Quantum Games

Diptiman Sen

Indian Institute of Science, Bengaluru

Email: diptiman@iisc.ac.in

Outline

- Introduction to qubits and game theory
- Game I: coin flipping
- Quantum error correction codes
- Game II: cats and dogs questions
- Clauser-Horne-Shimony-Holt and Bell inequalities
- Game III: the prisoner's dilemma
- Nash equilibrium and Pareto optimality
- Quantum version of the prisoner's dilemma

Qubit

A qubit (quantum bit) is a two-state quantum mechanical system. Denote the two basis states as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The most general possible state of the system is a superposition

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where we always fix $|\alpha|^2 + |\beta|^2 = 1$ so that the total probability to find the system in one of the two states is 1

Example: a spin-1/2 object, where the two basis states correspond to the object having the component of its spin in the z direction being equal to $\hbar/2$ and $-\hbar/2$ respectively

Game theory

von Neumann and Morgenstern, Theory of Games and Economic Behavior (1944)

Consider a game with two players 1 and 2 and many possible strategies for each player

For each strategy $S_i = (s_{i1}, s_{i2})$, there is a pay-off to the players $a = 1, 2$ given by (p_{i1}, p_{i2}), where the p_{ia} are real numbers

Each player tries to maximise their pay-off

The players may or may not cooperate with each other to maximise their pay-offs (cooperative or non-cooperative games)

A zero sum game is one in which $p_{i1} + p_{i2} = 0$ for all strategies S_i

Game theory $\cdots$

We will consider situations where a game is played a large number of times. The pay-off will then be taken to be the average of the play-offs in all the games

Classically, this allows us to consider mixed strategies where the players can randomly change their strategies from one game to the next, instead of considering only pure strategies where they make the same move in every game

Quantum mechanically also, playing games a large number of times and then taking the average makes sense since quantum mechanics is a probabilistic theory in which successive observations can give different results

Game I: coin flipping

Meyer, Phys. Rev. Lett. 82, 1052 (1999)

The game is as follows. There are two players, C (classical) and Q (quantum)

A coin is placed head up inside a box

First Q does something to the coin without looking at it

Then C either flips or does not flip the coin without looking at it

Finally Q does something to the coin without looking at it

Now the box is opened. If the coin is head up, Q wins, otherwise C wins. This is clearly a zero sum game

Turns out that Q has a quantum strategy which always wins regardless of what C does

Coin flipping $\cdots$

Key points:

(i) the coin is a quantum object. If head is $H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and tail is $T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the coin can, in general, be in the state

$$\alpha H + \beta T = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

(ii) Q is allowed to make quantum moves on the coin, i.e., act on the state of the coin with an arbitrary unitary matrix U

Note that the classical player C can only flip or not flip the coin. These correspond to acting on the coin with either

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x \quad \text{or} \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Since these two matrices commute with each other, $C's$ strategies are purely classical

Coin flipping $\cdots$

$Q's$ winning strategy is as follows:

First, Q acts on the coin with the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This changes the state of the coin from H to $U\,H$

Now C can act on this with either F or N, thereby producing the state $F\,U\,H$ or $N\,U\,H$

Finally, Q acts on the coin with the matrix $U^\dagger$, thereby producing either $U^\dagger\,F\,U\,H$ or $U^\dagger\,N\,U\,H$

Given the forms of $F = \sigma^x$, $N = I$, and U, we find that $U^\dagger\,F\,U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z$ and $U^\dagger\,N\,U = I$.

Both of these acting on H gives H

So the coin always ends with head up, and Q wins !

Coin flipping $\cdots$

Basically, the Hadamard matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

transforms from the σ^x basis to the σ^z basis, and it corrects a flip from up to down in the σ^z basis (called a “bit flip”)

This is the simplest example of a quantum error correction code

There are more complicated codes which correct a “sign flip” or a combination of both bit and sign flips

Roffe, arXiv:1907.11157

Game II: cats and dogs

There are two players, 1 and 2

They are each asked either one of two questions with probability $1/2$ each (they are informed of these probabilities at the beginning):

(i) do you like cats, or (ii) do you like dogs ?

Each of them must answer either yes or no

If either or both of them is asked the cat question, they win 1 point each if they give opposite answers, and get no points if they give the same answer

If they are both asked the dog question, they win 1 point each if they give the same answer, and get no points if they give opposite answers

They can decide on a strategy right at the beginning,
but not after the game begins

Cats and dogs · · ·

Example of a strategy: they decide to always give opposite answers, e.g., player 1 always answers yes and player 2 always answers no

Then if they are randomly asked the cat and dog questions, they win $3/4$ of the time and lose $1/4$ of the time

So their expected pay-off in the long run is $3/4$

This turns out to be the best possible expected pay-off if they are restricted to classical strategies

But there is a cooperative quantum strategy which does better !

Cats and dogs · · ·

Quantum strategy: before each game, a friend produces a singlet state using two spin-1/2 objects, and gives the two objects to the two players

A singlet made from two spin-1/2 objects has a wave function which is a superposition of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

This is an entangled state; a measurement of the component of the spin in any direction for the first object will give either up or down with probability $1/2$ each, and this will automatically imply that a measurement of the spin in the same direction for the second object will give the opposite value (down or up)

More generally, if player 1 measures the component of the spin in a particular direction and finds some value, and player 2 measures the spin in some other direction which is at an angle θ with respect to the first direction, then the second player will find the opposite value of the spin with probability $\cos^2(\theta/2)$

Cats and dogs $\cdots$

Explanation for the factor of $\cos^2(\theta/2)$:

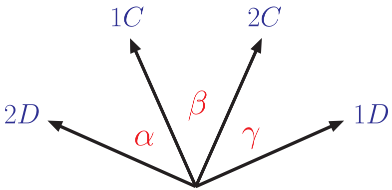
If a direction in three dimensions has the polar coordinates (θ, ϕ) (θ is the angle between this direction and the z axis), then the wave function of a spin-1/2 object whose spin component in that direction is given by $\hbar/2$ is

$$\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i\phi} \end{pmatrix}$$

From this we can show that if one of the two objects in a singlet has its spin pointing along the z direction, the probability that the spin of the other object will point in the same direction as (or opposite to) (θ, ϕ) is given by $\sin^2(\theta/2)$ (or $\cos^2(\theta/2)$)

Cats and dogs $\cdots$

The above facts motivate the following quantum strategy:



If player 1 is asked the cat (dog) question, he measures the spin of his spin-1/2 object in the direction $1C$ ($1D$) respectively

If player 2 is asked the cat (dog) question, she measures the spin of her spin-1/2 object in the direction $2C$ ($2D$) respectively

The four directions are taken to lie in a plane with relative angles $\alpha,\ \beta,\ \gamma$ as shown, and they are fixed by the players in the beginning

In all cases, each player answers yes if they find the component of their spin to be $\hbar/2$ and no if the component is $-\hbar/2$

Cats and dogs $\cdots$

With the above strategy, their expected pay-off in the long run is found to be

$$\frac{1}{4}\left[\ \cos^2(\alpha/2)\ +\ \cos^2(\beta/2)\ +\ \cos^2(\gamma/2)\ +\ \sin^2((\alpha+\beta+\gamma)/2)\ \right]$$

(The factor of $1/4$ is because each player can be asked the cat or dog question with probability $1/2$ each)

Maximising this expression with respect to $\alpha,\ \beta,\ \gamma$ gives $\alpha\ =\ \beta\ =\ \gamma\ =\ \pi/4,$ and the expected pay-off is $\cos^2(\pi/8)\ =\ \frac{1}{2}+\frac{\sqrt{2}}{4}\ \simeq\ 0.8536$

This is better than the best classical strategy which gives $3/4$

It turns out that the pay-off of $\cos^2(\pi/8)$ is the best that a quantum strategy can give

Clauser, Horne, Shimony and Holt, Phys. Rev. Lett. 23, 880 (1969)

Cats and dogs $\cdots$

This game shows that two players who possess two parts of a quantum entangled state have access to some kind of shared information which has no classical counterpart

Note that we have two different inequalities here:

Classical strategies have a pay-off which must be $\leq 3/4$

Quantum strategies can violate this inequality but they must be $\leq \cos^2(\pi/8) \simeq 0.8536$

The first inequality, where quantum mechanics can violate a classical bound, is called a Bell inequality

The second inequality, which even quantum mechanics cannot violate, is called a Tsirelson bound

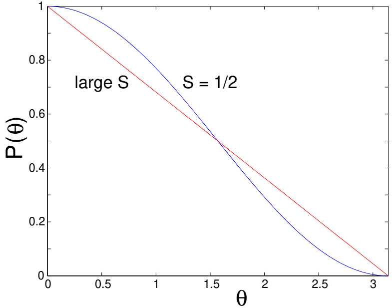
Cats and dogs $\cdots$

We can consider a generalisation of the quantum strategy in which the singlet is formed out of two spin-S objects (where S can take values $1, 3/2, 2, \cdots$), and each player gets one of the two spin-S objects

Then, as before, they measure the component of the spin along certain directions which depend on whether they are asked the cat or dog question, and they answer yes or no depending on whether they find the component of the spin in that direction to be positive or not

One finds that in the limit $S \rightarrow \infty,$ the expected pay-off becomes $3/4$ which is the same as the best classical pay-off. This is another way of seeing that $S \rightarrow \infty$ is the classical limit of a quantum spin

Cats and dogs $\cdots$



Plot of probability of the second spin pointing opposite to the direction θ when the first spin points in the z direction, for a singlet made up of two spin-S objects

$$
\begin{aligned}
P(\theta) &= \cos^2(\theta/2) \quad \text{for spin} - 1/2 \\
&= 1 - \frac{\theta}{\pi} \quad \text{for} \quad S \to \infty
\end{aligned}
$$

Bell inequality

Consider two observers, Alice and Bob (A and B), who independently measure certain properties of two objects. Each of them can measure either property 1 or property 2.

We will call these properties a_1, a_2, b_1, b_2

Let's assume that each of these properties can only take values in the range $[-1, 1]$. The properties are measured many times, and the average values of the four possible quantities are calculated, denoted as $E(a_1, b_1)$, $E(a_1, b_2)$, $E(a_2, b_1)$ and $E(a_2, b_2)$

If all these quantities are classical, and the quantities measured by A and B are uncorrelated, then $E(a_1, b_1) = E(a_1)\, E(b_1)$, etc. Then, using $-1 \leq E(a_i), E(b_j) \leq 1$, one can easily show that

$$\begin{aligned} & E(a_1, b_1) + E(a_1, b_2) + E(a_2, b_1) - E(a_2, b_2) \\ = \; & E(a_1)\,(E(b_1) + E(b_2)) + E(a_2)\,(E(b_1) - E(b_2)) \end{aligned}$$

has a magnitude which is ≤ 2

Violation of Bell inequality

However, suppose that A and B are given the two spin-1/2 objects forming a singlet state $|\psi\rangle$ as before, and the properties they measure are the expectation values of $\vec{a}_1 \cdot \vec{\sigma}_A,\ \vec{a}_2 \cdot \vec{\sigma}_A,\ \vec{b}_1 \cdot \vec{\sigma}_B$ and $\vec{b}_2 \cdot \vec{\sigma}_B,$ where $\vec{a}_i$ and $\vec{b}_j$ are some unit vectors

Then we find that $E(\vec{a}_i, \vec{b}_j) = \langle\psi|\, \vec{a}_i \cdot \vec{\sigma}_A\, \vec{b}_j \cdot \vec{\sigma}_B \,|\psi\rangle = -\ \vec{a}_i \cdot \vec{b}_j$

The calculation of $E(\vec{a}_1, \vec{b}_2) + E(\vec{a}_1, \vec{b}_2) + E(\vec{a}_2, \vec{b}_1) - E(\vec{a}_2, \vec{b}_2)$ then reduces exactly to the one we did for the cats and dogs game due to the identity $\cos\alpha = 2\cos^2(\alpha/2) - 1$

We find that, depending on the choices of $\vec{a}_i$ and $\vec{b}_j,$ the magnitude of the above quantity can exceed 2 (violation of Bell inequality) but cannot exceed $2\sqrt{2}$ (Tsirelson bound)

Game III: the prisoner's dilemma

Consider a game between Alice and Bob (A and B) with strategies C and D and the pay-off table

	$B\ C$	$B\ D$
$A\ C$	$(3,3)$	$(0,5)$
$A\ D$	$(5,0)$	$(1,1)$

The story:

subtract the numbers in the above table from 5 and think of them as the number of years that the two persons would be in prison

	$B\ C$	$B\ D$
$A\ C$	$(2,2)$	$(5,0)$
$A\ D$	$(0,5)$	$(4,4)$

Two prisoners, interrogated independently by police, can either cooperate with each other (C) or defect from each other (D)

Game III: the prisoner's dilemma

We return to the pay-off table

	$B\ C$	$B\ D$
$A\ C$	$(3, 3)$	$(0, 5)$
$A\ D$	$(5, 0)$	$(1, 1)$

What is so strange about this problem?

In the language of game theory, strategies (AC, BC), (AC, BD) and (AD, BC) are Pareto optimal, while (AD, BD) is at a Nash equilibrium

Pareto optimality

Vilfredo Pareto (1848 - 1923) was an Italian sociologist and economist

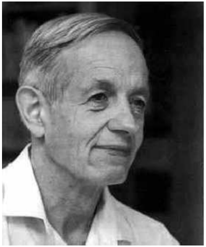
- Turned economics into a quantitative field of research
- Made a $80 - 20$ observation called the Pareto principle

A strategy S_i is called Pareto optimal if there is no other strategy in which at least one player has a higher pay-off and no one has a lower pay-off

A given game may have no Pareto optimal strategy or may have more than one

Nash equilibrium

Concept introduced by John Nash (1928-2015)



A strategy S_i is said to be at a Nash equilibrium if, for each player a, there is no other strategy which increases the pay-off p_{ia}, assuming that the other player b holds their strategy fixed at p_{ib}

A game may have no Nash equilibrium or may have more than one

The prisoner's dilemma $\cdots$

	$B\ C$	$B\ D$
$A\ C$	$(3,3)$	$(0,5)$
$A\ D$	$(5,0)$	$(1,1)$

Strategies (AC, BC), (AC, BD) and (AD, BC) are Pareto optimal because any deviation from any of these strategies makes at least one player worse off. Strategy (AD, BD) is not Pareto optimal because there is another strategy (AC, BC) where both players are better off

On the other hand, only (AD, BD) is at a Nash equilibrium.
For any one of the other three strategies, one player can increase their pay-off by unilaterally changing their strategy, assuming that the other player does not change their strategy

The prisoner's dilemma $\cdots$

	$B\ C$	$B\ D$
$A\ C$	$(3,3)$	$(0,5)$
$A\ D$	$(5,0)$	$(1,1)$

So if each player independently found their best strategy, each would choose D. Hence the two together would choose (AD, BD)

But the players would agree that the cooperative strategy (AC, BC) gives both of them a better pay-off than (AD, BD)

This is a game where cooperating with each other gives a better result than each player choosing a selfish strategy !

The prisoner's dilemma $\cdots$

Can a quantum version of this game have a different solution?

Eisert, Wilkens and Lewenstein, Phys. Rev. Lett. 83, 3077 (1999)

Brief description:

- The state space is four-dimensional, with any superposition of the four strategies (AC, BC), (AC, BD), (AD, BC), and (AD, BD) being allowed
- Consider an initial state and introduce quantum entanglement in some way. The initial state must be such that if no quantum entanglement is introduced, one recovers the classical game
- Each of the player then independently makes an arbitrary unitary transformation on the given state
- The final state and the pay-off table are used to compute the expected pay-off of each of the players

The prisoner's dilemma $\cdots$

Details of the quantum game:

The basis states are $C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Denote the strategies of cooperating or defecting by the matrices

$$\hat{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{D} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Each player can act with a unitary matrix which can be taken to be of the form

$$U(\theta, \phi) = \begin{pmatrix} \cos(\theta/2)\, e^{i\phi} & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2)\, e^{-i\phi} \end{pmatrix}$$

Some special cases of this are are $\hat{C}$, $\hat{D}$ and $\hat{Q} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

The combined strategy of the two players is given by a direct product

$$U_A(\theta, \phi) \otimes U_B(\theta', \phi')$$

The prisoner's dilemma $\cdots$

We begin with an initial state in which the two players cooperate with each other: $C \otimes C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Quantum entanglement is introduced by acting on this with a matrix of the form

$$\mathcal{J} = \exp\ [\ i\ (\gamma/2)\ \hat{D} \otimes \hat{D}\ ] = \cos(\gamma/2)\ I \otimes I + i\ \sin(\gamma/2)\ \hat{D} \otimes \hat{D}$$

If $\gamma = 0$, no entanglement is introduced and we get the classical game. The entanglement is maximal if $\gamma = \pi/2$

Next, the players act with a transformation $U_A \otimes U_B$

Finally we act with $\mathcal{J}^\dagger$

The prisoner's dilemma $\cdots$

The final state is $|\psi\rangle = \mathcal{J}^{\dagger}\, U_A \otimes U_B\, \mathcal{J} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The probabilities that the players will use the four possible strategies, denoted as P_{CC}, P_{CD}, P_{DC} and P_{DD}, are obtained by taking inner products of $C \otimes C$, $C \otimes D$, $D \otimes C$ and $D \otimes D$ with $|\psi\rangle$

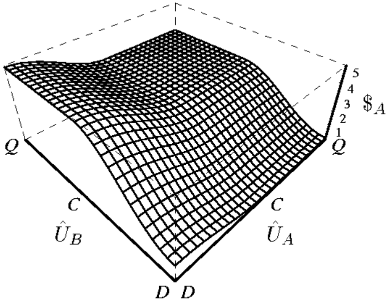
The pay-offs S_A and S_B of the two players are then calculated using the table

	$B\ C$	$B\ D$
$A\ C$	$(3,3)$	$(0,5)$
$A\ D$	$(5,0)$	$(1,1)$

We find that

$$
\begin{aligned}
S_A &= 3\, P_{CC} + 5\, P_{DC} + P_{DD} \\
S_B &= 3\, P_{CC} + 5\, P_{CD} + P_{DD}
\end{aligned}
$$

The prisoner's dilemma · · ·

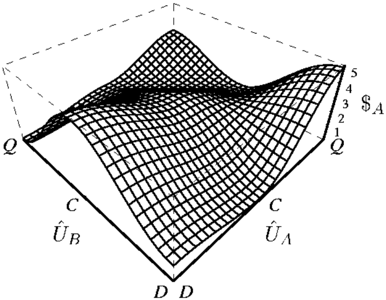


unentangled (classical) game corresponding to $\gamma = 0$

$D \otimes D$ is a Nash equilibrium but is not Pareto optimal

The pay-off is 1 for each player

The prisoner's dilemma $\cdots$



maximally entangled game corresponding to $\gamma = \pi/2$

$Q \otimes Q$ is a Nash equilibrium as well as Pareto optimal

The pay-off is 3 for each player

Eisert, Wilkens and Lewenstein, Phys. Rev. Lett. 83, 3077 (1999)

Summary

- Coin flipping game: there is a quantum strategy which always wins. This acts like an error correction code

- Cats and dogs game: there is a quantum strategy, based on a two-spin entangled state, which does better than the best classical strategy. This is related to the CHSH and Bell inequalities

- Prisoner's dilemma: this is a game in which Pareto optimal strategies are not at Nash equilibrium and vice versa. A quantum version of this game which includes some entanglement can resolve this situation, giving a strategy which is both Pareto optimal and a Nash equilibrium

References

1. Clauser, Horne, Shimony and Holt, Phys. Rev. Lett. 23, 880 (1969)
2. Meyer, Phys. Rev. Lett. 82, 1052 (1999)
3. Eisert, Wilkens, and Lewenstein, Phys. Rev. Lett. 83, 3077 (1999)
4. Cleve, Hoyer, Toner, and Watrous, arXiv:quant-ph/0404076
5. Landsburg, Notices of the AMS 51 (2004) 394

   http://www.ams.org/notices/200404/fea-landsburg.pdf

— a simple introduction to quantum game theory by an economist

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