebra homomorphism $\pi : M \to A$ where A is a $C^\star$ algebras such that $\pi(a^\star) = (\pi(a))^\star$.*

By a **non-trivial reprsentation** of $T_n(\tau)$ we mean a $C^\star$ representation π such that $\pi(e_i) \neq 0$ for some $i \in \{1, 2, \cdots, n-1\}$.

First we define **Jones-Wenzl idempotents** in $T_n(\tau)$. See [Wen].

Define a sequence of polynomials recursively by

$$P_0(\lambda) = 1 = P_1(\lambda)$$
$$P_k(\lambda) = P_{k-1}(\lambda) - \lambda P_{k-2}(\lambda), \text{for } k \geq 2$$

The basic properties of $P_k(\lambda)$ are summarised in the following proposition.

Proposition 3. *Let k be a non-negative integer and let $m = [\frac{k}{2}]$. Then*

1. *The polynomial P_k is of degree m. It's leading coefficient is $(-1)^m$ if $k = 2m$ and $(-1)^m(m+1)$ if $k = 2m+1$.*
2. *The polynomial P_k has m distinct roots given by $\{\frac{1}{4}\sec^2(\frac{\pi j}{k+1}) : j = 1, 2, \cdots, m\}$.*
3. *Assume $k \geq 1$. Let $\lambda \in \mathrm{R}$ be such that $\frac{1}{4}\sec^2(\frac{\pi}{k+2}) < \lambda < \frac{1}{4}\sec^2(\frac{\pi}{k+1})$. Then $P_i(\lambda) > 0$ for $i \in \{1, 2, \cdots, k\}$ and $P_{k+1}(\lambda) < 0$*

Proof. For a proof, we refer to [GHJ]. □

Let $TL(\tau) = \bigcup_n T_n(\tau)$. Then $TL(\tau)$ is a $\star$ algebra generated by $1, e_1, e_2, \ldots.$ When $\tau > 0$, e_i's are self adjoint.

Proposition 4. *Let τ be a nonzero complex number such that $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n$. Define f_k in $TL(\tau)$ recursively as follows.*

$$f_0 = 1 = f_1$$
$$f_{k+1} = f_k - \frac{P_{k-1}(\tau)}{P_k(\tau)} f_k e_k f_k, \ 1 \leq k \leq n.$$

Then,

1. $f_k \in T_k(\tau)$ *for* $1 \leq k \leq n+1$.
2. $1 - f_k$ *is in the algebra generated by* $\{e_1, e_2, \cdots, e_{k-1}\}$ *for* $2 \leq k \leq n+1$.
3. $(e_k f_k)^2 = \frac{P_k(\tau)}{P_{k-1}(\tau)} e_k f_k$, $(f_k e_k)^2 = \frac{P_k(\tau)}{P_{k-1}(\tau)} f_k e_k$ *for* $1 \leq k \leq n+1$.
4. f_k *is an idempotent for* $1 \leq k \leq n+1$.
5. $f_k e_i = 0$, $e_i f_k = 0$ *if* $i \leq k-1$ *where* $1 \leq k \leq n+1$
6. $tr(f_k) = P_k(\tau)$ *for* $1 \leq k \leq n+1$.

When $\tau > 0$, f_k *is selfadjoint.*

Proof. This is due to Wenzl and we include a proof here for completeness. The proof is by induction on k. $1, 2 \cdots, 6$ are clearly true for $k \leq 2$. Now assume that $1, 2 \cdots, 6$ are true for $1 \leq k \leq l$ where $l \geq 2$. We will show the result is true for $k = l+1$.

Since f_l is in the algebra generated by $1, e_1, e_2, \cdots, e_{l-1}$ by definition it follows that f_{l+1} is in the algebra generated by $1, e_1, e_2, \cdots, e_l$. Hence $f_{l+1} \in T_{l+1}(\tau)$. Since $1 - f_l$ is in the algebra genrated by $e_1, e_2, \cdots, e_{l-1}$, by definition, it follows that $1 - f_{l+1}$ is in the algebra generated by $e_1, e_2, \cdots, e_l$.

Now note that $f_{l+1} f_l = f_{l+1}$ and $f_l f_{l+1} = f_{l+1}$ since f_l is an idempotent. Since $f_l \in T_l(\tau)$, e_{l+1} commutes with f_l. Hence we have,

$$\begin{aligned} e_{l+1} f_{l+1} e_{l+1} &= e_{l+1} f_l - \frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_{l+1} e_l e_{l+1} f_l \\ &= \frac{P_{l+1}(\tau)}{P_l(\tau)} e_{l+1} f_l \end{aligned}$$

Hence $(e_{l+1} f_{l+1})^2 = \frac{P_{l+1}(\tau)}{P_l(\tau)} e_{l+1} f_{l+1}$.

The proof that $(f_{l+1}e_{l+1})^2 = \frac{P_{l+1}(\tau)}{P_l(\tau)} f_{l+1}e_{l+1}$ is similar. Now

$$\begin{aligned} f_{l+1}^2 &= f_l^2 - 2\frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l + \left(\tfrac{P_{l-1}(\tau)}{P_l(\tau)}\right)^2 f_l e_l f_l e_l f_l \\ &= f_l^2 - 2\frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l + \left(\tfrac{P_{l-1}(\tau)}{P_l(\tau)}\right)^2 \frac{P_l(\tau)}{P_{l-1}(\tau)} f_l e_l f_l \\ &= f_l - \frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l = f_{l+1} \end{aligned}$$

Hence f_{l+1} is an idempotent. Since $f_{l+1}e_i = f_{l+1}f_l e_i$, it follows that $f_{l+1}e_i = 0$ if $i \leq l-1$. Now $f_{l+1}e_l = f_l e_l - \frac{P_{l-1}(\tau)}{P_l(\tau)}(f_l e_l)^2$. But $(f_l e_l)^2 = \frac{P_l(\tau)}{P_{l-1}(\tau)} f_l e_l$. Hence $f_{l+1}e_l = 0$. Hence $f_{l+1}e_i = 0$ for $i \leq l$. Similarly $e_i f_{l+1} = 0$. Now

$$\begin{aligned} tr(f_{l+1}) &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(f_l e_l f_l) \\ &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(\epsilon_l(f_l e_l f_l)) \\ &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(f_l \epsilon_l(e_l) f_l) \\ &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(\tau f_l) \\ &= P_l(\tau) - \tau P_{l-1}(\tau) = P_{l+1}(\tau) \end{aligned}$$

If $\tau > 0$ then $P_k(\tau)$ is real. Hence by induction it follows that $f_k's$ are self-adjoint. □

The idempotents described in the previous proposition are called **Jones-Wenzl idempotents**.

Let τ be positive. The following result due to Wenzl restricts the values of τ for which $TL(\tau)$ has a nontrivial $C^\star$ representation. The proof can be found in [Wen]. We include the proof for completeness.

Theorem[Wenzl]. *Let τ be a positive real number. If $TL(\tau)$ has a non-trivial $C^\star$ representation, then $\tau \leq \frac{1}{4}$ or $\tau = \frac{1}{4}\sec^2(\frac{\pi}{n+1})$ for some $n \geq 2$.*

We begin the proof with the following lemma.

Lemma 4. *Let τ be such that $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4}sec^2(\frac{\pi}{n+1})$ for some $n \in \mathrm{N}$, with $n \geq 2$. Suppose $\pi : TL(\tau) \to B(H)$ be a $\star$ homomorphism, where H is a Hilbert space. Let e_i^T denote the idempotents in $TL(\tau)$. Then the Jones-Wenzl idempotents f_k^T's are defined for $k = 1, 2, \cdots n+2$. Suppose $f_k = \pi(f_k^T)$ for $k \leq n+2$. Then*

(1) $1 - f_k = e_1 \vee e_2 \vee \cdots e_{k-1}$ *for* $k \leq n+2$.

(2) $e_{n+1} f_{n+1} = 0$.

(3) e_{n+1} *is orthogonal to* f_n.

Proof. Note that $P_k(\tau) > 0$ for $k = 1, 2, \cdots n$ and $P_{n+1}(\tau) < 0$. Hence the Jones-Wenzl idempotents are defined for $k = 1, 2, \cdots n+2$.

By proposition 4, it follows that $f_k e_i = 0$ for $i \leq k-1$. Hence we have $e_1 \vee e_2 \vee \cdots \vee e_{k-1} \leq 1 - f_k$. Since $1 - f_k$ is in the algebra generated by $e_1, e_2, \cdots, e_{k-1}$, it follows that $1 - f_k \leq e_1 \vee e_2 \vee \cdots e_{k-1}$. This proves (1).

Observe that $e_{n+1} f_{n+1} e_{n+1} = \frac{P_{n+1}(\tau)}{P_n(\tau)} e_{n+1} f_n$. But $e_{n+1} f_{n+1} e_{n+1}$ is positive and $e_{n+1} f_n$ is a projection. Since $P_{n+1}(\tau) < 0$, it follows that $e_{n+1} f_n = 0$ and $(f_{n+1} e_{n+1})^\star f_{n+1} e_{n+1} = 0$. Hence $f_{n+1} e_{n+1} = 0$ and e_{n+1} is orthogonal to f_n. By taking adjoints, we get $e_{n+1} f_{n+1} = 0$. This proves (2) and (3). □

Proposition 5. *Let* H *be a Hilbert space. Suppose* $e_1, e_2, \cdots$ *is a sequence of non-zero projections in* $B(H)$ *satisfying the following relation :*

$$\begin{aligned} e_i^2 =& e_i = e_i^\star \\ e_i e_j =& e_j e_i = 0 \qquad \textit{if} \ \ |i-j| \geq 2 \\ e_i e_j e_i =& \tau e_i \qquad\qquad \textit{if} \ \ |i-j| = 1 \end{aligned}$$

Then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{n+1}) : \ n \geq 2\}$.

Proof. There exists a nontrivial $C^\star$ representation of $TL(\tau)$ say π which is unital and for which $\pi(e_i^T) = e_i$ where e_i^T denote the idempotents in $TL(\tau)$. By taking norms on the third relation, it follows that $\tau \leq 1$. Suppose that τ is not in the set given in the proposition. Then there exists $n \geq 2$ such that $\frac{1}{4} sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4} sec^2(\frac{\pi}{n+1})$. Then $P_k(\tau) > 0$ for $k = 1, 2, \cdots n$ but $P_{n+1}(\tau) < 0$. Hence, the Jones Wenzl idempotents f_k^T's are defined for $k = 1, 2, \cdots n+2$. Let $f_k = \pi(f_k^T)$ for $k \leq n+2$.

From lemma 4, it follows that e_{n+1} is orthogonal to f_n. But e_{n+1} is orthogonal to $e_1 \vee e_2 \vee \cdots e_{n-1}$ which is, again by lemma 4, $1 - f_n$. Hence $e_{n+1} = e_{n+1} f_n + e_{n+1}(1 - f_n) = 0$ which is a contradiction. This completes the proof. □

Now we will prove the previous conclusion without the orthogality assumption of $e_i^\prime s$.

Proposition 6. *Let H be a Hilbert space. Suppose $e_1, e_2, \cdots$ is a sequence of non-zero projections in $B(H)$ satisfying the following relation :*

$$\begin{aligned} e_i^2 =& e_i = e_i^\star \\ e_ie_j =& e_je_i \quad \textit{if} \ \ |i-j| \geq 2 \\ e_ie_je_i =& \tau e_i \quad \textit{if} \ \ |i-j| = 1 \end{aligned}$$

Then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{n+1}) : \ n \geq 2\}$.

Proof. Suppose that τ is not in the set described above. Then there exists $n \geq 2$ such that $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4}sec^2(\frac{\pi}{n+1})$. From lemma 4, it follows that $e_{n+1}f_{n+1} = 0$. Also $e_if_{n+1} = 0$ for $i \leq n$. Hence $f_{n+1} \leq 1 - e_1 \vee e_2 \vee \cdots \vee e_{n+1} = f_{n+2}$. But $f_{n+2} \leq f_{n+1}$. Hence $f_{n+1} = f_{n+2}$. Let k be the least element in $\{2, 3, \cdots, n\}$ for which $f_{k+1} = f_{k+2}$. Let $g_i = e_{k+i}f_{k-1}$ for $i \geq 0$. We will derive a contradiction by showing that $g_i's$ satisfy the hypothesis of proposition 5.

Since e_{k+i} commutes with f_{k-1} for $i \geq 0$, it follows that g_i's are projections. For the same reason, $g_i's$ satisfy the third relation of proposition 5. First, we show that $g_0 \neq 0$. By the choice of k, $f_k \neq f_{k+1}$. Hence $f_ke_kf_k \neq 0$. Since $f_k \leq f_{k-1}$, it follows that $f_{k-1}e_k = g_0 \neq 0$.

Now we show that $g_ig_j = 0$ if $|i-j| \geq 2$. We begin by showing $g_0g_2 = 0$. Observe that since $f_{k+1} = f_{k+2}$, we have

$$e_{k+1}f_k = e_{k+1}(f_k - f_{k+1})e_{k+1} = e_{k+1}(\frac{P_{k-1}(\tau)}{P_k(\tau)}f_ke_kf_k)e_{k+1} = \tau\frac{P_{k-1}(\tau)}{P_k(\tau)}e_{k+1}f_k.$$

Since $P_{k+1}(\tau) \neq 0$, it follows that $e_{k+1}f_k = 0$. By premultiplying and postmultiplying by e_{k+2}, we see that $e_{k+2}f_k = 0$. Hence we have,

$$\begin{aligned} g_0g_2 &= e_ke_{k+2}f_{k-1} \\ &= e_ke_{k+2}(f_{k-1} - f_k)e_{k+2}e_k \\ &= e_{k+2}e_k(f_{k-1} - f_k)e_ke_{k+2} \\ &= e_{k+2}e_k(\frac{P_{k-2}(\tau)}{P_{k-1}(\tau)}f_{k-1}e_{k-1}f_{k-1})e_ke_{k+2} \\ &= \tau\frac{P_{k-2}(\tau)}{P_{k-1}(\tau)}g_0g_2 \end{aligned}$$

Since $P_k(\tau) \neq 0$, it follows that $g_0g_2 = 0$. Let $i \geq 2$. Let us consider the partial isometry $w = (\frac{1}{\tau})^{i-1}e_{k+i}e_{k+i-1}\cdots e_{k+2}$. Since w commutes with e_k and f_{k-1}, we_kf_{k-1} is a partial isometry. Note that $(we_kf_{k-1})^\star we_kf_{k-1} = g_0g_2 = 0$. Thus, $g_ig_0 = we_kf_{k-1}(we_kf_{k-1})^\star = 0$. Hence $g_ig_0 = 0$ if $i \geq 2$. Let i, j be such that $j \geq i+2$. Now let $u = (\frac{1}{\tau})^{i+1} e_{k+i} e_{k+i-1} \cdots e_k$. Then u is a partial isometry which commutes with f_{k-1} and e_{k+j}. Let $v = u e_{k+j} f_{k-1}$. Then v is a partial isometry such that $v^\star v = g_0 g_j$ and $v v^\star = g_i g_j$. Since $v^\star v = 0$, it follows that $v v^\star = 0$. Thus $g_i g_j = 0$. Therefore g_i 's satisfy the assumptions of proposition 5. Hence we have a contradiction. This completes the proof. □

Now Wenzl's theorem follows from proposition 6.

Chapter 3

Existence of C* representations of $T_n(\tau)$

In this chapter we will describe C* representations of $T_n(\tau)$ when the parameter $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{m+1}) : m \geq 2\}$. First we describe the basic construction for a pair of finite dimensional C* algebras due to Jones. We refer to [Jon] for most of the material in this chapter. But first let us recall some basic facts about finite dimensional C* algebras.

3.1 Finite dimensional C* algebras

Let M be a finite dimensional C* algebra. Then M is unital. Let $\{p_1, p_2, \cdots, p_s\}$ be the set of minimal central projections of M.
Let $p_iMp_i = \{x \in M : p_ix = xp_i = x\}$ and $\mu_i = \sqrt{dim\ p_iMp_i}$.
Then M is isomorphic to $M_{\mu_1}(\mathrm{C}) \oplus \cdots \oplus M_{\mu_s}(\mathrm{C})$ as C* algebras. The algebra M is called a **factor** if it's center is trivial. Let $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_s)$. The vector $\vec{\mu}$ is called the dimenstion vector of M.

Definition 6. *Let M be a C* algebra. A linear functional $\rho : M \to \mathrm{C}$ is said to be a trace if $\rho(ab) = \rho(ba)\ \ \forall a, b \in M$. The functional ρ is said to be positive if $\rho(x^\star x) \geq 0\ \ \forall x \in M$ and faithful if $\rho(x^\star x) = 0$ implies $x = 0$. If M is unital then ρ is said to be unital if $\rho(1) = 1$.*

Any trace on $M_n(\mathrm{C})$ is just a multiple of the usual matrix trace i.e. if $\rho : M_n(\mathrm{C}) \to \mathrm{C}$ is a trace then $\rho((a_{ij})) = \lambda \sum_{i=1}^{n} a_{ii}$. If p is a minimal projetion in $M_n(\mathrm{C})$ then $\rho(p) = \lambda$. Hence ρ is determined by it's value on any minimal projection.

Let M be a finite dimensional C* algebra. Let $\{p_1, p_2, \cdots, p_s\}$ be the set of minimal central projections of M and let $\vec{\mu}$ be the dimension vector of M. Suppose $\rho : M \to \mathrm{C}$ is a trace. Suppose e_i is a minimal projection in p_iMp_i

and let $t_i = \rho(e_i)$. Let $\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$. Then $\vec{t}$ is calles the trace vector associated to ρ. Then ρ is positive if and only if $t_i \geq 0 \ \ \forall i$. The trace ρ is faithful if and only if $t_i > 0 \ \ \forall i$ and it is unital if and only if $\vec{\mu}.\vec{t} = 1$.

Let N and M be finite dimensional C* algebras such that $N \subset M$. We always assume that the inclusion is unital i.e. $1_N = 1_M$. Let $\{p_1, p_2, \cdots, p_s\}$ and $\{q_1, q_2, \cdots, q_r\}$ be the minimal central projections of M and N respectively. Then $q_i p_j M q_i p_j$ and $q_i p_j N q_i p_j$ are factors. Define $\Lambda_{ij} = \sqrt{\frac{dim \ q_i p_j M q_i p_j}{dim \ q_i p_j N q_i p_j}}$ if $p_j q_i \neq 0$. If $p_j q_i = 0$ then define $\Lambda_{ij} = 0$. Then Λ is an $r \times s$ matrix such that $\vec{\mu} = \vec{\nu}.\Lambda$. The matrix Λ is called the inclusion matrix for the inclusion $N \subset M$.

Let $N \subset M$ be a unital inclusion with inclusion matrix Λ. Let ρ_M be a trace on M with trace vector $\vec{t}$ and ρ_N be a trace on N with trace vector $\vec{s}$. Then $\rho_M \mid_N = \rho_N$ if and only if $\Lambda.\vec{t} = \vec{s}$.

The inclusion $N \subset M$ can also be described by it's **Bratelli diagram**. Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras with inclusion matrix Λ. Let $\{q_1, q_2, \cdots, q_r\}$ and $\{p_1, p_2, \cdots, p_s\}$ be the minimal central projections of N and M respectively.The Bratelli diagram for the pair $N \subset M$ is a bipartite graph with verices $\{q_1, q_2, \cdots q_r\} \coprod \{p_1, p_2, \cdots, p_s\}$ where p_j is joined to q_i with Λ_{ij} bonds.

Let us recall the finite dimensional version of von Neumann's double commutant theorem whose proof can be found for instance in [GHJ]. Let H be a Hilbert space. Let $B(H)$ denote the space of bounded linear operators on H. For $S \subset B(H)$, it's commutant denoted by $S^{'}$ is defined as follows:

$$S^{'} := \{x \in B(H) : xs = sx \ \forall s \in S\}.$$

Note that $S \subset S^{''}$.

Theorem [von Neumann]. *Let H be a finite dimensional Hilbert space. Let $M \subset B(H)$ be a $\star$ closed algebra such that M contains the identity operator. Then $M^{''} = M$. If M is a factor then $M \otimes M^{'}$ is isomorphic to B(H) and Hence $dimM \ dimM' = (dim \ H)^2$.*

We end this section with the following lemma. Let $M \subset F$ be a unital inclusion of finite dimensional C* algebras with F as factor. Then the commutant of M in F is denoted by $C_F(M)$.

Lemma 5. *Let $M \subset F$ be a unital inclusion of finite dimensional C⋆ algebras. Assume that F is a factor. Suppose $q \in M \cup C_F(M)$ is a nonzero projection. Then*

(1) qFq is a factor.

(2) $C_{qFq}(qMq) = qC_F(M)q$.

Suppose $N \subset M$ be a unital inclusion of finite dimensional C⋆ algebras with the inclusion matrix Λ. Then the inclusion matrix for $C_F(M) \subset C_F(N)$ is Λ^t.

Proof. If $F = B(H)$ for some finite dimensional Hilbert space then $qFq = B(qH)$. Hence (1) is true.

Let us first consider the case when $q \in M$. Let $x \in M$ and $y \in C_F(M)$. Then $(qxq)(qyq) = qxyq = qyxq = (qyq)(qxq)$. Hence $qC_F(M)q \subset C_{qFq}(qMq)$. Now let $s \in C_{qFq}(qC_F(M)q)$ be given. Then $sq = qs = s$. Let $t \in C_F(M)$. Then $st = sqqt = sqtq = qtqs = tqqs = ts$. Hence $s \in C_F(C_F(M)) = M$. Hence $C_{qFq}(qC_F(M)q) \subset qMq$. Hence taking commutants and using von-Neumann's double commutant theorem $C_{qFq}(qMq) \subset qC_F(M)q$. Hence $C_{qFq}(qMq) = qC_F(M)q$. The case $q \in C_F(M)$ follows from von Neumann's double commutant theorem.

Suppose $N \subset M$ be a unital inclusion of finite dimensional C⋆ algebras with the inclusion matrix Λ. Let Γ be the inclusion matrix for $C_F(M) \subset C_F(N)$. Let $q_1, q_2, \cdots q_r$ be the minimal central projections of N and $p_1, p_2 \cdots, p_s$ be that of M. Since the center of $C_F(M)$ and M are the same, it follows that $p's$ and $q's$ are the minimal central projections of $C_F(M)$ and $C_F(N)$ respectively. Suppose $p_i q_j \neq 0$. Then

$$\begin{aligned} \Gamma_{ij}^2 &= \frac{dim\ p_i q_j C_F(N) p_i q_j}{dim\ p_i q_j C_F(M) p_i q_j} \\ &= \frac{dim\ C_{p_i q_j F p_i q_j}(p_i q_j N p_i q_j)}{dim\ C_{p_i q_j F p_i q_j}(p_i q_j M p_i q_j)} \end{aligned}$$

For $X = M$ or N, Since $p_i q_j X p_i q_j$ is a factor in $p_i q_j F p_i q_j$, it follows, from von Neumann's theorem, that $dim\ C_{p_i q_j F p_i q_j}(p_i q_j X p_i q_j) = \frac{dim\ p_i q_j F p_i q_j}{dim\ p_i q_j X p_i q_j}$. Hence $\Gamma_{ij}^2 = \Lambda_{ij}^2$. Hence $\Gamma = \Lambda^t$. This completes the proof. □

3.2 Basic construction

In this section, We describe the Jones' basic construction for a unital inclusion $N \subset M$ of finite dimensional C⋆ algebras with a faithful unital trace.

We refer to [Jon] for this section. But we include the proofs for completeness.

Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras. Suppose $tr : M \to \mathrm{C}$ is a faithful unital positive trace. Then for $x, y \in M$, define $\langle x, y\rangle = tr(y^\star x)$. Then $\langle , \rangle$ defines an inner product on M. We denote this Hilbert space by $L^2(M, tr)$. Let $E : M \to N$ be the orthogonal projection.

Proposition 7. *E is the unique trace preserving conditional expectation of M onto N. That is*

(1) $E(axb) = aE(x)b$ for $a, b \in N$ and $x \in M$.

(2) $E(n) = n$ for $n \in N$.

(3) $tr(E(x)) = tr(x)$.

Further (1), (2) *and* (3) *determine E uniquely.*

Proof. Let $a, b \in N$ and $x \in M$ be given. For $n \in N$, we have

$$\begin{aligned}\langle aE(x)b, n\rangle &= tr(n^\star aE(x)b)\\ &= tr(bn^\star aE(x))\\ &= \langle E(x), a^\star nb^\star\rangle\\ &= \langle x, a^\star nb^\star\rangle\\ &= tr(bn^\star ax) = tr(n^\star axb)\\ &= \langle axb, n\rangle = \langle axb, E(n)\rangle\\ &= \langle E(axb), n\rangle\end{aligned}$$

Hence $\langle aE(x)b, n\rangle = \langle E(axb), n\rangle$ for every $n \in N$. Thus $E(axb) = aE(x)b$. This proves (1). Since E is the orthogonal projection of M onto N, (2) is true. Let $x \in M$. Now $tr(E(x)) = \langle E(x), 1\rangle = \langle x, E(1)\rangle = \langle x, 1\rangle = tr(x)$. Hence (3) is true.

Let $E' : M \to N$ be linear such that (1), (2) and (3) are satisfied for E'. Let $x \in M$ be given. Then for $n \in N$, $\langle E'(x), n\rangle = tr(n^\star E'(x)) = tr(E'(n^\star x)) = tr(n^\star x)$. A similar calculation with E shows that $\langle E(x), n\rangle = tr(n^\star x)$. Hence $\langle E'(x), n\rangle = \langle E(x), n\rangle$ for every $n \in N$. Hence $E(x) = E'(x)$. Hence $E = E'$. □

We denote E by e_N when we think of E as an element in $B(L^2(M, tr))$. For $x \in M$, define $\pi_l(x)(y) = xy$ for $y \in M$ and $\pi_r(x)(y) = yx$ for $y \in M$. Then $\pi_l(x), \pi_r(x) \in B(L^2(M, tr))$ for $x \in M$. The map $\pi_l : M \to B(L^2(M, tr))$ is a faithful unital * homomorphism. But π_r is an anti homomorphism in the sense that $\pi_r(x^*) = (\pi_r(x))^*$ and $\pi_r(xy) = \pi_r(y)\pi_r(x)$.

Lemma 6. *The commutant of $\pi_r(M)$ in $B(L^2(M, tr))$ is $\pi_l(M)$.*

Proof. It is clear that $\pi_l(M)$ commutes with $\pi_r(M)$. Let $T \in \pi_r(M)^{'}$. Let $x = T(1)$. Now $T(y) = T\pi_r(y)(1) = \pi_r(y)(T(1)) = xy = \pi_l(x)(y)$. Hence $T = \pi_l(x) \in \pi_l(M)$. This completes the proof. □

Henceforth we identify M with $\pi_l(M)$. Now $\pi_r(N) \subset \pi_r(M)$. Note that $\pi_l(M) = \pi_r(M)^{'} \subset \pi_r(N)'$. Hence starting with a unital inclusion $N \subset M$ together with a unital faithful positive trace on M, we obtain another unital inclusion $M \subset \pi_r(N)^{'}$.

Definition 7. *Suppose $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras. Let tr be a faithful, unital, positive trace on M. Then the inclusion $M \subset \pi_r(N)^{'}$ is called the* **basic construction** *for the pair $(N \subset M, tr)$.*

The main properties of the basic construction are summarised in the following porposition.

Proposition 8. *Suppose $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras. Let tr be a faithful, unital, positive trace on M. Then,*

1. *The $C^\star$ algebra generated by M and e_N in $B(L^2(M, tr))$ is $\pi_r(N)^{'}$.*
2. *The central support of e_N in $\pi_r(N)^{'}$ is 1.*
3. *$e_N x e_N = E(x)e_N$ for $x \in M$.*
4. *If Λ is the inclusion matrix for $N \subset M$ then Λ^t is the inclusion matrix for $M \subset \pi_r(N)^{'}$.*

Proof. Let $\langle M, e_N\rangle$ denote the C* algebra generated by M and e_N. We prove that the commutant of $\langle M, e_N\rangle$ is $\pi_r(N)$. Let $T \in (\langle M, e_N\rangle)^{'}$. Since T commutes with e_N, T leaves N invariant. Let $x = T(1)$. Then $x \in N$. Now $T(y) = T\pi_l(y)(1) = \pi_l(y)T(1) = yx = \pi_r(x)(y)$. Hence $T \in \pi_r(N)$. This implies $\langle M, e_N\rangle^{'} \subset \pi_r(N)$ On the other hand, $\pi_r(N)$ commutes with M. Since N is invariant under $\pi_r(N)$, it follows that $\pi_r(N)$ commutes with e_N. Hence $\pi_r(N)$ commutes with $\langle M, e_N\rangle$. This implies $(\langle M, e_N\rangle)^{'} = \pi_r(N)$. By von Neumann's double commutant theorem, $(\langle M, e_N\rangle) = \pi_r(N)^{'}$.

Let $q_1, q_2, \cdots, q_r$ denote the minimal central projections in N. Then the minimal central projections of $(\pi_r(N))^{'}$ are $\pi_r(q_1), \pi_r(q_2), \cdots, \pi_r(q_r)$. Since $\pi_r(q_i)e_N(q_i^\star) = q_i^\star q_i$, we have $\pi_r(q_i)e_N \neq 0$. Thus the central support of e_N in $\langle M, e_N\rangle$ is 1.

Let $x \in M$ be given. On $N^\perp$, $e_N x e_N = 0 = E(x)e_N$. Let $n \in N$ be given. Then $e_N x e_N(n) = E(xn) = E(x)n = E(x)e_N(n)$. Hence $e_N x e_N = E(x)e_N$.

For a C* algebra A, Let A^{op} denote the C* algebra whose underlyind set and the invloution are that of A but the multiplication is changed to $x.y = yx$. Now the center of A^{op} is same as the center of A. Hence the minimal central projections of A^{op} are the same as that of A. Now $\pi_r : M^{op} \to B(L^2(M, tr))$ is a unital inclusion. Now the inclusion matrix of $N^{op} \subset M^{op}$ is the same as that of $N \subset M$ since the minimal central projections of N^{op} and M^{op} are the same as that of N and M. Now by Lemma 5, it follows that the inclusion matrix for $M = (\pi_r(M))' \subset (\pi_r(N))') = \langle M, e_N \rangle$ is Λ^t. This completes the proof. □

Definition 8. *Suppose $N \subset M$ is a unital inclusion of finite dimensional $C^\star$ algebras. Let $tr : M \to$ C be a faithful, unital, positive trace on M. Let $M \subset \langle M, e_N \rangle$ be the basic construction associated to the pair $(N \subset M, tr)$. Then tr is called a* **Markov trace** *of modulus τ if there exists a positive trace $Tr : \langle M, e_N \rangle \to$ C such that*

1. *$Tr(xe_N) = \tau tr(x)$ for $x \in M$.*

2. *$Tr(x) = tr(x)$ for $x \in M$.*

Proposition 9. *Let $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras with a faithful positive trace tr. Suppose that tr is a Markov trace of modulus τ.Then there exists a unique positive trace Tr on $\langle M, e_N \rangle$ satisfying (1) and (2) of definition 8.*

Proof. By definition, there exists a positive trace Tr on $\langle M, e_N \rangle$ such that (1) and (2) holds. Let Tr_1 be another trace for which (1) and (2) holds. Let $x, y \in M$. Now $Tr(xe_N y) = Tr(yxe_N) = \tau tr(yx) = Tr_1(yxe_N) = Tr_1(xe_N y)$. Consider the set $I = \{\sum_{i=1}^n x_i e_N y_i : \; x_i, y_i \in M \;, n \in \mathrm{N}\}$. Then proposition 8 implies that I is an ideal in $\langle M, e_N \rangle$ which contains e_N. Since the central support of e_N is 1, it follows that $I = \langle M, e_N \rangle$. The preceeding calculations show that $Tr_1 = Tr$ on I. Hence $Tr = Tr_1$. □

The following proposition determines when a trace for the pair $N \subset M$ is a Markov trace of modulus τ. Before that we need the following Lemma.

Lemma 7. *Let $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras with a faithful, unital, positive trace tr. Suppose $q_1, q_2, \cdots, q_r$ are the minimal central projections in N. Then $\pi_r(q_1), \pi_r(q_2), \cdots, \pi_r(q_r)$ are the minimal central projections in $\langle M, e_N \rangle$. If f is a minimal projection in $q_i N q_i$ then fe_N is minimal in $\pi_r(q_i)\langle M, e_N \rangle$.*

Proof. Since N commutes with e_N, the map $x \to xe_N$ from $N \to \langle M, e_N \rangle$ is a homomorphism. We assert that this map is 1-1 and it's range is $e_N \langle M, e_N \rangle e_N$. Suppose that $xe_N = 0$ for some $x \in N$. Then $\pi_l(x)e_N(1) = 0$.

Hence $x = 0$. Hence $x \to xe_N$ is 1-1. Let $T \in e_N\langle M, e_N\rangle e_N$ be given. Since T commutes with e_N, T leaves N invariant. Let $x = T(1)$. Then $x \in N$. Since $T(1 - e_N) = 0$ it follows that $T = 0$ on $N^\perp$. Hence $T = xe_N$ on $N^\perp$. Since T is right N linear, it follows that for $n \in N$, $T(n) = T(1)n$. Hence $T(n) = xe_N(n)$ for $n \in N$. Hence $T = xe_N$ on N. Hence $T = xe_N$. It is clear that the map $x \to xe_N$ has range in $e_N\langle M, e_N\rangle e_N$. This proves the assertion.

Let f be a minimal projection in q_iNq_i. Note that $\pi_r(q_i)e_N = \pi_l(q_i)e_N$. Note that $fe_N\pi_r(q_i) = fq_ie_N = fe_N$. Hence $fe_N \le \pi_r(q_i)$. Let p be a nonzero projection in $\langle M, e_N\rangle$ such that $p \le fe_N$. Now $p = fe_Npfe_N = e_Nfpfe_N$. Hence $p = xe_N$ for some $x \in N$. By the 1-1 ness of the map $x \to xe_N$, it follows that x is a nonzero projection. Now $xe_N = xe_Nfe_N = xfe_N$. Thus $x = xf$. Similarly $x = fx$. Hence by the minimality of f, it follows that $x = f$ and hence $p = fe_N$. Therefore fe_N is minimal. This completes the proof. □

Proposition 10. *Suppose $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras with a faithful, unital, positive trace tr. Let Λ be the inclusion matrix for $N \subset M$. Let $\vec{\mu}$ and $\vec{\nu}$ be the dimension vectors for M and N respectively. Suppose $\vec{r}$ and $\vec{s}$ are the trace vectors for $tr|_N$ and $tr|_M$ respectively. Then tr is a Markov trace of modulus τ if and only if $\Lambda^t\Lambda\vec{s} = \frac{1}{\tau}\vec{s}$ and $\Lambda\Lambda^t\vec{r} = \frac{1}{\tau}\vec{r}$.*

Proof. Let tr be Markov of modulus τ and Let Tr be the corresponding trace on $\langle M, e_N\rangle$. Let $\vec{t}$ be the trace vector for Tr on $\langle M, e_N\rangle$. By lemma 7, we have $\vec{t} = \tau\vec{r}$. Since the traces are consistent, we have $\vec{r} = \Lambda\vec{s} = \Lambda\Lambda^t(\vec{t}) = \Lambda\Lambda^t(\tau\vec{r}) = \tau\Lambda\Lambda^t(\vec{r})$. Also, $\vec{s} = \Lambda^t(\vec{t}) = \Lambda^t(\tau\vec{r}) = \tau\Lambda^t\Lambda(\vec{s})$.

Suppose the inclusion matrix satisfies the condition in the proposition. Define Tr on $\langle M, e_N\rangle$ by letting it's trace vector be $\vec{t} = \tau\vec{r}$. Then $\Lambda^t(\vec{t}) = \tau\Lambda^t(\vec{r}) = \tau\Lambda^t\Lambda\vec{s} = \vec{s}$. Hence $Tr(x) = tr(x)$ for $x \in M$. Also by definition of Tr, it follows that $Tr(pe_N) = \tau tr(p)$ for every minimal projection p in N and hence $Tr(xe_N) = \tau tr(x)$ for $x \in N$. Let $x \in M$. Now $Tr(xe_N) = Tr(e_Nxe_N) = Tr(E(x)e_N) = \tau tr(E(x)) = \tau tr(x)$. This proves that tr is a Markov trace of modulus τ. □

Corollary 1. *Let $N \subset M$ be a unital inclusion of finite dimensional $C^\star$ algebras with a faithful, unital, positive trace tr. Suppose that tr is a Markov trace of modulus τ. Then the unique trace Tr on $\langle M, e_N\rangle$ which extends tr and for which $Tr(xe_N) = \tau tr(x)$ is a Markov trace of modulus τ for the pair $M \subset \langle M, e_N\rangle$.*

Proof. Let $\vec{r}, \vec{s}, \vec{t}$ be as in proposition 10. Let Λ be the inclusion matrix for the pair $N \subset M$. Then $\vec{t} = \tau\vec{r}$. Now $\Lambda\Lambda^t\vec{t} = \tau\Lambda\Lambda^t\vec{r} = \tau\frac{1}{\tau}(\vec{r}) = \frac{1}{\tau}(\vec{t})$. Hence by proposition 10, it follows that Tr is a Markov trace of modulus τ. $\square$

We end this section with a lemma which characterises the basic construction for a pair $N \subset M$ whose proof can be found in [JS].

Lemma 8. *Let $A \subset B$ be a unital inclusion of finite dimensional $C^\star$ algebras with a faithful, unital, positive trace tr. Let E be the unique trace preserving conditional expectation of B onto A. Let $B_1 = \langle B, e \rangle$ denote the result of the basic construction. Let $B \subset C$ be a unital inclusion of finite dimensional $C^\star$ algebras. Suppose C contains a projection f satisfying*

(1) $C = \langle B, f \rangle$;

(2) $fbf = E(b)f$ for $b \in B$; and

(3) f commutes with A and $a \to af$ is an injective $*$ homomorphism of A into C.

(4) The central support of f in C is 1.

Then there exists a unique isomorphism $\Psi : B_1 \to C$ such that $\Psi(b) = b$ for $b \in B$ and $\Psi(e) = f$.

3.3 Jones Tower

Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras with a faithful, unital, positive trace tr. Suppose that tr is Markov of modulus τ. Then there exists a unique faithful, positive trace which extends tr which we continue to denote by tr such that $tr(xe_N) = \tau tr(x)$ for $x \in M$. Then tr is a Markov trace of modulus τ for the pair $M \subset \langle M, e_N \rangle$. Let $e_1 = e_N$.

Iterating the basic construction for the pair $M \subset \langle M, e_1 \rangle$, we get a tower of finite dimensional C* algebras $N \subset M \subset \langle M, e_1 \rangle \subset \langle M, e_1, e_2 \rangle \subset \cdots$ with faithful, unital, positive trace on $\bigcup_n \langle M, e_1, e_2, \cdots, e_n \rangle$ which we again denote by tr. This tower is called the Jones tower. Let $M_0 = N$, $M_1 = M$ and $M_n = \langle M, e_1, e_2, \cdots, e_{n-1} \rangle$. M_{n+1} is obtained by the basic construction for the pair $(M_{n-1} \subset M_n, tr)$. Let $E_{n-1} : M_n \to M_{n-1}$ be the corresponding conditional expectation. Then we have the following,

(1) $tr(x) = tr(E_{n-1}(x))$ if $x \in M_n$.

(2) $tr(xe_n) = \tau tr(x)$ if $x \in M_n$.

(3) e_n commutes with M_{n-1}.

(4) $e_n x e_n = E_{n-1}(x) e_n$ if $x \in M_n$.

Now $tr(E_n(e_n)x) = tr(E_n(e_n x)) = tr(e_n x) = \tau tr(x) = tr(\tau x)$ for $x \in M_n$. Since tr is faithful, $E_n(e_n) = \tau$.

The next proposition says that the sequence of projections e_n satisfy the TL relations.

Proposition 11. *Suppose $N \subset M$ is a unital inclusion of finite dimensional $C^\star$ algebras and Let tr be a Markov trace of modulus τ. If $\{e_n\}$ denote the sequence of projections in the Jones tower, then*

$$
\begin{aligned}
e_i^2 = e_i = e_i^\star & \quad \forall\ i \in \mathrm{N} \\
e_i e_j = e_j e_i & \quad if \quad |i - j| \geq 2 \\
e_i e_j e_i = \tau e_i & \quad if \quad |i - j| = 1
\end{aligned}
$$

Proof. Only the third relation requires proof. Let $n \in \mathrm{N}$ be given. Now $e_{n+1} e_n e_{n+1} = E_n(e_n) e_{n+1} = \tau e_{n+1}$. Consider the previous relation in M_{n+2}. Then, $\frac{e_{n+1}e_n}{\sqrt{\tau}}$ is a partial isometry.
Hence $(\frac{e_{n+1}e_n}{\sqrt{\tau}})^\star \frac{e_{n+1}e_n}{\sqrt{\tau}} = \frac{e_n e_{n+1} e_n}{\tau}$ is a projection. Clearly $\frac{e_n e_{n+1} e_n}{\tau} \leq e_n$. Now $tr(\frac{e_n e_{n+1} e_n}{\tau}) = tr(e_n)$. Since tr is faithful, it follows that $\frac{e_n e_{n+1} e_n}{\tau} = e_n$. This completes the proof. □

3.4 Jones quotient

We will describe a $C^\star$ quotient for $TL(\tau)$ called the Jones quotient for every $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{m+1} : \ m \geq 2\}$.

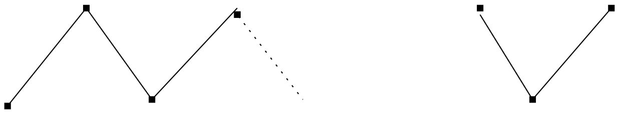
First we show that for $\tau \in \{\frac{1}{4} sec^2(\frac{\pi}{m+1} : \ m \geq 2\}$ there exists an inclusion $N \subset M$ of finite dimensional $C^\star$ algebras which admits a Markov trace of modulus τ. We need the following proposition for that. We say that the inclusion $N \subset M$ is connected if the Bratelli diagram for the inclusion $N \subset M$ is connected.

Proposition 12. *Let $N \subset M$ be a unital inclusion which is connected. Then there exists a unique Markov trace of modulus τ if and only if $\tau = \| \Lambda \|^{-2}$.*

For a proof we refer to [GHJ] □

Let $\tau = \frac{1}{4} sec^2 \frac{\pi}{n+1}$. It is enough to exhibit a Bratelli diagram or a bipartite graph whose corresponding matrix Λ satisfies $||\Lambda|| = \frac{1}{\sqrt{\tau}}$. First suppose that n is even, say $n = 2l$. Note that the norm of a matrix won't change by changing rows and columns. Consider the following bipartite graph with

$2l = l + l$ vertices.



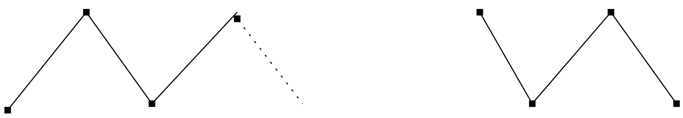
Let Λ be the corresponding matrix. Let $Y = \begin{pmatrix} 0 & \Lambda \\ \Lambda^t & 0 \end{pmatrix}$. Then Y is the adjacency matrix of the following path with $2l$ vertices.

Then

$$Y = \begin{pmatrix} 0 & 1 & 0 & . & 0 & 0 \\ 1 & 0 & 1 & . & 0 & 0 \\ 0 & 1 & 0 & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & 0 & 1 \\ 0 & 0 & 0 & . & 1 & 0 \end{pmatrix}$$

For $j = 1, 2, \cdots n$, one checks that $Y\xi_j = \lambda_j \xi_j$ where $\lambda_j = 2cos(\frac{j\pi}{n+1})$, $\xi_j = \left(sin(\frac{jk\pi}{n+1})\right)_{1 \leq k \leq l}$. Since Y is symmetric, it follows that $||Y|| = 2cos(\frac{\pi}{n+1})$. Now note that $YY^t = \begin{bmatrix} \Lambda\Lambda^t & 0 \\ 0 & \Lambda^t\Lambda \end{bmatrix}$. Hence $||Y||^2 = ||YY^t|| = ||\Lambda||^2$. Hence $||\Lambda||^2 = \frac{1}{\tau}$.

When n is odd say $n = 2l + 1$, considering the following bipartite graph with $2l + 1 = l + (l + 1)$ vertices and arguing as above will do the job.



We now define the Jones quotient $J_n(\tau)$ for $\tau \in \{\frac{1}{4}sec^2(\frac{\pi}{m+1} : m \geq 2\}$. Suppose $\tau \in \{\frac{1}{4}sec^2(\frac{\pi}{m+1} : m \geq 2\}$. Let $N \subset M$ be an inclusion of finite dimensional $C^\star$ algebras which admits a Markov trace of modulus τ. Let $M_0 \subset M_1 \subset M_2 \subset \cdots$ be the Jones tower. Let $J_n(\tau) \subset M_n$ be the $C^\star$ algebra generated by $1, e_1, e_2, \cdots, e_{n-1}$. We set $J_i(\tau) = \mathrm{C}$ for $i = 0, 1$. Then $E_{n-1}(J_n(\tau)) \subset J_{n-1}(\tau)$. Then we have a tower $J_n(\tau) \subset J_{n+1}(\tau)$ of finite dimensional $C^\star$ algebras and a faithful unital positive trace on $\bigcup_n J_n(\tau)$. We refer to [Jon] for the Bratelli diagram of the tower $J_n(\tau) \subset J_{n+1}(\tau)$. From the Bratelli diagram it follows that the tower $J_n(\tau) \subset J_{n+1}(\tau)$ together with the conditional expectations E_{n-1} and the trace depends only on τ and is independent of the initial inclusion $N \subset M$.

Let $\tau < \frac{1}{4}$. It is shown in [Jon] that, in this case, there exists a unital inclusion of type II_1 factors with index τ^{-1}, and that here too, just as in the finite dimensional case, one may, by iterated basic construction, obtain the Jones' tower $N \subset M \subset \langle M, e_1 \rangle \subset \langle M, e_1, e_2 \rangle$ of type II_1 factors and conditional expectations $E_n : M_{n+1} \to M_n$ where $M_0 = N$,$M_1 = M$ and $M_n = \langle M, e_1, e_2, \cdots e_{n-1} \rangle$. The tower $M_n \subset M_{n+1}$ has a faithful positive trace tr on $\bigcup_n M_n$.

Then we have the following,

(1) $tr(x) = tr(E_{n-1}(x))$ if $x \in M_n$.

(2) $tr(xe_n) = \tau tr(x)$ if $x \in M_n$.

(3) e_n commutes with M_{n-1}.

(4) $e_n x e_n = E_{n-1}(x) e_n$ if $x \in M_n$.

Also the $e_n' s$ satisfy the TL relations. Now $J_n(\tau)$ is defined as in the finite dimensional case. As in the finite dimensional case, the tower $J_n(\tau) \subset J_{n+1}(\tau)$ together with the conditional expectations $E_n : J_{n+1}(\tau) \to J_n(\tau)$ and the trace depends only on τ and is independent of the initial inclusion $N \subset M$. We refer to [JS] for the definition of type II_1 factors and the basic construction for type II_1 factors.

From now on, Let $e_1^T, e_2^T, \cdots e_{n-1}^T$ denote the idempotents in $T_n(\tau)$ and $e_1^J, e_2^J, \cdots e_{n-1}^J$ denote the 'Jones' projections in $J_n(\tau)$. Suppose ϵ_n^T and ϵ_n^J denote the corresponding conditional expectation and let $T_i(\tau) = \mathrm{C}$ for $i = 0, 1$. By the universal property of $T_n(\tau)$ there exists a unique map $\phi_n : T_n(\tau) \to J_n(\tau)$ such that ϕ_n is unital and $\phi_n(e_i^T) = e_i^J$. Note that $\phi_{n+1}(a) = \phi_n(a)$ if $a \in T_n(\tau)$. Hence we can and will denote the maps ϕ_n by ϕ. The algebra $J_n(\tau)$ is called the Jones quotient of $T_n(\tau)$

Note the following properties of ϕ:

(1) The map ϕ is $*$ preserving.

(2) $\phi(\epsilon_n^T(a)) = \epsilon_n^J(\phi(a))$ if $a \in T_{n+1}(\tau)$.

(3) $\phi(tr^T(a)) = tr^J(\phi(a))$ if $a \in T_n(\tau)$.

(1),(2) and (3) can be proved by induction on n and by noting the fact that $\{x + \sum_{i=1}^r x_i e_n^T y_i \ : x, x_i, y_i \in T_n(\tau) \ and \ r \in \mathrm{N}\} = T_{n+1}(\tau)$.

Recall the polynomials $P_k(\lambda)$ and the Jones Wenzl projections f_k^T defined in chapter 2. Let $f_k^J = \phi(f_K^T)$.

Proposition 13. *If $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n-1$ then $f_k^J = 1 - \bigvee_{i=1}^{k-1} e_i$ for $2 \leq k \leq n$.*

Proof. Let $k \geq 2$. Since $f_k^J e_i^J = 0$ for $i \in \{1, 2, \cdots, k-1\}$, it follows that $1 - f_k^J \geq e_1^J \vee e_2^J \vee \cdots \vee e_{k-1}^J$. But $1 - f_k^J$ is in the algebra generated by $e_1, e_2, \cdots e_{k-1}$. Thus $1 - f_k^J \leq e_1^J \vee e_2^J \vee \cdots \vee e_{k-1}^J$.
Hence $1 - f_k^J = e_1^J \vee e_1^J \vee \cdots \vee e_{k-1}^J$. This completes the proof. □

We refer to [Jon] for the following proposition.

Proposition 14. *If $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots n-1$ then dim $J_k(\tau) = \frac{1}{k+1}\binom{2k}{k}$ for $k = 1, 2, \cdots, n-1$. Hence $\phi : T_k(\tau) \to J_k(\tau)$ is an isomorphism for $k = 1, 2, \cdots, n-1$.*

Hence if $\tau \leq \frac{1}{4}$, any C* representation of $T_k(\tau)$ is a C* representation of $J_k(\tau)$. In the next chapter, we will prove that if $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$, any C* representation π for which $\pi(e_1^T) \vee \pi(e_2^T) \cdots \vee \pi(e_{k-1}^T) = 1$ factors through $J_k(\tau)$ when $k \geq n$.

Let us recall the Murray von Neumann equivalence. Let M be a finite dimensional C* algebra. Let p, q be projections in M. We say p is Murray von Neumann equivalent to q if there exists $w \in M$ such that $w^\star w = p$ and $ww^\star = q$. Note that in $J_n(\tau)$ all the $e_i^{'}s$ are Murray von Neumann equivalent.

Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Then $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots n-1$ but $P_n(\tau) = 0$. Note that $tr^J(f_n^J) = P_n(\tau) = 0$. Since tr is faithful, $f_n^J = 0$. Hence $e_1^J \vee e_2^J \vee \cdots \vee e_{k-1}^J = 1$ in $J_k(\tau)$ for $k \geq n$. We will prove in the next chapter that the kernel of the map $\phi : T_k(\tau) \to J_k(\tau)$ is the ideal generated by f_n^T in $T_k(\tau)$ for $k \geq n$. We need the following proposition for that.

Proposition 15. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ for some $n \geq 2$. Then $J_{k+1}(\tau)$ together with e_k^J is the basic construction of the pair $(J_{k-1}(\tau) \subset J_k(\tau), tr)$ for $k \geq n-1$. That is, if $\langle J_k(\tau), e\rangle$ denotes the basic construction then there exists a unique isomorphism $\Psi : \langle J_k(\tau), e\rangle \to J_{k+1}(\tau)$ such that $\Psi(a) = a$ if $a \in J_k(\tau)$ and $\Psi(e) = e_k^J$.*

Proof. Let $k \geq n-1$ be given. We apply Lemma 8 with $f = e_k^J$ to prove this. ϵ_{k-1}^J is the unique trace preserving conditional expectation of $J_k(\tau)$

onto $J_{k-1}(\tau)$. Clearly $(1), (2)$ of lemma 8 are true. Also, e_k^J commutes with $J_{k-1}(\tau)$. Now let $xe_k^J = 0$ for some $x \in J_{k-1}(\tau)$. Then $yxe_k^J = 0$ for every $y \in J_{k-1}(\tau)$. Hence for $y \in J_{k-1}(\tau)$, $\tau tr(yx) = tr(yxe_k^J) = 0$. Hence $tr(yx) = 0$ for every $y \in J_{k-1}(\tau)$. Since tr is faithful, it follows that $x = 0$. Hence (3) of lemma 8 is satisfied.

Let p be a central projection in $J_{k+1}(\tau)$ such that $p \geq e_k^J$. Let $i \in \{1, 2, \cdots, k\}$ be given. Let $w \in J_{k+1}(\tau)$ be such that $w^*w = e_k^J$ and $ww^* = e_i^J$. Now $e_i^J p = ww^*p = wpw^* = we_k^J pw^* = we_k^J w^* = ww^\star = e_i^J$. Hence $p \geq e_i^J$ for every $i \in \{1, 2, \cdots, k\}$. Hence $p \geq e_1^J \vee e_2^J \vee \cdots \vee e_k^J \geq 1 - f_n^J = 1$ by the observation preceeding this proposition. Hence (4) of lemma 8 is satisfied. The proof is complete by applying lemma 8. $\square$

Chapter 4

Maximal C* quotient of $T_n(\tau)$

4.1 Maximal C* qoutient of a $\star$ algebra

Let A be a unital C algebra. For $a \in A$, it's spectrum, denoted $\sigma_A(a)$ is defined by $\sigma_A(a) = \{\lambda \in \mathrm{C} : a - \lambda 1 \textit{ is not invertible in } A\}$. Let B be a unital finite dimensional C algebra. Let $\pi : A \to B$ be a unital algebra homomorphism. Then $\sigma_B(\pi(a)) \subset \sigma_A(a)$ for $a \in A$.

Suppose A is a unital finite dimensional C algebra. For $a \in A$, let $\pi_l(a)$ be defined by $\pi_l(a)(b) = ab$. Let $End(A)$ denote the space of C linear endomorphisms of A. Then $\pi_l : A \to End(A)$ is a unital algebra homomorphism which is 1-1. Since $\sigma_{End(A)}(\pi_l(a))$ is nonempty, it follows that $\sigma_A(a)$ which contains $\sigma_{End(A)}(\pi_l(a))$ is nonempty. Now we will show that $\sigma_A(a)$ is finite by showing $\sigma_A(a)$ is contained in the set of zeros of the characteristic polynomial of $\pi_l(a)$.

Lemma 9. *Let A be a unital finite dimensional* C *algebra. Let $a \in A$. Then $\sigma_A(a)$ is nonempty and finite.*

Proof. We have already shown that $\sigma_A(a)$ is nonempty. Now for a polynomial $p(x)$ over C, $p(\pi_l(a)) = \pi_l(p(a))$. Since $\pi_l(a)$ satisfies it's characteristic polynomial, it follows that $\exists$ a polynomial $p(x)$ over C such that $p(a) = 0$. Now we show that $\lambda \in \sigma_A(a)$ implies $p(\lambda) = 0$. Let $\lambda \in \mathrm{C}$ be such that $p(\lambda) \neq 0$. Then $p(x) - p(\lambda) = (x - \lambda)q(x)$ for some polynomial q. Now $-p(\lambda) = p(a) - p(\lambda) = (a - \lambda)q(a) = q(a)(a - \lambda)$. Hence $\frac{-q(a)}{p(\lambda)}$ is the inverse of $a - \lambda$. Thus $\lambda \notin \sigma_A(a)$. Therefore $\sigma_A(a)$ is contained in the zero set of p. As a result we conclude that $\sigma_A(a)$ is finite. $\square$

Let A be a finite dimensional unital $\star$ algebra. Let $\pi : A \to B$ be a C* representation where B is a C* algebra. Then for $a \in A$,

$$\begin{aligned} ||\pi(a)||^2 = ||\pi(a^\star a)|| &\leq sup\{|\lambda| : \lambda \in \sigma_B(\pi(a^\star a))\} \\ &\leq sup\{|\lambda| : \lambda \in \sigma_A(a^\star a)\} \quad \text{since } \sigma_B(\pi(a^\star a)) \subset \sigma_A(a^\star a)\ . \end{aligned}$$

For $a \in A$, define

$||a|| := sup\{||\pi(a)|| : \pi : A \to B$ is a * algebra homomorphism where B is a C* algebra$\}$

Then $||a|| < \infty \; \forall a \in A$. Let $I = \{a \in A : ||a|| = 0\}$. Then I is an ideal in A.
For $a \in A$, note that $||a + I|| = ||a||$ depends only on $a + I$. Then A/I becomes a C* algebra with the above norm. Let $q : A \to A/I$ be the qoutient map.
A/I has the following universal property:

Let B be a C* algebra and let $\pi : A \to B$ be a $\star$ homomorphism. Then $\exists$ a unique $\star$ homomorphism $\tilde{\pi} : A/I \to B$ such that $\tilde{\pi} \circ q = \pi$.

Definition 9. *Let A be a unital finite dimensional $\star$ algebra. A C* algebra B together with a $\star$ algebra homomorphism $q : A \to B$ is said to be a maximal C* quotient of A if it has the following universal property:*
Given a $\star$ homomorphism $\pi : A \to C$ where C is a C* algebra, $\exists$ a unique $\star$ homomorphism $\tilde{\pi} : B \to C$ such that $\tilde{\pi} \circ q = \pi$.

Note that maximal C* quotient of a unital finite dimensional $\star$ algebra exists and is unique upto a unique isomorphism.

Let $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{n+1}) : n \geq 2\}$. Now if $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots n-1$ then the natural map $\phi : T_k(\tau) \to J_k(\tau)$ is a $\star$ isomorphism. Hence if $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n-1$ then $(J_k(\tau), \phi)$ is the maximal C* quotient of $T_k(\tau)$ for $k = 1, 2, \cdots, n-1$. In particular, if $\tau \leq \frac{1}{4}$ then $(J_k(\tau), \phi)$ is the maximal C* quotient of $T_k(\tau) \; \forall \; k \geq 1$.

Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Let $\tilde{1} : T_k(\tau) \to$ C be the $*$ homomorphism defined by $\tilde{1}(e_i^T) = 0$ for $i \leq k-1$ and $\tilde{1}(1) = 1$ (which exists by the universal property of $T_k(\tau)$). We will prove that $(J_k(\tau) \oplus \mathrm{C}, \phi \oplus \tilde{1})$ is the maximal C* quotient of $T_k(\tau)$ when $k \geq n$. This requires the determination of the kernel of the map $\phi : T_k(\tau) \to J_k(\tau)$ when $k \geq n$. We need the following lemma for that.

Lemma 10. *Let $N \subset M$ be a unital inclusion of finite dimensional C* algebras with a faithful, unital, positive trace tr. Then M is a $N-N$ bimodule. Let $\langle M, e_N \rangle$ denote the basic construction. Then the $M-M$ bimodule homomorphism $\Psi : M \otimes_N M \to \langle M, e_N \rangle$ defined by $\Psi(x \otimes y) = xe_N y$ is an isomorphism.*

Proof. The map Ψ is well defined since e_N commutes with N. Consider M as a right N module. Then $\langle M, e_N \rangle$ is just the space of right N linear maps of M. Let $E : M \to N$ be the unique trace preserving conditional expectation. Let M^* dente the space of right N linear maps from M to N. Then M^* is a left N module. For $b \in M$, define $E_b(x) = E(bx)$ for $x \in M$. Then $E_b \in M^*$. Define $\theta : M \to M^*$ by $\theta(b) = E_b$. Clearly θ is left N linear.

Assertion: θ is an isomorphism.

Suppose $\theta(b) = 0$ for some $b \in M$. Then $tr(bx) = tr(E(bx)) = tr(E_b(x)) = 0\ \forall x \in M$. Since tr is faithful,we have $b = 0$. Hence θ is one one. Now let $\sigma \in M^*$ be given. Then $tr \circ \sigma$ is a linear functional on M. Since M is a Hilbert space, $\exists\ b \in M$ such that $tr \circ \sigma = \langle\ ,\ b^\star\rangle$. Hence $tr(\sigma(x)) = tr(bx)\ \forall x \in M$. Hence $tr(\sigma(x)n) = tr(\sigma(xn)) = tr(bxn) = tr(E(bxn)) = tr(E(bx)n)$ for $x \in M, n \in N$. Since tr is faithful on N , $\sigma(x) = E(bx)\ \forall x \in M$. Hence $\sigma = \theta(b)$. Therefore, θ is onto. This proves the assertion.

Since C* algebras are semisimple, M as a right N module is semisimple. M is also finitely generated as an N module. Hence M is finitely genrerated projective and hence flat. Hence $id \otimes \theta : M \otimes_N M \to M \otimes_N M^*$ is an isomorphism. Since M is finitely generated and projective, the canonical map $\chi : M \otimes_N M^\star \to End_N(M)$ given by $\chi(x \otimes y^\star)(m) = xy^\star(m)$ is one one. Hence $\chi \circ id \otimes \theta$ is one one.

Assertion: $\Psi = \chi \circ (id \otimes \theta)$. Let $x, y, m \in M$ be given. Now

$$(\chi \circ (id \otimes \theta))(x \otimes y)(m) = x\theta(y)(m) = xE(ym) = xe_N y(m).$$

Hence $\chi \circ (id \otimes \theta) = \Psi$. This proves the assertion. Hence Ψ is one one.

The image of Ψ is clearly an ideal which contains e_N. Since the central support of e_N in $\langle M, e_N\rangle$ is 1, it follows that Ψ is onto. Hence Ψ is an isomorphism. $\square$

Now We compute the kernel of the map $\phi : T_k(\tau) \to J_k(\tau)$ for $k \geq n$ when $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. The proof of the following proposition can be found in [JR]. We include the proof for completeness.

Proposition 16. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Then the kernel of the natural map $\phi : T_k(\tau) \to J_k(\tau)$ for $k \geq n$ is the ideal generated by f_n^T in $T_k(\tau)$ for $k \geq n$.*

Proof. By induction, $\tilde{1}(f_k^T) = 1$ for $0 \leq k \leq n$. Hence $f_n^T \neq 0$. We will write T_k for $T_k(\tau)$.

Let $A_k = T_k$ for $0 \leq k \leq n-1$. Let $A_k = A_{k-1}e_{k-1}^T A_{k-1}$ for $k \geq n$. Then $A_k \subset T_k$.
Assertion: For every $k \geq 0$,

(1) A_k is a subalgebra of T_k.

(2) $\epsilon^T_{k-1}(A_k) \subset A_{k-1}$.

(3) A_k is a $A_{k-1} - A_{k-1}$ bimodule.

We prove this by induction on k. Clearly (1), (2) and (3) holds for $k \leq n-1$. Now assume (1), (2) and (3) holds for k. Let $x, y, z, w \in A_k$. Now $(xe_k y)(ze_k w) = x\epsilon^T_{k-1}(yz)e_k w$. Now (1), (2), (3) for A_k implies $x\epsilon^T_{k-1}(yz) \in A_k$. Hence $(xe^T_k y)(ze^T_k w) \in A_{k+1}$. Hence A_{k+1} is a subalgebra of T_{k+1}. Let $x, y \in A_k$. Then $\epsilon^T_k(xe_k y) = \tau xy \in A_k$ since A_k is a subalgebra of T_k. Hence $\epsilon^T_k(A_{k+1}) \subset A_k$. Since A_k is a subalgebra of T_k, it follows that A_{k+1} is a $A_k - A_k$ bimodule. This proves the assertion.

Assertion : The map $\phi : A_k \to J_k$ is an isomorphism.

We prove the assertion by induction on k. The map $\phi : A_k \to J_k$ is an isomorphism for $k \leq n-1$ is exactly proposition14. Now assume that ϕ is an isomorphism for $0 \leq l \leq k$. Let $\phi \otimes \phi$ denote the isomorphism from $A_k \otimes_{A_{k-1}} A_k$ to $J_k \otimes_{J_{k-1}} J_k$ when one identifies A_l with J_l when $l \leq k$ via ϕ. Let $\chi : A_k \otimes_{A_{k-1}} A_k \to A_{k+1}$ be defined by $\chi(x \otimes y) = xe^T_k y$. Let Ψ be the map of Lemma 10 where $N = J_{k-1}$, $M = J_k$ and the projection $e_N = e^T_k$. Now $\Psi \circ \phi \otimes \phi = \phi \circ \chi$. By induction hypothesis, $\phi \otimes \phi$ is an isomorphism. Since Ψ is also an isomorphim, it follows that $\phi \circ \chi$ is an isomorphism. By definition, χ is onto. Hence ϕ is one-one. Since $\phi \circ \chi$ is onto, ϕ is onto. Hence $\phi : A_{k+1} \to J_{k+1}$ is an isomorphism. This proves the assertion.

For $k \geq n$, Let I_k denote the ideal in $T_k(\tau)$ generated by f^T_n. Clearly $I_k \subset I_{k+1}$. Observe that $T_k e^T_k T_k$ is an ideal in T_{k+1} which contains e^T_k. Since $e^T_{k-1} = \frac{1}{\tau}(e^T_{k-1}e^T_k e^T_{k-1})$ it follows that $T_k e^T_k T_k$ contains e^T_{k-1}. Similarly it contains $e^T_1, e^T_2, \cdots, e^T_{k-2}$. Hence $1 - f^T_n \in T_k e^T_k T_k$ for $k \geq n-1$. Hence $I_{k+1} + T_k e^T_k T_k = T_{k+1}$ for $k \geq n-1$. We claim that $I_k + A_k = T_k$ for $k \geq n$. We prove this by induction on k. We have just proved that the claim is true for $k = n$. Now assume the claim is true for k. Since $T_{k+1} = I_{k+1} + T_k e^T_k T_k$, it is enough to show that if $x, y \in T_k$ then $xe^T_k y \in I_{k+1} + A_{k+1}$. By induction hypotheis, $\exists z, w \in I_k$ and $u, v \in A_k$ such that $x = z + u$ and $y = w + v$. Now $xe^T_k y = ze^T_k w + ue^T_k w + ze^T_k v + ue^T_k v$. Since $I_k \subset I_{k+1}$, it follows that $ze^T_k w + ue^T_k w + ze^T_k v \in I_{k+1}$. By definition $ue^T_k v \in A_{k+1}$. Hence $I_{k+1} + A_{k+1} = T_{k+1}$. Thus completes the induction and proves the claim.

Now we prove that the kernel of the map ϕ is I_k for $k \geq n$. Let $k \geq n$ be given. Since $f^J_n = 0$, it follows that $I_k \subset ker(\phi)$. Now let $x \in Ker(\phi)$ be given. Let $z \in I_k$ and $w \in A_k$ be such that $x = z + w$. Then $0 = \phi(w)$. Since $\phi : A_k \to T_k$ is an isomorphism, it follows that $w = 0$. Hence $x \in I_k$. Thus $ker(\phi) \subset I_k$. Therefore $ker(\phi) = I_k$. This completes the proof. $\square$

Now We prove the much promised fact that when $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$, $(J_k(\tau) \oplus \mathrm{C}, \phi \oplus \tilde{1})$ is the maximal C* quotient of $T_k(\tau)$ when $k \geq n$. We begin with the following theorem.

Theorem 2. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Let $k \geq n$. Let A be a $C^\star$ algebra. Let $\pi : T_k(\tau) \to A$ be a $\star$ algebra homomorphism such that $\bigvee_{i=1}^{k-1} \pi(e_i) = 1$. Then $\exists$ a unique $\star$ algebra homomorphism $\tilde{\pi} : J_k(\tau) \to T_k(\tau)$ such that $\tilde{\pi} \circ \phi = \pi$.*

Proof. It is enough to show that $\pi = 0$ on $ker(\phi)$. Since $ker(\phi)$ is the ideal generated by f_n^T, it is enough to show that $\pi(f_n^T) = 0$.

Assertion: $\pi(f_n^T)\pi(e_i^T) = 0$ for $1 \leq i \leq k-1$.

Note that $f_n^T e_i^T = 0$ for $1 \leq i \leq n-1$. Hence if $k = n$ then we are done. Hence assume $k > n$. Now

$$\begin{aligned} e_n^T f_n^T e_n^T &= e_n^T f_{n-1}^T - \frac{P_{n-2}(\tau)}{P_{n-1}(\tau)} f_{n-1}^T e_n^T e_{n-1}^T e_n^T f_{n-1}^T \\ &= \frac{P_n(\tau)}{P_{n-1}(\tau)} e_n^T f_{n-1}^T \\ &= 0 \end{aligned}$$

Hence $\pi((e_n^T f_n^T)(e_n^T f_n^T)^\star)) = 0$. Hence $\pi(e_n^T f_n^T) = 0$. Hence taking adjoints $\pi(f_n^T e_n^T) = 0$. Now let i be such that $n < i \leq k$. Let $w_i = e_i^T e_{i-1}^T \cdots e_{n+1}^T$. Then $w_i e_n^T w_i^\star = \tau^{n-i} e_i^T$. But w_i commutes with T_n. Hence we have $\pi(f_n^T e_i^T) = \frac{1}{\tau^{n-i}} \pi(w_i)\pi(f_n^T e_n^T)\pi(w_i^\star) = 0$. This proves the assertion.
Since $\bigvee_{i=1}^{k-1} \pi(e_i^T) = 1$, it follows that $\pi(f_n^T) = 0$ which completes the proof.
□

Theorem 3. *Let $\tau = \frac{1}{4}sec^2(\frac{\pi}{n+1})$ where $n \geq 2$. Let $k \geq n$. Then the maximal $C^\star$ quotient of $T_k(\tau)$ is $(J_k(\tau) \oplus \mathrm{C}, \phi \oplus \tilde{1})$.*

Proof. We will show that $(J_k(\tau) \oplus \mathrm{C}, \phi \oplus \tilde{1})$ satisfies the universal property of the maximal C* quotient. Suppose A be a C* algebra and Let $\pi : T_k(\tau) \to A$ be a $\star$ algebra homomorphism. By considering the image of π, if necessary, we can assume that π is onto. Then π is unital. Let $p = \bigvee_{i=1}^{k-1} \pi(e_i^T)$. Then p is a central projection in A. Let $\pi_1 : T_k(\tau) \to pA$ be defined by $\pi_1(a) = p\pi(a)$. Then $\bigvee_{i=1}^{k-1} \pi_1(e_i^T) = 1$. Hence by Theorem 2, $\exists$ a map $\tilde{\pi_1} : T_k(\tau) \to pA$ such that $\tilde{\pi_1} \circ \phi = \pi_1$. Now define $\tilde{\pi} : J_k(\tau) \oplus \mathrm{C} \to A$ by

$\tilde{\pi}(a, \lambda) = \tilde{\pi_1}(a) + \lambda(1 - p)$. Since 1 together with nonempty reduced words form a basis for $T_k(\tau)$, it follows that $\pi(a)(1 - p) = \tilde{1}(a)(1 - p)$. Hence $\tilde{\pi} \circ (\phi \oplus \tilde{1}) = \pi$. That such a map is unique follows from the ontoness of $\phi \oplus \tilde{1}$. This completes the proof. $\square$

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