GAPS BETWEEN THE ZEROS OF EPSTEIN’S ZETA-FUNCTIONS ON THE CRITICAL LINE

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Abstract. It is proved that Epstein’s zeta-function \( \zeta_Q(s) \) related to a positive definite integral binary quadratic form has a zero \( \frac{1}{2} + i\gamma \) with \( T \leq \gamma \leq T + T^{5/11 + \varepsilon} \) for sufficiently large positive numbers \( T \). This improves a classical result of H. S. A. Potter and E. C. Titchmarsh.

To Professor K. Ramachandran on his seventieth birthday

1. Introduction

Let \( Q(x, y) = ax^2 + bxy + cy^2 \) be a positive definite integral binary quadratic form and denote by \( r_Q(n) \) the number of solutions of the equation \( Q(x, y) = n \) in integers \( x \) and \( y \). Epstein’s zeta-function for the form \( Q \) is defined by the series

\[
\zeta_Q(s) = \sum_{n=1}^{\infty} r_Q(n)n^{-s},
\]

with \( s = \sigma + it \), in the half-plane \( \sigma > 1 \). It is well-known (see e.g. [6]) that \( \zeta_Q(s) \) can be analytically continued to a meromorphic function satisfying the functional equation

\[
(\sqrt{\Delta}/2\pi)^s \Gamma(s)\zeta_Q(s) = (\sqrt{\Delta}/2\pi)^{1-s} \Gamma(1-s)\zeta_Q(1-s)
\]

with \( \Delta = 4ac - b^2 > 0 \). Its only singularity is a simple pole at \( s = 1 \) with residue \( 2\pi \Delta^{-1/2} \). We suppose throughout the paper that \( \Delta \) is not a square, so that \( \sqrt{\Delta} \) is irrational; other cases like \( \Delta = 4 \) related to the form \( Q(x, y) = x^2 + y^2 \) are easier or well-known.

If \( -\Delta \) is a fundamental discriminant and the class number of the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-\Delta}) \) equals one, then \( \zeta_Q(s) \) is a multiple of the Dedekind zeta-function \( \zeta_K(s) \) of \( K \). The Dedekind zeta-function has an Euler product and it belongs to that class of Dirichlet series for which Riemann’s hypothesis can be reasonably expected. In contrast, if the class number \( h(-\Delta) \) of the quadratic forms with discriminant \( -\Delta \) exceeds one, then \( \zeta_Q(s) \) has no Euler product and Riemann’s hypothesis fails to hold for it. Indeed, S. M. Voronin [12] proved that if \( h(-\Delta) > 1 \), then the number of zeros of \( \zeta_Q(s) \) in the rectangle \( \sigma_1 \leq \sigma \leq \sigma_2, |t| \leq T \), with \( 1/2 < \sigma_1 < \sigma_2 \leq 1 \) and \( T \) sufficiently large, exceeds \( c(\sigma_1, \sigma_2)T \), where \( c(\sigma_1, \sigma_2) > 0 \).

Though \( \zeta_Q(s) \), in general, has thus infinitely many zeros off the critical line \( \sigma = 1/2 \), it has nevertheless infinitely many zeros on the critical line, and in fact

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it was proved by H. S. A. Potter and E. C. Titchmarsh [7] that there is a zero 
\(1/2 + i\gamma\) of \(\zeta_Q(s)\) with \(\gamma \in [T, T + T^{1/2+\varepsilon}]\), for any fixed \(\varepsilon > 0\) and \(T \geq T(Q, \varepsilon)\). As a slight sharpening, A. Sankaranarayanan [9] showed the same for intervals of the type \([T, T + cT^{1/2}\log T]\). Our goal is to improve the exponent of \(T\) below \(1/2\).

\textbf{THEOREM.} Let \(Q\) be a positive definite binary integral quadratic form. Then, for any fixed \(\varepsilon > 0\) and \(T \geq T(\varepsilon, Q)\), there is a zero \(1/2 + i\gamma\) of the corresponding Epstein zeta-function \(\zeta_Q(s)\) with

\[
|\gamma - T| \leq T^{5/11 + \varepsilon}.
\]

The exponent \(5/11\) comes from the estimation of certain double exponential sums, and it could be improved by more sophisticated arguments (see Remark 2 in Sec. 4). But the point is that any nontrivial estimate gives an exponent less than \(1/2\), whereas a trivial estimate leads just to the old exponent \(1/2\). Therefore, for our present purpose of breaking the one half "barrier", it suffices to estimate only one of the two summations nontrivially, by the classical one-dimensional van der Corput method.

For comparison, an analogue of our theorem for a cusp form \(L\)-function, with the exponent \(1/3 + \varepsilon\) in place of \(5/11 + \varepsilon\), was proved in [3]. The key result was a nontrivial estimate for a "critical" Dirichlet polynomial, and for that purpose this Dirichlet polynomial was first transformed by a certain summation formula of the Voronoi type. Likewise, in the present case, we end up with the problem of estimating sums of the type

\[
\sum_n \eta(n)r_Q(n)n^{-1/2-it},
\]

where \(\eta(n)\) is a smooth weight function of support \([T\sqrt{\Delta}/2\pi - K, T\sqrt{\Delta}/2\pi + K]\) and \(t\) lies close to \(T\). If this sum is small in a certain sense, then we may conclude that there is a zero \(1/2 + i\gamma\) of \(\zeta_Q(s)\) with \(|T - \gamma| \leq H = T^{1+\varepsilon}/K\), for \(T\) sufficiently large. Thus the parameter \(K\) should be maximized. Following [3], we first transform the sum (1.3) by a Voronoi formula from [5], and the transformed sum is then estimated nontrivially. The Voronoi formula in question involves an additive character related to a rational number, and a suitable rational number is chosen among the convergents of the continued fraction expansion of the quadratic irrational \(1/\sqrt{\Delta}\). An important property of this continued fraction is its periodicity which allows one to control the denominators of the convergents because the partial quotients are bounded.

The proof of the theorem proceeds in three main steps and follows [3] with certain modifications. First, in Sec. 2, we recall how the basic problem concerning zeros on the critical line can be reduced to the estimation of sums of the type (1.3). Next, in Sec. 3, we work out a transformation formula analogous to those in [4] for such sums. Finally, in Sec. 4, the transformed sum is estimated by van der Corput’s method.

As a notational convention, we are going to keep the form \(Q\) fixed, and the constants implied by the symbols \(O(\cdots), \ll \) or \(\gg\) may depend on \(Q\) (or \(\Delta\)). Otherwise we use standard notation. In particular, we write \(e(\alpha) = e^{2\pi i\alpha}\), \(e_k(\alpha) = e(\alpha/k)\), and the relation \(A \asymp B\) means that \(A \ll B\) and \(B \ll A\). The numbers \(\varepsilon, \delta\) with or without suffixes denote small positive constants.
2. The zero-detecting method

The idea of this method goes back to Hardy and Littlewood. Following [7], define the functions

\[ f(s) = e^{\frac{1}{2} \pi i (\frac{1}{2} - s)} \left( \frac{\sqrt{\Delta}}{T} \right)^s \Gamma(s) \zeta_Q(s) \]

and

\[ W(t) = f \left( \frac{1}{2} + it \right) ; \]

the latter is an analogue of Hardy’s function \( Z(t) \) in the theory of Riemann’s zeta-function. Then, by the functional equation (1.1), we see that \( W(t) \) is real for real \( t \). Thus the zeros of \( \zeta_Q(s) \) on the critical line correspond to the real zeros of \( W(t) \).

Suppose now that \( W(t) \) has no zero in the interval \( [-T + H, T + H] \) with \( T^{2\epsilon} \leq H \leq T^{1/2} \), put \( H_0 = HT^{-\epsilon} \), and consider the integral

\[ I = \int_{-H}^{H} W(T + u)e^{-(u/H_0)^2} du. \]

We are going to estimate \(|I|\) from below and from above to derive eventually a contradiction if \( H = T^{5/11 + \epsilon} \). Since the integrand is of constant sign, we have

\[ |I| = \int_{-H}^{H} |W(T + u)|e^{-(u/H_0)^2} du. \]

First, as an estimate from below, we have

\[ |I| \gg H_0/\log T \]

as a direct analogue of Lemma 4 in [3]. Alternatively, this follows, even in a stronger form, from \( \Omega \)-results of K. Ramachandra (see [8], Chapter II).

Turning to the estimation from above, write

\[ I = \int_{-H}^{H} e^{\frac{1}{2} \pi (T + u)} \left( \frac{\sqrt{\Delta}}{2\pi} \right)^{\frac{1}{2} + i(T + u)} \Gamma \left( \frac{1}{2} + i(T + u) \right) \]

\[ \times \zeta_Q \left( \frac{1}{2} + i(T + u) \right) e^{-(u/H_0)^2} du, \]

and substitute an approximate formula for \( \zeta_Q \left( \frac{1}{2} + i(T + u) \right) \) from the following lemma. It is an analogue of Lemma 3 in [3]; the only new feature is the fact that, unlike the cusp form \( L \)-function, the function \( \zeta_Q(s) \) has a pole at \( s = 1 \), and this accounts for an extra explicit term on the right of (2.3).

**LEMMA 1.** Let \( t \geq 2 \) and \( t^2 \ll X \ll t^4 \), where \( A \) is an arbitrarily large positive constant. Then we have

\[ \zeta_Q \left( \frac{1}{2} + it \right) = \sum_{n \leq X} r_Q(n)n^{-1/2-it} \]

\[ + (\log 2)^{-1} \sum_{X < n \leq 2X} r_Q(n) \log(2X/n)n^{-1/2-it} \]

\[ + (\log 2)^{-1} 2\pi \Delta^{-1/2} \left( \frac{1}{2} - it \right)^{-2} - \frac{X^{1/2-it} + O(tX^{-1/2})}. \]

We apply this with $X = T^3$, for $t \approx T$, so that the last two terms on the right are $\ll T^{-1/2}$. The contribution of these terms to the integral (2.2) is $\ll H_0 T^{-1/2}$.

We are now left with the two sums on the right of (2.3), and from the first one we extract a weighted sum of the type (1.3). We suppose that the weight function $\eta(x)$ satisfies $\eta(x) = 1$ for $|x - T\sqrt{\Delta}/2\pi| \leq K/2$ and $\eta(x) = 0$ for $|x - T\sqrt{\Delta}/2\pi| \geq K$. The parameters $H$ and $K$ are connected by the relation
\begin{equation}
H K = T^{1+2\epsilon}.
\end{equation}

If the critical sum (1.3) involving the weight function $\eta(x)$ satisfying the condition $|t - T| \leq H$ is pulled out, then the remaining terms in (2.3) give rise to oscillating integrals in (2.2) which are negligibly small. As in [3], Sec. 8, this can be seen by invoking Stirling’s formula for the gamma-function and deforming the path of integration appropriately. Or alternatively one may apply repeated integration by parts with respect to the oscillating factors in each of these integrals.

It now remains to choose the parameter $K$ as large as possible under the condition that
\begin{equation}
\sum_n \eta(n) r_Q(n) n^{-1/2} e_k(nh) \ll (\log T)^{-2}
\end{equation}
for $|t - T| \leq H$. Then by (2.2) and the preceding remarks we have $|I| \ll H_0 (\log T)^{-2}$, which contradicts (2.1).

The weight function $\eta(x)$ can be chosen to be of the type $\eta_J(x)$ considered in [4]. This arises as a result of a certain $J$-fold averaging procedure with respect to the limits of summation or integration. The smoothing goes as follows: if the original range of summation is the interval $[a, b]$, we consider sums over $[a + u, b - u]$ with $u = u_1 + \cdots + u_J$ and take the average over all $u_j \in [0, U]$ for a suitable parameter $U$.

3. Summation and transformation formulae

Let $S$ be the sum on the left of (2.5). Introducing an additive character (mod $k$), we may write it formally as
\begin{equation}
S = \sum_n \eta(n) r_Q(n) n^{-1/2} e_k(nh) \cdot e_k(-nh);
\end{equation}
as in [4], the purpose of the extra exponential factor is to damp the oscillations of the original exponential sum before an application of the Voronoi summation. Thus $S$ is of the general form
\begin{equation}
\sum_n r_Q(n) f(n) e_k(hn),
\end{equation}
and a Voronoi summation formula for such sums was given in [5], Eq. (28). To state it, we need some notation. The Gauss sums related to the form $Q$ and additive characters are
\begin{equation}
G_Q(k, h) = \sum_{x,y (mod k)} e_k(hQ(x, y)),
\end{equation}
and it holds (see [10], Lemma 1)
\begin{equation}
|G_Q(k, h)| \leq (\Delta, k) k.
\end{equation}
Further, the summation formula involves an integral positive definite quadratic form $Q^*(x, y)$ depending on $Q$ and $k$, and the discriminant of $Q^*$ is at most $\Delta$ in
absolute value. Also, there occurs an arithmetic function (corresponding to \( r_Q \) (n) in [5]) of the form

\[
\rho(n) = \rho(n; Q, h/k) = \sum_{Q^*(x,y)=n} \alpha(x,y),
\]

where \(|\alpha(x,y)| \leq (\triangle, k) \leq \triangle\) and \(\alpha(x,y)\) depends only on the classes of \(x\) and \(y\) (mod \(\triangle\)) for given \(Q\) and \(h/k\). In this notation, a slightly simplified version of the summation formula (28) in [5] can be stated as follows.

**LEMMA 2.** Suppose that the function \( f \in C^1(\mathbb{R}) \) has a compact support in the positive real numbers. Then for \((h, k) = 1\) and \(k \geq 1\), we have

\[
\sum_n r_Q(n)e_k(nh)f(n) = 2\pi \triangle^{-1/2}k^{-2}G_Q(k, h) \int f(x) \, dx
\]

\[
+2(k^2)^{-1} \sum_{n=1}^{\infty} \rho(n)e_{k_1}(-\bar{h}\bar{\triangle}_0\delta_1^{-1}n) \int J_0 \left(4\pi \frac{nx/\triangle_0}{k}\right) f(x) \, dx,
\]

where \(\rho(n) = \rho(n, Q; h/k)\) is of the form (3.3), and \(\triangle_0|x, y\), \(\delta_1\delta_1|\triangle, k\), \(k_1 = k/\delta_1\), \((\triangle_0, k_1) = 1\), and \(\delta_1\delta_1|\triangle_0\), the bar indicating a multiplicative inverse (mod \(k_1\)).

**REMARK 1.** More specifically (see [5], Eqs. (10) and (11))

\[
\delta_1 = \prod_{ord_p, \triangle > order_k} p^{ord_p, k},
\]

and \(\triangle_0 = \triangle/\delta_0\), with \(\delta_0\delta_1 = (\triangle, k)\). However, these formulae will be irrelevant for our purposes, as will the explicit structure of \(\alpha(x,y)\). A noteworthy property of the form

\[
Q^*(x, y) = a^*x^2 + b^*xy + c^*y^2
\]

is that its coefficients are limited, for given \(Q\), by bounds depending on \(Q\) but not on \(k\).

Lemma 2 should be compared with Theorem 1.7 in [4] containing analogous formulae with \(r_Q(n)\) replaced by the divisor function \(d(n)\) or the Fourier coefficients of a cusp form. In view of this analogy, we may state, *mutatis mutandis*, a transformation formula for the sum (3.1), where \(\eta(n) = \eta_J(n)\) in the notation of [4]. The main difference is that in the treatment of the integrals involving the Bessel function \(J_0\), besides using the appropriate trigonometric approximation for that function, at some places we replace \(h\) and \(k\) formally by \(h\sqrt{\triangle_0}\) and \(k\sqrt{\triangle_0}\), respectively. Since \(\triangle\) is fixed in our argument, this does not cause any new complications. Let \(\chi(s)\) be as in the functional equation \(\zeta(s) = \chi(s)\zeta(1-s)\), thus \(\chi(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)\Gamma(1-s)\). The following Lemma 3 is a direct analogue of Theorem 4.3 in [4].

**LEMMA 3.** Let \(t\) be a large positive number, \(r = h/k\) a positive rational number with \((h, k) = 1\), and suppose that the positive numbers \(M_1\) and \(M_2\) satisfy

\[
M_1 < \frac{t}{2\pi r} < M_2
\]

\[
M_j = \frac{t}{2\pi r} + (-1)^j m_j, \quad m_1 < m_2,
\]

\[
1 \leq k \ll M_1^{1/2-\delta_1},
\]

\[
t^{\delta_1} \max(t^{1/2-1}, hk) \ll m_1 \ll M_1^{1-\delta_3}
\]
for some small positive constants $\delta_j$. Further, let
\[ U \gg r^{-1/2 + \delta_4} \]
and let $J$ be a fixed positive integer exceeding a certain bound which depends on $\delta_4$. Write
\[ M'_j = M_j + (-1)^{j-1} JU = \frac{t}{2\pi r} + (-1)^j m'_j, \]
supposing that $m_j \simeq m'_j$, and let
\[ n_j = \triangle_0 h^2 m_j^{-1}, \quad n'_j = \triangle_0 h^2 (m_j')^{-1}. \]
Then for a certain weight function $\eta \in C^{1-1}(\mathbb{R})$ with support $[M_1, M_2]$ and satisfying $\eta(x) = 1$ for $x \in [M'_1, M'_2]$, we have
\[
\sum_{n=1}^{\infty} \eta(n) rQ(n) n^{-1/2} = \left( 2\pi \triangle^{-1/2} k^{-2} G_Q(k,-h)n^{-1/2} + \pi^{1/4} i (2hkt \triangle_0)^{-1/4}(\triangle_0/\triangle)^{1/2} \sum_{j=1}^{2} (-1)^j x(n_j) \rho(n) \right. \\
\times e \left( n \left( 2hkt \triangle_0 \right) \right) n^{-1/4} \left( 1 + \frac{\pi n}{2hk \triangle_0} \right)^{-1/4} \\
\times \exp \left( i(-1)^{j-1} \left( 2t \phi \left( \frac{\pi n}{2hk \triangle_0} \right) + \frac{\pi}{4} \right) \right) \left\{ \frac{1}{2} + it \right\} \\
+ O(h^2 k^{-1} m_1^{1/2} t^{-3/2} U \log t),
\]
where
\[ \phi(x) = \text{arsinh}(x^{1/2}) + (x + x^2)^{1/2}, \]
$w_j(n) = 1$ for $n < n'_j$, $w_j(n) \ll 1$ for $n \leq n_j$, $w_j(y)$ and $w'_j(y)$ are piecewise continuous in the interval $(n'_j, n_j)$ with at most $J - 1$ jumps, and
\[ w'_j(y) \ll (n_j - n'_j)^{-1} \quad \text{for } n'_j < y < n_j \]
whenever $w'_j(y)$ exists.

4. Proof of the theorem

In view of the relation (2.4), it now remains to show that (2.5) holds for
\[ K = T^{6/11-\epsilon}. \]
However, to begin with, we keep $K$ unspecified and suppose only that
\[ T^{1/2 + \delta} \ll K \ll T^{1-\delta} \]
for some small number $\delta > 0$ and choose
\[ U = T^{1/2 + \delta/2} \]
in Lemma 3. In the notation of that lemma, we have
\[ M_1 = \frac{T \sqrt{\triangle}}{2\pi} - K, \quad M_2 = \frac{T \sqrt{\triangle}}{2\pi} + K. \]

We choose a convergent $r = h/k$ for $1/\sqrt{\triangle}$ in such a way that
\[ \frac{K}{T} \ll \left| \frac{1}{\sqrt{\triangle}} - \frac{h}{k} \right| \ll \frac{\pi K}{T \triangle}. \]
and

\[(4.3) \quad k \asymp \sqrt{T/K}.
\]

By the recursion formula for the convergents and their well-known approximation properties, this is possible, because the continued fraction of \(1/\sqrt{\Delta} \) is periodic and its partial quotients are thus bounded. It is now easy to check that the conditions of Lemma 3 are fulfilled for \(|t - T| \leq H\). We observe that in Lemma 3, \( r = h/k \) behaves like a constant and \( m_1 \sim m_2 \sim K \). The error term in (3.6) is \( \ll T^{-1/2 + \varepsilon} \), which is negligible. Also, the leading term is small by (3.2) and our choice of \( k \).

Consider finally the sums of length \( n_j \) in (3.6). Note that \( n_j \asymp K \) by (3.5). By the definition (3.3) of \( \rho(n) \), we may replace \( n \) by \( Q^*(x, y) \) if we restrict \( x \) and \( y \) temporarily to some classes \((\text{mod} \ \Delta)\); since \( \Delta \) is fixed, our sums will be decomposed into finitely many parts which can be estimated individually.

By partial summation and the properties of the weight functions \( w_j \), it suffices to estimate sums of the form

\[
K^{1/4}N^{-1/4}T^{-1/2} \sum_{N \leq Q^*(x, y) \leq N'} e\left( Q^*(x, y) \left( \frac{h \Delta_0}{k} - \frac{1}{2hk\Delta_0} \right) \right) \times \exp\left( 2ti \phi \left( \frac{\pi Q^*(x, y)}{2hk\Delta_0} \right) \right),
\]

where \( 1 \leq N < N' \leq 2N \) and \( N \ll K \), with \( x \) and \( y \) restricted \((\text{mod} \ \Delta)\) as mentioned above.

Here we apply the following classical estimate for exponential sums (see [11], Theorem 5.11).

**LEMMA 4.** Let \( f(x) \) be real and have continuous derivatives up to the third order, and let \( \lambda_3 \leq f'''(x) \leq h\lambda_3 \), or \( \lambda_3 \leq -f'''(x) \leq h\lambda_3 \), and \( b - a \geq 1 \). Then

\[
\sum_{a < n \leq b} e(f(n)) \ll h^{1/2}(b - a)\lambda_3^{1/6} + (b - a)^{1/2}\lambda_3^{-1/6}.
\]

Because the third partial derivatives of \( Q^*(x, y) \) vanish, the relevant oscillating factor in (4.4) is

\[
\exp\left( 2ti \phi \left( \frac{\pi Q^*(x, y)}{2hk\Delta_0} \right) \right) = e^{iF(x, y)},
\]

say. Note that for small \( u \)

\[
\phi(u) = 2u^{1/2} \left( 1 + \frac{1}{6}u + \cdots \right) \approx 2u^{1/2}.
\]

Since

\[
\pi Q^*(x, y) \over 2hk\Delta_0 \ll NK^{-2} \ll K^2T^{-2},
\]

the approximation (4.6) makes sense in (4.5). Now, in the notation (3.4) and with \( \Delta^* = 4a^*c^* - b^* \), we have

\[
\frac{\partial^2(Q^*(x, y)^{1/2})}{\partial x^3} = -3 \frac{\Delta^* y^2(2a^* x + b^* y)}{8 \ (Q^*(x, y))^{5/2}},
\]

\[
\frac{\partial^3(Q^*(x, y)^{1/2})}{\partial y^2} = -3 \frac{\Delta^* x^2(2e^* y + b^* x)}{8 \ (Q^*(x, y))^{5/2}}.
\]
For a point $(x_0, y_0)$ in (4.4), we have $|x| + |y| = N^{1/2}$, and then at least one of the derivatives (4.7) and (4.8) is $\asymp N^{-1}$ in absolute value, for the norm of the linear mapping with the matrix

$$
\begin{pmatrix}
2a^* & b^* \\
b^* & 2c^*
\end{pmatrix}
$$

is $\asymp 1$. Moreover, since the corresponding derivatives of the function $Q^*(x, y)^{k/2}$ are $\ll N^{-1+(k-1)/2}$, for $k = 1, 3, 5, \ldots$, at least one of the derivatives $F_{xxx}(x_0, y_0)$ and $F_{yyy}(x_0, y_0)$ is $\asymp K^{1/2}N^{-1}$. By continuity, the same holds in a certain square with side $\asymp N^{1/2}$ containing $(x_0, y_0)$. Restricting the summation temporarily to such a square, we apply Lemma 4 to one of the sums over $x$ or $y$ with $\lambda_3 \asymp K^{1/2}N^{-1}$, and the other sum is estimated trivially. In this way, we see that the corresponding part of the sum (4.4) is

$$
\ll K^{1/4}N^{1/4}T^{-1/2}N^{1/3}K^{1/12} + N^{5/12}K^{-1/12} \ll K^{11/12}T^{-1/2},
$$

which is small if we choose $K$ as in (4.1). Of course it was unimportant that $x$ and $y$ run over progressions rather than over consecutive integers. This completes the proof of the theorem.

REMARK 2. The preceding estimation of the sum (4.4) can be slightly sharpened by the method of exponent pairs. Referring to Sec. 2.3 in [2], consider exponential sums of the form

$$
S = \sum_{B < n \leq B+h} e(f(n)) \quad (B \geq 1, 1 < h \leq B).
$$

Suppose that the derivatives of $f(x)$ for $B \leq x \leq 2B$ satisfy

$$
|f^{(r)}(x)| \geq AB^{1-r} \quad (r = 1, 2, \ldots),
$$

where the constants implied by the notation depend only on $r$. Then, if $(\kappa, \lambda)$ is an exponent pair, we have

$$
S \ll A^\kappa B^\lambda.
$$

In our case, we have $A = K^{1/2}$ and $B = N^{1/2}$. Hence the sum (4.4) is

$$
\ll K^{1/4+\kappa/2}N^{1/4+\lambda/2}T^{-1/2} \ll K^{1/2+\kappa+\lambda}T^{-1/2}.
$$

The exponent pair which minimizes $\kappa + \lambda$ has been determined by R. A. Rankin (see [2], p. 77), and it is $(\frac{1}{3}\alpha + \epsilon, \frac{1}{3}\alpha + \epsilon)$ with $\alpha = 0.3290213568\ldots$. Hence the preceding estimate is $\ll K^{3/4+\alpha/2+\epsilon}T^{-1/2}$. We choose now $K = T^\beta$ with $\beta$ slightly less than $2(3 + 2\alpha)^{-1}$, and then $H = T^{0.4532595\ldots}$, whereas $5/11 = 0.454545\ldots$.

REMARK 3. An ideal class zeta-function for an imaginary quadratic field is essentially an Epstein zeta-function, so that our theorem can be interpreted in the language of quadratic fields. An interesting problem now is to extend the result to real quadratic fields. The above mentioned gap theorem of Sankaranarayanan holds also in this case, and this is the difficult part of his paper [9]. An analogue of the Potter-Titchmarsh theorem was previously shown by Bruce C. Berndt [1].

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Gaps between the zeros of Epstein’s zeta-function on the critical line

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