A Dynamical View of Different Solution Paradigms in Two-Person Symmetric Games: Nash Versus Co-action Equilibria

V. Sasidevan and Sitabhra Sinha

Abstract The study of games and their equilibria is central to developing insights for understanding many socio-economic phenomena. Here we present a dynamical systems view of the equilibria of two-person, payoff-symmetric games. In particular, using this perspective, we discuss the differences between two solution concepts for such games—namely, those of Nash equilibrium and co-action equilibrium. For the Nash equilibrium, we show that the dynamical view can provide an equilibrium refinement, selecting one equilibrium among several possibilities, thereby solving the issue of multiple equilibria that appear in some games. We illustrate in detail this dynamical perspective by considering three well known 2-person games namely the Prisoner's Dilemma, game of Chicken and the Stag-Hunt. We find that in all of these cases, co-action equilibria tends to correspond to 'nicer' strategies than those corresponding to Nash equilibria.

1 Introduction

Games represent strategic interactions between entities generally referred to as agents. Here, the term "agents" could refer to a variety of entities, ranging from human beings or animals to computer programs or robots. In games, each agent receives a payoff depending upon the strategy choice made by all agents including herself. Thus, an agent who wants to optimize her payoff should consider not only the payoff structure of the game, but also the decision making processes of other agents. The choice of strategy by each agent in such an interaction leads to a collective outcome that may or may not be globally optimal. In this context, it is imperative to understand how two agents facing a game situation, who have to make a strategic decision, will go about doing it, since the strategic interaction between agents is

V. Sasidevan (✉) · S. Sinha
The Institute of Mathematical Sciences, CIT Campus, Taramani,
Chennai 600113, India
e-mail: sasidevan@imsc.res.in

S. Sinha
e-mail: sitabhra@imsc.res.in

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the basis of the collective behavior in a system comprising such agents. Financial markets, for example, may be viewed as the collective outcome of strategic interactions between a large number of people participating in it. Another example is that of evolution, where one may view evolution by natural selection as a result of the interaction between competing genes. Cooperation and conflict is at the heart of such systems and forms the subject matter of the study of games. In games, in general, each agent should have a behavior model of other agents so that she has a way to describe the decision making process of other agents. In this regard, standard game theory makes several assumptions about the agent's behavior. It assumes that agents are fully rational and would like to optimize their payoff and they are perfect in execution of their strategies (see for e.g. [1] for a detailed discussion). While the applicability of these assumptions in any particular situation is open to criticism, they form an important benchmark for optimal behavior. In fact, these assumptions form an important part of modern economic theory in which the participating agents are often assumed to be fully rational.

The simplest of games consists of the strategic interaction between two agents in a single play of the game. In fact, 2-person games like Prisoners Dilemma, Stag-Hunt etc., describe very general socio-economic scenarios, towards the analysis of which considerable effort has been devoted. A key concept in the study of games is that of an "equilibrium". It refers to a state of affairs where each agent has decided her strategy for the game at hand. How the agents pick their equilibrium strategy is given by a solution concept. A solution concept thus is a formal rule for predicting how a game will be played between agents and employs certain assumptions regarding agent's behavior. An important solution concept for non-cooperative games is that of Nash equilibrium. Informally, it is a state where after every agent has selected their 'Nash' strategies, none of the agents can improve their payoff by unilaterally deviating from it. It is to be noted that a game may have more than one Nash equilibrium.

In this article, we show that the equilibria of a game may be viewed as the "fixed-point" equilibria of a dynamical system. In particular, we present a dynamical view of the equilibria obtained by two different solution concepts, viz., Nash [2] and co-action [3], the latter being a concept that makes use of the symmetry between the agents for payoff-symmetric games. The vector flow diagrams on the strategy space that is generated using the dynamics approach makes the differences between the equilibria obtained in the two solution concepts visually apparent. For the Nash equilibrium, we argue that a dynamical perspective may be regarded as an equilibrium refinement selecting one equilibrium out of several possible ones, thus solving the multiplicity issue. We illustrate these points by considering three well known examples of 2-person games, namely the Prisoners Dilemma, Game of Chicken and Stag-Hunt.

2 A Dynamical Framework for Analysing 2-Person Games

Here we describe a dynamical perspective for analyzing games, focusing on 2-person single-stage games in which two agents interact only once. No communication is allowed between the agents. Furthermore, we consider the simple case where each agent has to choose one of two possible actions (say, Action 1 and Action 2) available to her. Each agent receives a payoff according to the pair of choices made by them, such that the game may be represented by a payoff matrix that specifies all possible outcomes (Fig. 1). We consider situations where the game is payoff symmetric, i.e., on exchanging the identities of the players (A, B), the payoff matrix remains unchanged. Note that most 2-person games that are studied in the literature fulfil the above criteria. Given the payoff matrix, an agent can have a mixed strategy, where she chooses Action 1 with some probability p and Action 2 with probability $(1 - p)$. If p is either 0 or 1, it is called a pure strategy. Given a game, represented by a matrix containing the numerical values of R, S, T and P (or a hierarchical relation among them), Nash equilibrium is defined as a state—i.e., a set of the choices made by all the agents—where no agent can increase her payoff by unilaterally deviating from the Nash state. A Nash equilibrium comprising pure strategies may be found by a search procedure, whereby each possible state is explicitly examined for the above criterion. Note that a given game can have more than one Nash equilibrium, possibly involving mixed strategies. In such cases, the choice of a particular equilibrium will have to involve additional refinement criteria, which is an important area of research in game theory [4].

We now illustrate a dynamical perspective on Nash equilibria by first defining payoff functions for all possible mixed strategies of the two agents. Assuming that agent A (B) chooses Action 1 with probability p_1 (p_2) and Action 2 with probability $1 - p_1$ ($1 - p_2$, respectively), the expected payoffs of the agents are

$$W_A = p_1 p_2 R + p_1(1 - p_2)S + (1 - p_1)p_2 T + (1 - p_1)(1 - p_2)P, \qquad (1)$$

$$W_B = p_1 p_2 R + p_1(1 - p_2)T + (1 - p1)p_2 S + (1 - p1)(1 - p2)P. \qquad (2)$$

As the payoffs are continuous functions of p_1 and p_2, they can be represented as two-dimensional surfaces (Fig. 2) analogous to fitness landscapes in biology or energy landscapes in physics. However, unlike the latter, there are two distinct surfaces

		Agent B	
		Action 1	Action 2
Agent A	Action 1	(R,R)	(S,T)
	Action 2	(T,S)	(P,P)

Fig. 1 A generic representation of the payoff matrix for a 2-person symmetric game where each agent has two actions available to her

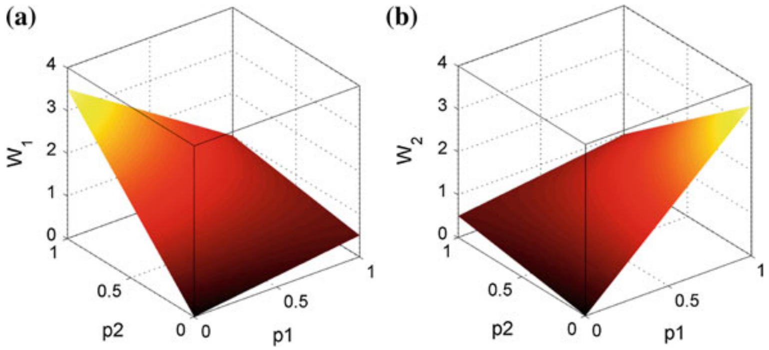

Fig. 2 The payoff functions W_A and W_B for two agents playing the game of Chicken, shown as functions of p_1 and p_2, i.e., the probability of each agent to choose Action 1. The payoffs (in terms of the terminology given in Fig. 1) are $T = 3.5$, $R = 1$, $S = 0.5$ and $P = 0$

for the two agents, and each of them would like to achieve the maximum of their respective payoff functions, a goal that may not be mutually compatible. By contrast, the evolution of the state of a physical system can be seen as a convergence process to a minimum of a single function, e.g., the free energy that describes the entire system.

Given the payoff function surfaces we can now proceed to find the strategy pairs (p_1^*, p_2^*) that correspond to a Nash equilibrium. Note that while the Nash solution is usually not defined in terms of a dynamical perspective, one can view (p_1^*, p_2^*) as an equilibrium point for flow dynamics in the $p_1{-}p_2$ plane, as described below. The initial condition for this dynamical system can be any arbitrary point in this plane. Each agent is then allowed to change its strategy infinitesimally (i.e., $p_1 \to p_1 + dp_1$, $p_2 \to p_2 + dp_2$) in order to improve their respective payoffs, taking into consideration that the other agent would also be doing the same. A sequence of such incremental changes, which will be manifested as a flow in the p_1–p_2 plane would eventually converge to an equilibrium point (p_1^*, p_2^*). Note that, while such a strategy would correspond to a stable equilibrium of the flow dynamics, there may also be unstable equilibria.

The dynamical equations governing the flow can be derived by considering the change in the payoffs (dW_A, dW_B) of the two agents as a result of the infinitesimal change in their strategies dp_1, dp_2:

$$\begin{aligned} \frac{\partial W_A}{\partial p_1} &= p_2(R-T) + (1-p_2)S, \\ \frac{\partial W_B}{\partial p_2} &= p_1(R-T) + (1-p_1)S. \end{aligned} \tag{3}$$

Thus, on any point in the p_1–p_2 plane, the magnitude and direction of the flow can be obtained by a vector sum of the two components given by Eq. 3. The resulting flow diagram will describe the trajectory in strategy space starting from any arbitrary strategy pair (p_1, p_2). This will be illustrated with specific examples of 2-person games in the next section.

As mentioned earlier, Nash equilibrium is not the only possible solution of a payoff symmetric game. Recently, an alternative paradigm referred to as co-action equilibrium for solving such games has been introduced in the specific context of minority game [3]. Here we study this novel solution concept in the context of generic 2-person games with symmetric payoff from a dynamical perspective. The key notion of co-action equilibrium is: as the two agents are aware that they face an exactly symmetric situation, the choice made by agent A should be identical to the choice of agent B, assuming that they are equally rational (for a detailed discussion see Ref. [5]). Thus, in terms of the flow dynamics introduced above, in this solution concept, each agent will take into account in her calculation for revising her strategy that the other agent is not only using the same strategy (i.e., $p_1 = p_2$) but will also make exactly the same infinitesimal change, i.e., $dp_1 = dp_2$. Then the change in the payoffs of the two agents, as a result of changing p_1, p_2 (analogous to Eq. 3 for Nash equilibrium) is:

$$\frac{\partial W_A}{\partial p_1} = 2p_1 R + (1 - 2p_1)(T + S), \tag{4}$$
$$\frac{\partial W_B}{\partial p_2} = 2p_2 R + (1 - 2p_2)(T + S).$$

Note that the above equations hold not only when $p_1 = p_2 = p$ (so that the dynamics is confined to the diagonal line in the $p_1 - p_2$ plane), but also for situations where the two agents initially start with different strategies ($p_1 \neq p_2$), believing however that the other agent is using exactly the same strategy.

The co-action solution yields results that differ remarkably from those obtained using the concept of Nash equilibrium, some of which will be described in the next section in the context of specific 2-person games. An important distinction is that while there could be multiple Nash equilibria for a game, the corresponding co-action equilibrium is unique. The dynamical perspective allows us to also distinguish between Nash and co-action solutions for 2-person symmetric games in that a stable mixed strategy equilibrium is possible for the latter unlike in the former (Nash) where a mixed strategy equilibrium, if it exists, is always unstable.

Note that while the flow diagrams produced by the dynamical process presented here may resemble the trajectories generated by solving replicator equations [6], the two approaches are essentially distinct. In particular, the latter approach is based on the concept of evolutionary stable strategies, which is an equilibrium refinement of the Nash solution. Also, instead of being stages in the evolutionary progression of a population, the sequence of infinitesimal changes in strategies in the dynamical approach presented here, can be interpreted as steps in the deductive reasoning of the two agents, at the end of which they choose the strategy corresponding to the equilibrium they converge to. When our approach is applied to study the Nash solution of a game, it can also be viewed as an equilibrium refinement as, if there are multiple

Nash equilibria, it allows agents to choose a particular equilibrium depending on the arbitrarily chosen initial state. Thus, in an ensemble of many realizations of a game, the fraction of cases where agents will converge to a particular equilibrium is proportional to the size of its basin of attraction. An unstable equilibrium (if it exists) will lie on the separatrix that demarcates the basins of different stable equilibria.

3 Examples

We now illustrate the approach outlined in the previous section using three well-studied 2-person symmetric games, viz., Prisoner's Dilemma, Game of Chicken and Stag-Hunt, each of which can be connected to real-life scenarios involving interactions between a pair of agents who have to choose between two possible actions. Each of these games is defined in terms of a specific hierarchical relationship between the payoffs R, S, T and P (using the terminology of the payoff matrix in Fig. 1). Without loss of generality, we can set $P = 0$ and $R = 1$ (thereby fixing the origin and the scale for the payoffs), leaving only S and T as free parameters. In the following subsections, we discuss each of these games in turn, describing the meaning of the different choices available to the agents (viz. Action 1 and Action 2) in a particular game, and exploring the different equilibria obtained by using Nash and co-action solution concepts.

3.1 Prisoner's Dilemma

Prisoner's Dilemma (PD) [7] can be regarded as one of the most well-known games in the literature. It has evoked great interest among researchers from a multitude of disciplines ranging from social sciences and politics to biology and physics, from the 1950s onwards and continues to do so at present (a good place to read about historical developments in PD is the corresponding entry in the online Stanford Encyclopedia of Philosophy [8]). The game represents the strategic interaction between two players who have to choose between cooperation (Action 1) and defection (Action 2). The different payoffs are interpreted as follows: R is a "reward" for both players cooperating, P is a "punishment" for both players defecting, while, in the event that one agent defects while the other cooperates, T and S are the "temptation" received by former and the "sucker's payoff" of the latter. In PD, the hierarchical relation between the payoffs is $T > R > P > S$, which makes achieving mutual cooperation non-trivial as each player will benefit more by defecting (assuming that the other will cooperate).

It is easy to see that mutual defection is the only Nash equilibrium for PD. As Action 1 represents cooperation, p_1 (p_2) corresponds to the probability that agent A (B) will choose cooperation. As discussed in the previous section, we can associate a vector with each point in the (p_1, p_2) plane for the game which describes the flow from that point. Figure 3 shows the resulting flow diagrams obtained using the Nash

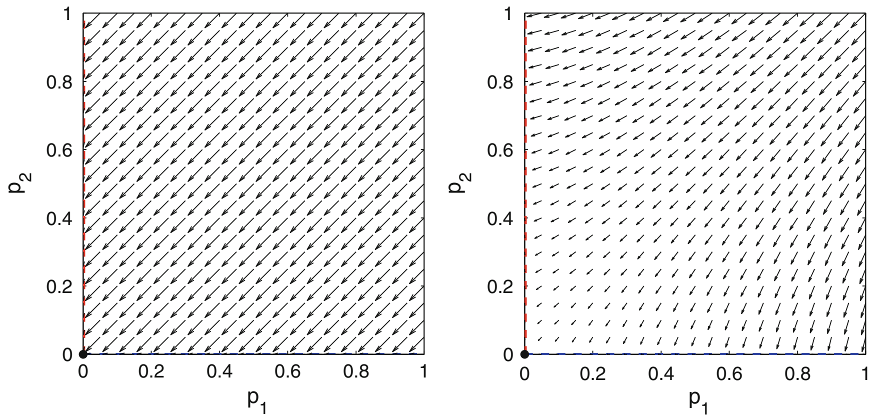

Fig. 3 Vector flow diagram representation of the Nash solution of the 2-person Prisoner's Dilemma game for temptation payoffs (*left*) $T = 1.5$ and (*right*) $T = 3.5$. The abscissae and ordinate correspond to the probabilities (p_1 and p_2) that players 1 and 2, respectively, choose to cooperate. The *broken lines* represent the best response (or reaction) correspondence of the players (*red* for player 1, *blue* for player 2). The intersection of the lines, represented by a *filled circle*, represent the single Nash equilibrium corresponding to both players defecting (i.e., $p_1 = 0$, $p_2 = 0$)

solution concept for two different values of the temptation payoff T (keeping S fixed at -0.5). In both cases, the system converges to the pure strategy $p_1 = p_2 = 0$ (mutual defection), which is the Nash equilibrium for PD.

By contrast, using the co-action solution concept, for low values of T we observe mutual cooperation (i.e., $p_1 = p_2 = 1$) as the stable equilibrium of the system (Fig. 4, left). For larger values of T, the stable equilibrium corresponds to a mixed strategy, $0 < p_1 = p_2 < 1$ (Fig. 4, right). Thus, as discussed in detail in Ref. [5], using the co-action concept for solving PD we can show that selfish agents trying to maximize their individual payoffs can also achieve the state of maximum collective benefit. This resolves a contentious aspect associated with the Nash solution of PD, where the agents end up worse off in trying to optimize their individual payoffs [9].

3.2 Chicken

The Game of Chicken (also referred to as Snowdrift) [2] is another well-studied 2-person game which is relevant in the context of social interactions [10] as well as evolutionary biology [11] (where it is also known as Hawk-Dove). The game represents a strategic interaction between two players, who are driving towards each other in a potential collision course, and have the choice between "chickening out", i.e., swerving away from the path of the other (Action 1) or continuing on the path (Action 2). Thus, the choices correspond to being docile or aggressive, respectively. If both players decide to swerve away, they receive the payoff R, while if one swerves

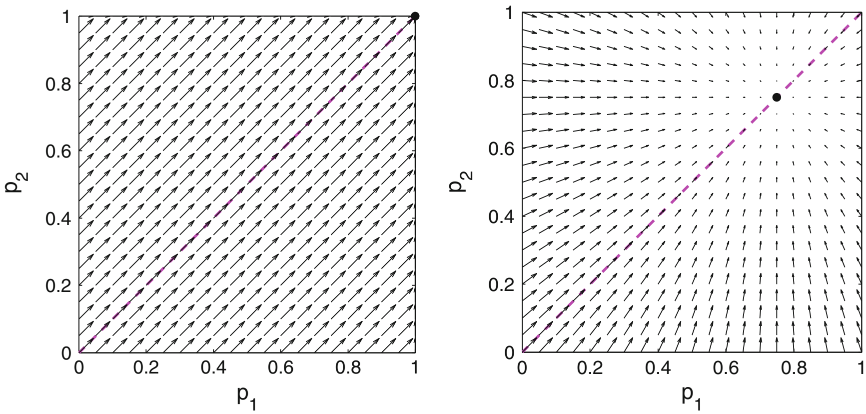

Fig. 4 Vector flow diagram representation of the co-action solution of the 2-person Prisoner's Dilemma game for temptation payoffs (*left*) $T = 1.5$ and (*right*) $T = 3.5$. The abscissae and ordinate correspond to the probabilities (p_1 and p_2) that players 1 and 2, respectively, choose to cooperate. The *broken line* represents the situation where the two agents have the same probability of cooperation. The *filled circles* represent the unique co-action equilibrium for each value of T corresponding to the players cooperating with equal probability ($=1$ for $T = 1.5$ and $= 0.75$ for $T = 3.5$)

and the other continues on the path, the former loses face (getting the payoff S) and the latter wins (payoff T). However, the worst possible outcome corresponds to when both players continue on the path, eventually resulting in a collision which is associated with payoff P. The hierarchical relation between the payoffs in Chicken is $T > R > S > P$, which suggests that a player will benefit from being aggressive as long as the other is docile, but is better off being docile if it is sure that the other will play aggressively, as the cost of mutually aggressive behavior is high.

Analyzing this game using the dynamical perspective described earlier yields the flow diagram shown in Fig. 5 (obtained for two different values of T, with $S = 0.5$) on using the Nash solution concept. As can be seen, two of the multiple Nash equilibria are stable, corresponding to the pure strategies (i) $p_1 = 1$, $p_2 = 0$ and (ii) $p_1 = 0$, $p_2 = 1$ (i.e., when one player is aggressive, the other is docile). The remaining equilibrium is an unstable mixed strategy located on the $p_1 = p_2$ line (which defines the separatrix demarcating the basins of attraction of the two stable equilibria). With increasing T, the unstable equilibrium—which dynamically corresponds to a saddle point in the p_1–p_2 plane—moves closer to $p_1 = 0$, $p_2 = 0$ corresponding to mutual aggression.

Using the co-action solution concept gives rise to a qualitatively different solution, as seen in the flow diagrams in Fig. 6. When T is low, the system has a stable equilibrium at $p_1 = 1$, $p_2 = 1$, i.e., both agents choose docile behavior to avoid the potential damages associated with mutual aggression. For higher values of T the stable equilibrium is a mixed strategy $0 < p_1 = p_2 < 1$. As in PD, the co-action paradigm yields a single, stable solution of the game.

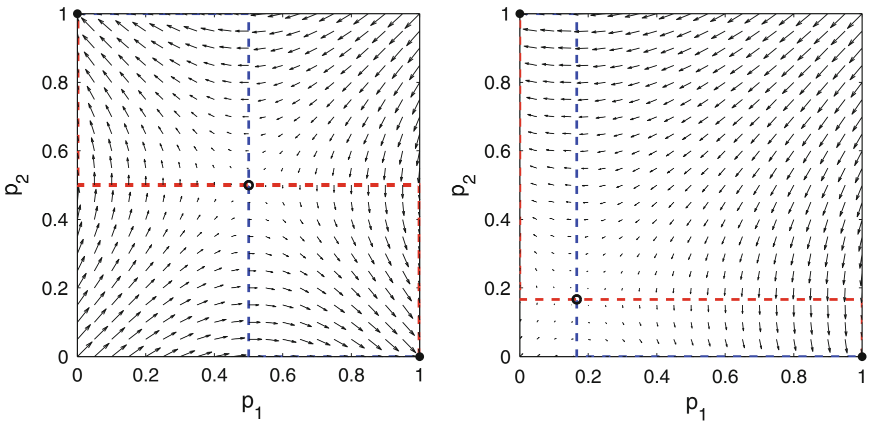

Fig. 5 Vector flow diagram representation of the Nash solution of the 2-person Chicken game for "temptation" payoffs (*left*) $T = 1.5$ and (*right*) $T = 3.5$. The abscissae and ordinate correspond to the probabilities (p_1 and p_2) that players 1 and 2, respectively, choose to be docile (i.e., non-aggressive). The *broken lines* represent the best response (or reaction) correspondence of the players (*red* for player 1, *blue* for player 2). The intersections of the lines, represented by unfilled and *filled circles*, represent the unstable and stable Nash equilibria respectively. The stable equilibria correspond to the pure strategy combination corresponding to one player being aggressive, the other being docile, while the unstable equilibrium in each case corresponds to a mixed strategy

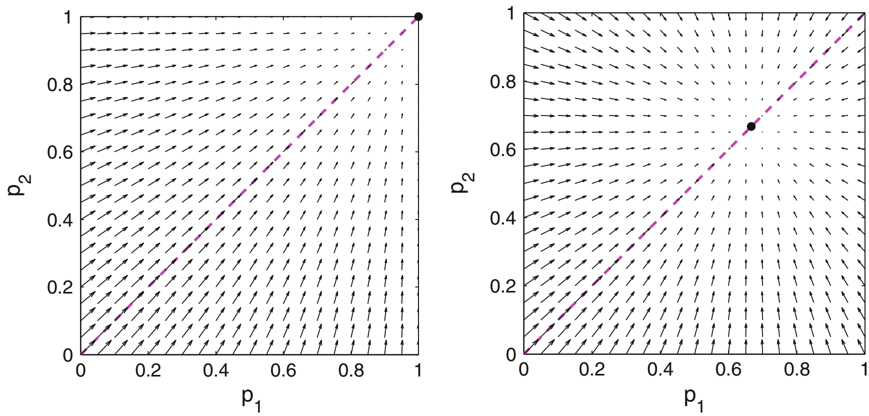

Fig. 6 Vector flow diagram representation of the co-action solution of the 2-person Chicken game for "temptation" payoffs (*left*) $T = 1.5$ and (*right*) $T = 3.5$. The abscissae and ordinate correspond to the probabilities (p_1 and p_2) that players 1 and 2, respectively, choose to be docile (i.e., non-aggressive). The *broken line* represents the situation where the two agents have the same probability of being docile. The *filled circles* represent the unique co-action equilibrium for each value of T corresponding to the players choosing to be docile with equal probability ($=1$ for $T = 1.5$ and $= 2/3$ for $T = 3.5$)

3.3 Stag-Hunt

The last of our examples, Stag-Hunt is a 2-person game that has been studied in the context of emergence of coordination in social interactions [12]. The game represents a strategic interaction between two players who have to choose between hunting stag (Action 1) or hunting hare (Action 2). A hare may be caught by a single agent but is worth less than a stag. On the other hand, hunting a stag is successful only if both agents hunt for it. Thus, if both agents cooperate by hunting stag they receive the highest payoff R. On the other hand, if they both choose to hunt hare, they receive the payoff P. However, if one chooses to hunt hare while the other goes for a stag, then the former receives the payoff T while the latter receives the worst possible payoff S. Thus, in Stag-Hunt, the hierarchical relation between the payoffs is $R > T \geq P > S$, which suggests that while choosing to hunt hare may be the safer option, there is a possibility of doing much better by choosing to hunt stag if one is confident that the other will also do the same.

The vector flow diagrams for Nash and co-action solution concepts in the Stag-Hunt are shown in Fig. 7 (obtained for $T = 0.5$ and $S = -0.5$). For Nash, as in the game of Chicken, there are three equilibria (Fig. 7, left), of which the pure strategies, corresponding to (i) $p_1 = 1$, $p_2 = 1$ and (ii) $p_1 = 0$, $p_2 = 0$ are stable (i.e., when both players hunt for stag or when both players hunt hare). The remaining equilibrium is an unstable mixed strategy located on the $p_1 = p_2$ line which again defines the separatrix demarcating the basins of attraction of the two stable equilibria.

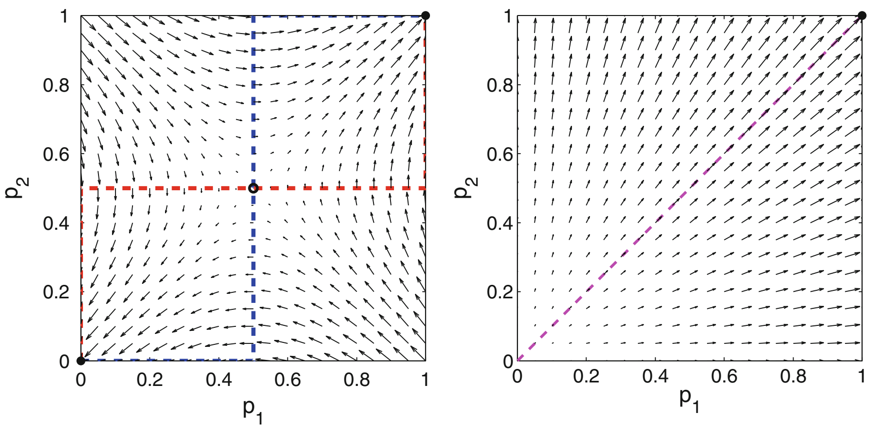

Fig. 7 Vector flow diagram representation of the (*left*) Nash and (*right*) co-action solutions of the 2-person Stag-Hunt game for $T = 0.5$ and $S = -0.5$. The abscissae and ordinate correspond to the probabilities (p_1 and p_2) that players 1 and 2, respectively, choose 'Stag' instead of 'Hare'. (*left*) The *broken lines* represent the best response (or reaction) correspondence of the players (*red* for player 1, *blue* for player 2). The *broken line* represents the situation where the two agents have the same probability of choosing 'stag'

The co-action solution for the games (Fig. 7, right) is a simple one in which both agents always choose hunting stag. i.e., $p_1 = p_2 = 1$. Thus, under the co-action concept, the players always converge to the best possible outcome. In this case, there is no mixed strategy equilibrium for any value of the parameters.

4 Conclusions

In this article we have shown that using a dynamical perspective allows us a visually appealing way to differentiate between two solution concepts, viz., Nash and co-action, for 2-person, symmetric games which lead to spectacularly different conclusions. To illustrate these differences in details we looked at three examples of such games in detail: Prisoners Dilemma, Chicken and Stag-Hunt. In all of these games, one action—in particular, Action 1 in the terminology used here—corresponds to the players being "nicer" to each other (e.g., cooperating in PD, etc.) compared to the other action. The vector flow diagrams generated by the approach presented here clearly show that co-action more often results in nicer strategies being converged at by the agents than in the case for Nash. Our results are intriguing in view of the experimental literature on 2-person games (see discussion in Ref. [9]), in particular PD, which seems to suggest that when these games are played between real human individuals they tend to be far nicer than suggested by the Nash solution.

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