Game Theory and Statistical Mechanics in the Self-organizing Dynamics of People, Birds, and Networks

Venkat Venkatasubramanian

Center for the Management of Systemic Risk
Department of Chemical Engineering
Columbia University
New York, NY 10027

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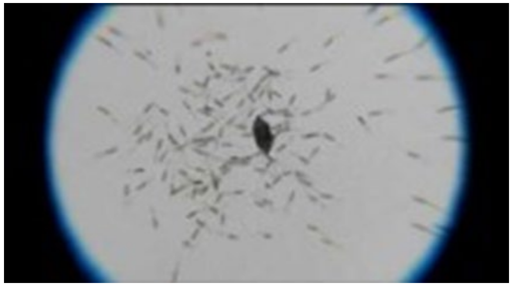
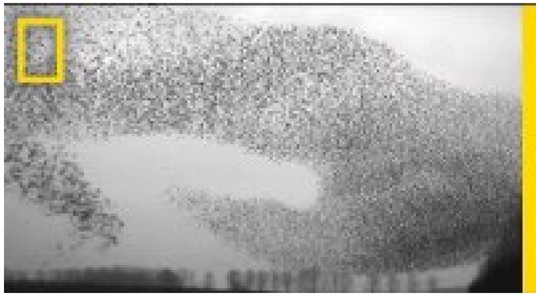
Self-organizing Dynamics and Emergent Order



Bacterial Chemotaxis

Ant Crater Formation



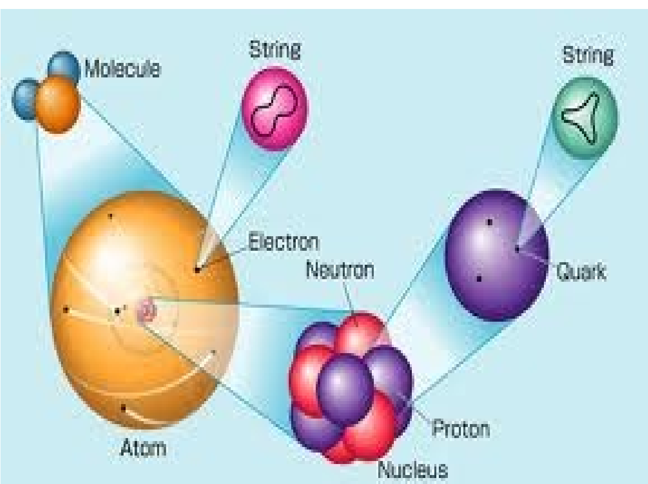
Birds flocking

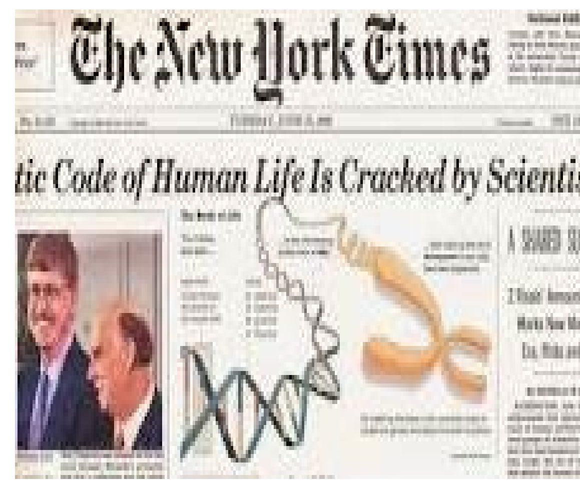
Wikipedia

- Could there be unifying organizing principle(s) behind such collective behavior?
- Different species, different scales, different phenomena
- But nature generally uses only a handful of tricks in her design!
- In physics, it is symmetry!
- Key insight from physics: Look for invariants!

Reductionist Science

- 20th Century Science was largely Reductionist
  - Quantum Mechanics and Elementary Particle Physics
  - Molecular Biology, Double Helix, Sequencing Human Genome
  - 600+ Nobel Prizes



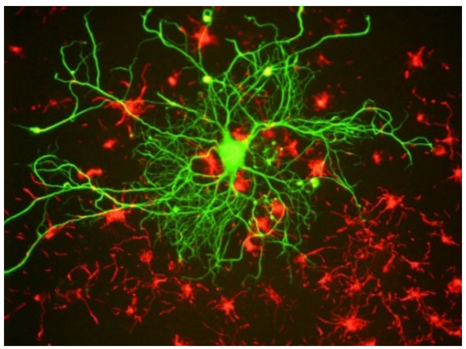


Wikipedia

Complex Self-organizing Systems

- But can reductionism answer the following question?
- Given the properties of a neuron, can we predict the collective behavior of a system of 100 billion neurons?

Wikipedia

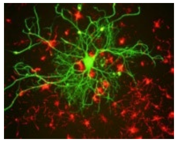


- From Neuron → Brain → Mind
- How do you go from Parts to System?

Reductionism cannot answer this!
There is nothing left to "reduce"!

From the Parts to the Whole: Statistical Teleodynamics

- Individual agent properties → Emergent properties of millions of agents
- Agents are goal-free (e.g., Molecules) → System (e.g., Gas)
  - Statistical Thermodynamics
- What if the agents are goal-driven?
  - e.g., neurons, bacteria, ants, birds, people
- Can we generalize statistical thermodynamics?



- Statistical Thermodynamics (goal-free agents) → Statistical Teleodynamics (goal-driven agents)
- Telos means goal in Greek

Self-organizing Emergent Phenomena

Biology

Ecology

Sociology

Economics

Self-organizing Dynamics in a Free Market

- People keep switching jobs to increase their utility
- Just like drivers keep switching lanes to reduce travel time
- Suddenly, a lane opens up – you switch!
- Other drivers also switch!
- Soon, the new lane is just as slow
- Drivers stop switching
  - Travel time is the same in all lanes
  - $t_1 = t_2 = t_3 = \ldots = t^*$
- **Arbitrage Equilibrium**

Wikipedia

Self-organizing Dynamics in a Free Market

- Similarly, people stop switching jobs if there is no arbitrage opportunity to increase their utility
  - **Arbitrage Equilibrium**
- Every worker enjoys the same effective utility
  - $h_{ij} = h^*$
- Is there an income distribution with constant h^*?
- Can the free market find it via the self-organizing dynamics?

Behavioral Microeconomics Model:
Effective Utility

- What is the utility of a job?
- It is a complicated function depending on a number of variables and parameters
- But for most people it's dominated by two pragmatic features
  - Pay bills now
  - Path for a better future and upward mobility
- Simple behavioral economics-like model: Ideal gas-like

- Effective utility (h_{ij}) of agent j at salary level *i*
  - Utility from income (u_{ij})
  - Disutility from contribution (v_{ij})
  - Utility of future prospects (w_{ij})

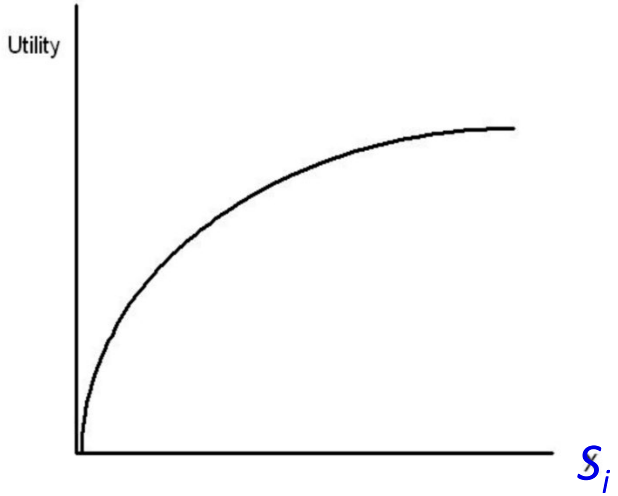
$$\boxed{h_{ij} = u_{ij} - v_{ij} + w_{ij}}$$

Modeling u_{ij}

- Utility from income
- Diminishing marginal utility

$$u_{ij} = \alpha_j \ln S_i$$

Wikipedia



Modeling v_{ij}

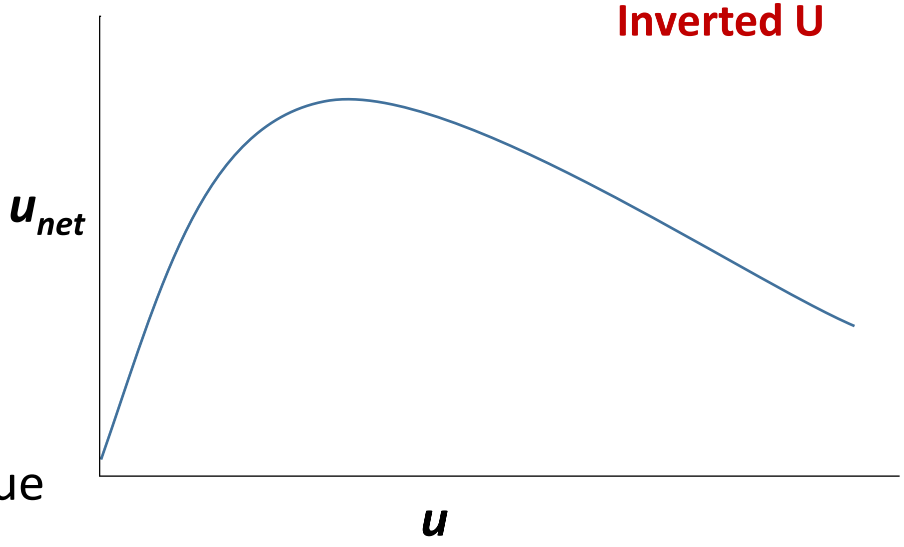
- Net benefit u_{net}

  Utility from Income (u) - Disutility of Effort (v)

- u_{net} initially increases as u increases

- However, it begins to decrease soon due to the increasing cost of personal sacrifices
  - Working overtime
  - Missing time with family
  - Job stress causing poor health
  - Relocation

Inverted U



$$\boxed{u_{\text{net}} = u - v = au - bu^2}$$

Since $u_{ij} \sim \ln S_i$

$$\boxed{v_{ij} = \beta_j \left(\ln S_i\right)^2}$$

Utility of Future Prospects w_{ij}

- N_i - Number of employees competing for partnership worth $\$Q$ in the future
- Employee's chance of winning: $1/N_i$
- Expected Value: Q/N_i
- Utility (w_i): $ln(Q/N_i)$
- Cost of competition

$$w_{ij} = -\gamma_j \ln N_i$$

Effective Utility of Agents

- Effective Utility h_{ij} of agent j at salary level i

$$h_{ij} = u_{ij} - v_{ij} + w_{ij}$$

$$h_{ij} = \alpha_j \ln S_i - \beta_j \left(\ln S_i\right)^2 - \gamma_j \ln N_i$$

- α_j, β_j, γ_j are parameters that weight an agent j's utility preferences
- Can vary from agent to agent
- Assumption: Same for all agents - 1-class society, Utopia - Ideal Gas

$$\boxed{h_i = \alpha \ln S_i - \beta \left(\ln S_i\right)^2 - \gamma \ln N_i}$$

- Note that the effective utility already accounts for the contribution made by an agent

Population Games: For Large *N*
Potential Game

- *N* employees competing for jobs
- There exists a potential $\boxed{\phi(\mathbf{x})}$, s.t.

$$\boxed{\begin{array}{l} h_i(\mathbf{x}) \equiv \partial \phi(\mathbf{x}) / \partial x_i \\ \text{where } x_i = N_i / N \end{array}}$$

$$h_i = \alpha \ln S_i - \beta \left(\ln S_i\right)^2 - \gamma \ln N_i$$

$$\phi(\mathbf{x}) = \phi_u + \phi_v + \phi_w \qquad \begin{aligned} \phi_u &= \alpha \sum_{i=1}^{n} x_i \ln S_i \\ \phi_v &= -\beta \sum_{i=1}^{n} x_i (\ln S_i)^2 \\ \phi_w &= \frac{\gamma}{N} \ln \frac{N!}{\prod_{i=1}^{n} (N x_i)!} \end{aligned}$$

Is there an Equilibrium Distribution?

- There exists a Nash Equilibrium, when $\phi(\mathbf{x})$ is maximized
- NE is unique if the potential is strictly concave

$$\boxed{\partial^2 \phi(\mathbf{x}) / \partial x_i^2 = -\gamma / x_i < 0}$$

$$\phi(\mathbf{x}) = \phi_u + \phi_v + \phi_w \qquad \begin{aligned} \phi_u &= \alpha \textstyle\sum_{i=1}^{n} x_i \ln S_i \\ \phi_v &= -\beta \textstyle\sum_{i=1}^{n} x_i (\ln S_i)^2 \\ \phi_w &= \frac{\gamma}{N} \ln \frac{N!}{\prod_{i=1}^{n} (N x_i)!} \end{aligned}$$

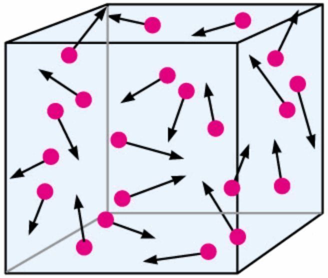
- But what is the equilibrium distribution?

Key insight: ϕ_w is **Entropy** in Statistical Mechanics

Thermodynamics as a Game of "Passive" Agents

$$h_i(E_i, N_i) = -\beta E_i - \ln N_i$$

$$\phi(\mathbf{x}) = -\frac{\beta}{N} E + \frac{1}{N} \ln \frac{N!}{\prod_{i=1}^{n} (N x_i)!}$$



$$L = \phi + \lambda(1 - \sum_{i=1}^{n} x_i). \qquad \partial L / \partial x_i = 0$$

$$E = N \sum_{i=1}^{n} x_i E_i \qquad \sum_{i=1}^{n} x_i = 1$$

Equilibrium: Gibbs-Boltzmann distribution at Maximum Entropy

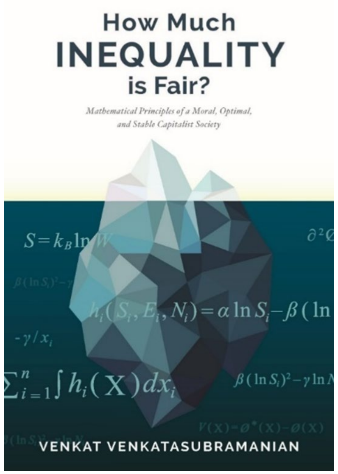
$$x_i = \frac{\exp(-\beta E_i)}{\sum_{j=1}^{n} \exp(-\beta E_j)}$$

- Potential Game agrees with Statistical Mechanics in the limiting case of purpose-free, agents, namely, molecules
- Thermodynamic Equilibrium = Nash Equilibrium
- $\phi(\mathbf{x})$ is Helmholtz Free Energy: Thermodynamic Potential
- Molecular "Utility" h_i is similar to Negative Chemical Potential

Surprising & Deep Connection

Income Game

- *N* employees competing for jobs
- Nash Equilibrium exists at maximum potential
- Equilibrium distribution : Lognormal

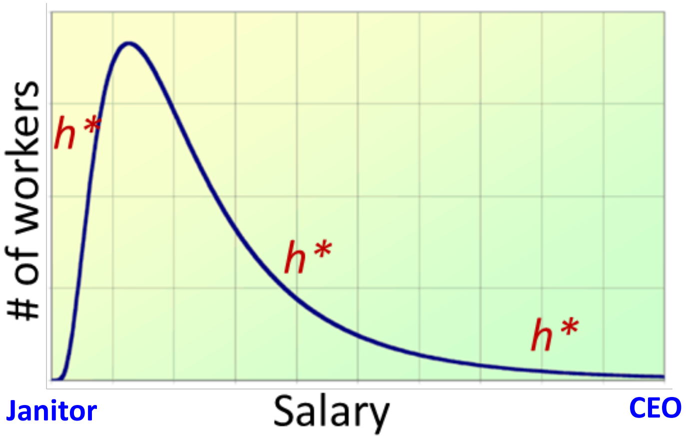


$$\boxed{x_i = \frac{1}{S_i Z} \exp\left[-\frac{\left(\ln S_i - \frac{\alpha+\gamma}{2\beta}\right)^2}{\gamma/\beta}\right]}$$

- At equilibrium, all agents have the same utility *h**
- Fairest distribution of income
- Bottom-up perspective: Parts-to-Whole
- Given the individual utility h_i

$$h_i = \alpha \ln S_i - \beta (\ln S_i)^2 - \gamma \ln N_i$$

- Can predict the system-level outcome: Lognormal



*Curve of Constant Effective Utility h**

Predictions for Different Countries

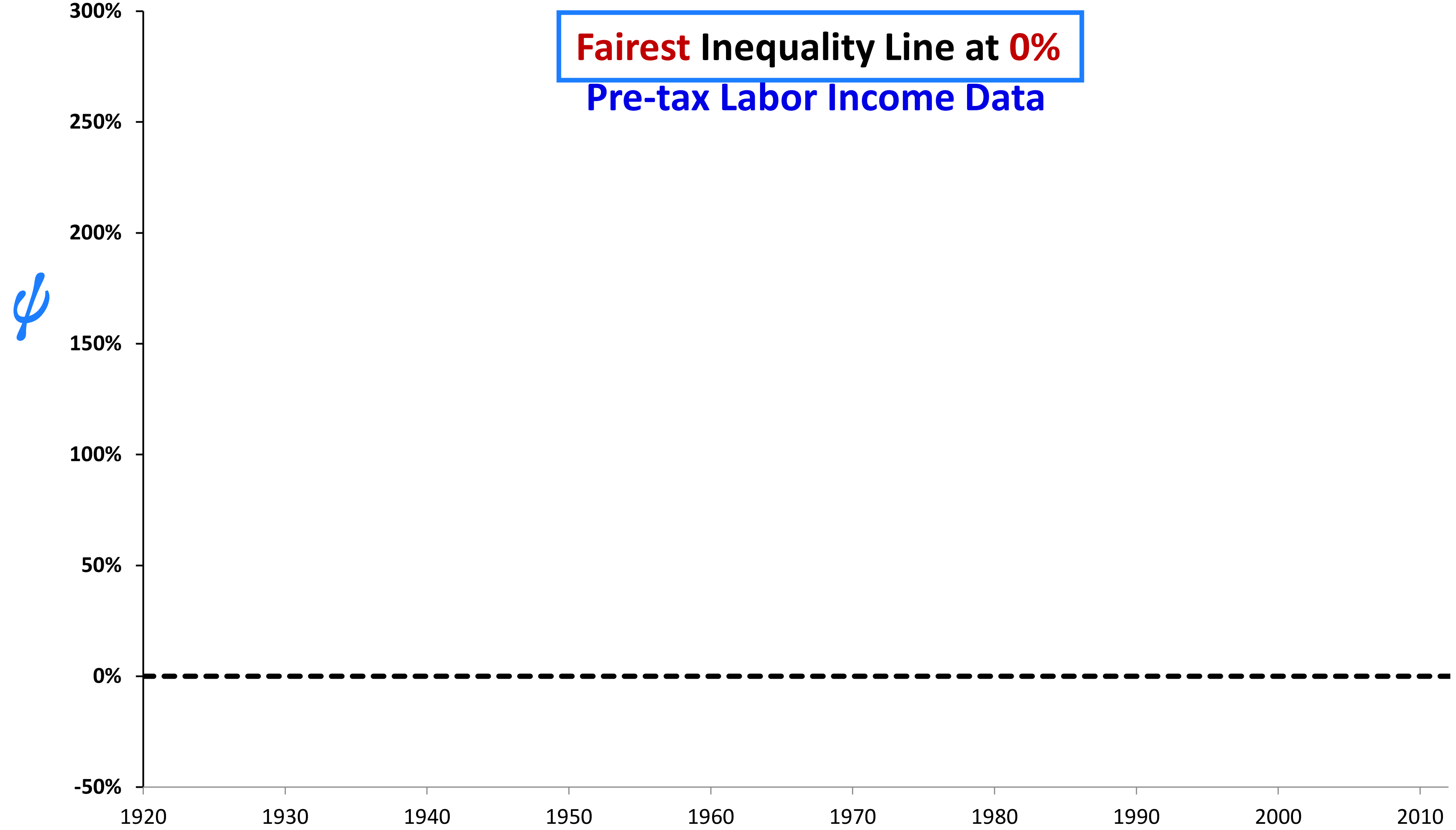
- How does our theory perform in practice?

- Our theory estimates lognormal-based income shares for Top 1%, Top 10-1%, and Bottom 90% for ideally fair societies

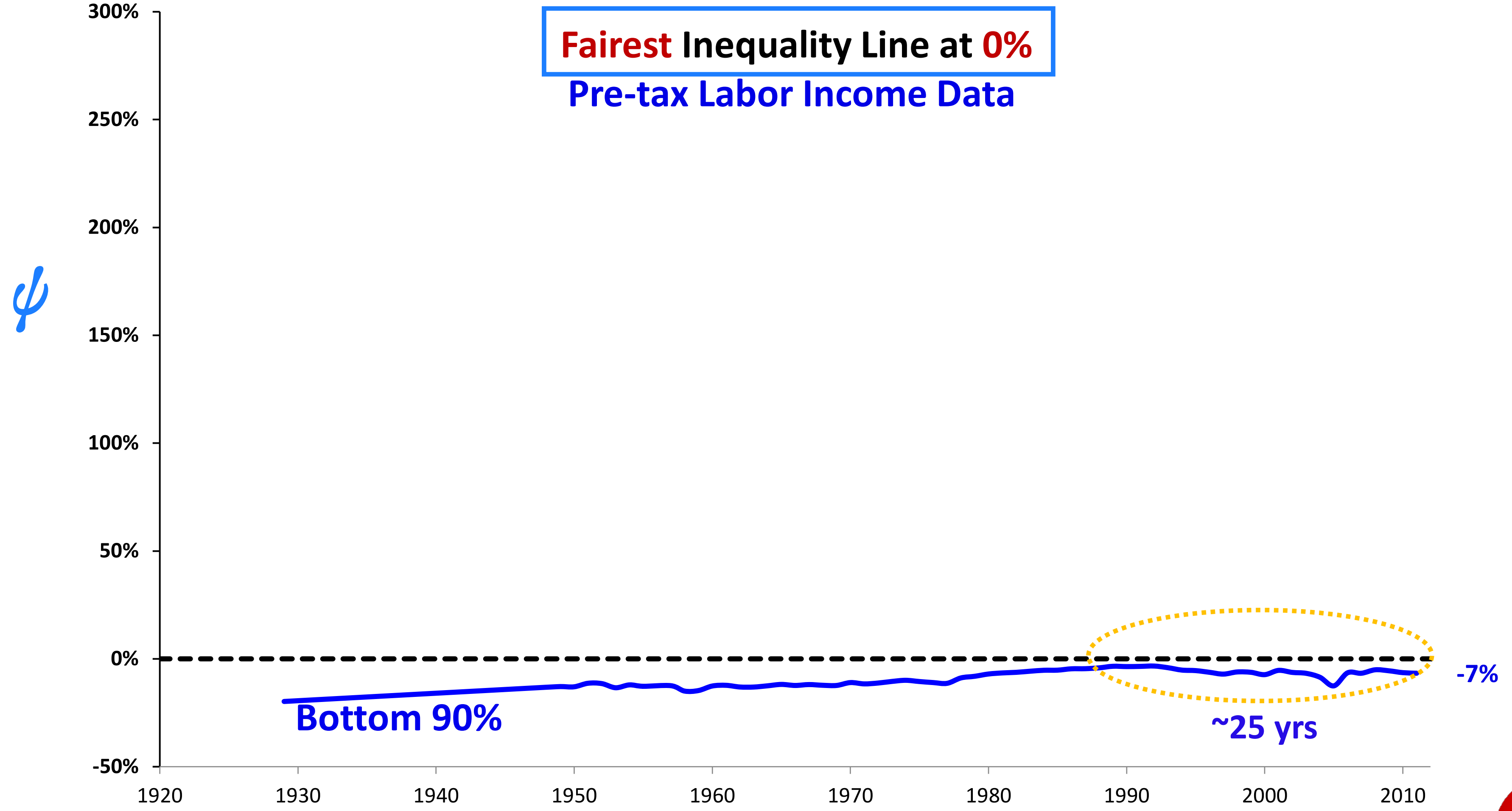
Non-ideal Inequality Coefficient $\psi = \frac{Actual\ share}{Ideal\ share} - 100\%$

$\psi = 0$ *Fairest Inequality*; $\psi \neq 0$ *Unfair Inequality*

Norway: Non-ideal Inequality ψ



Norway: Non-ideal Inequality ψ



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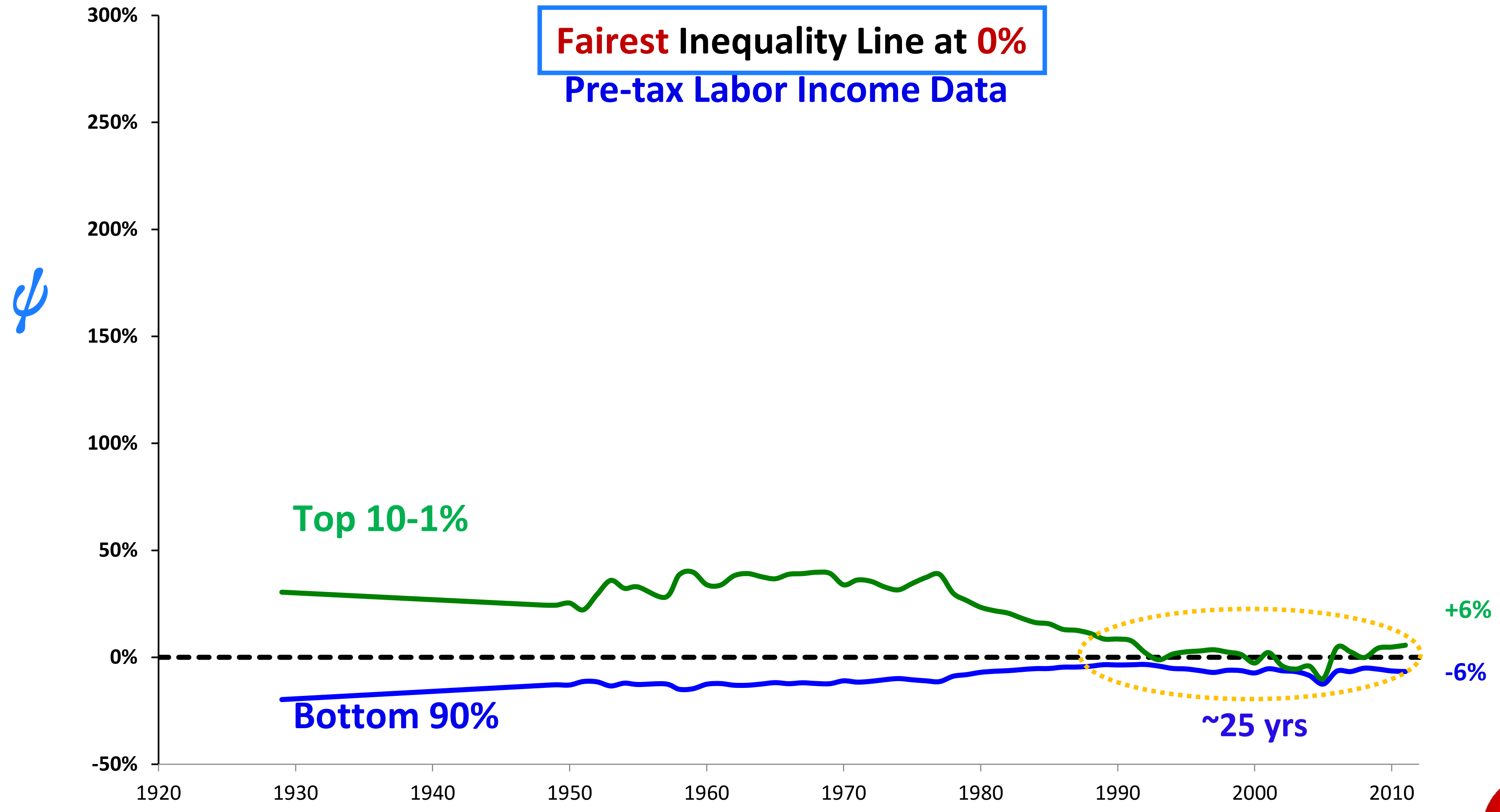
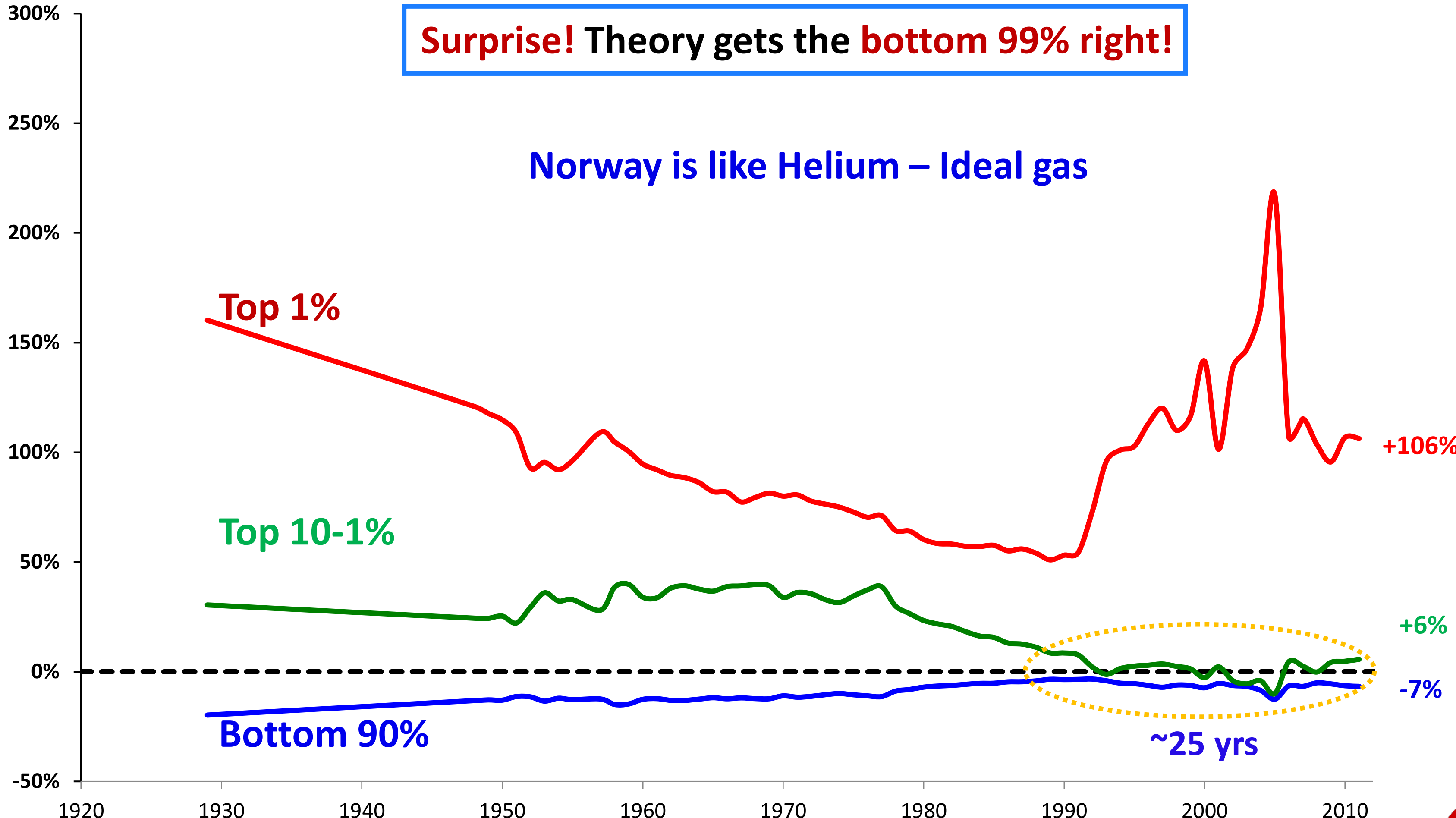
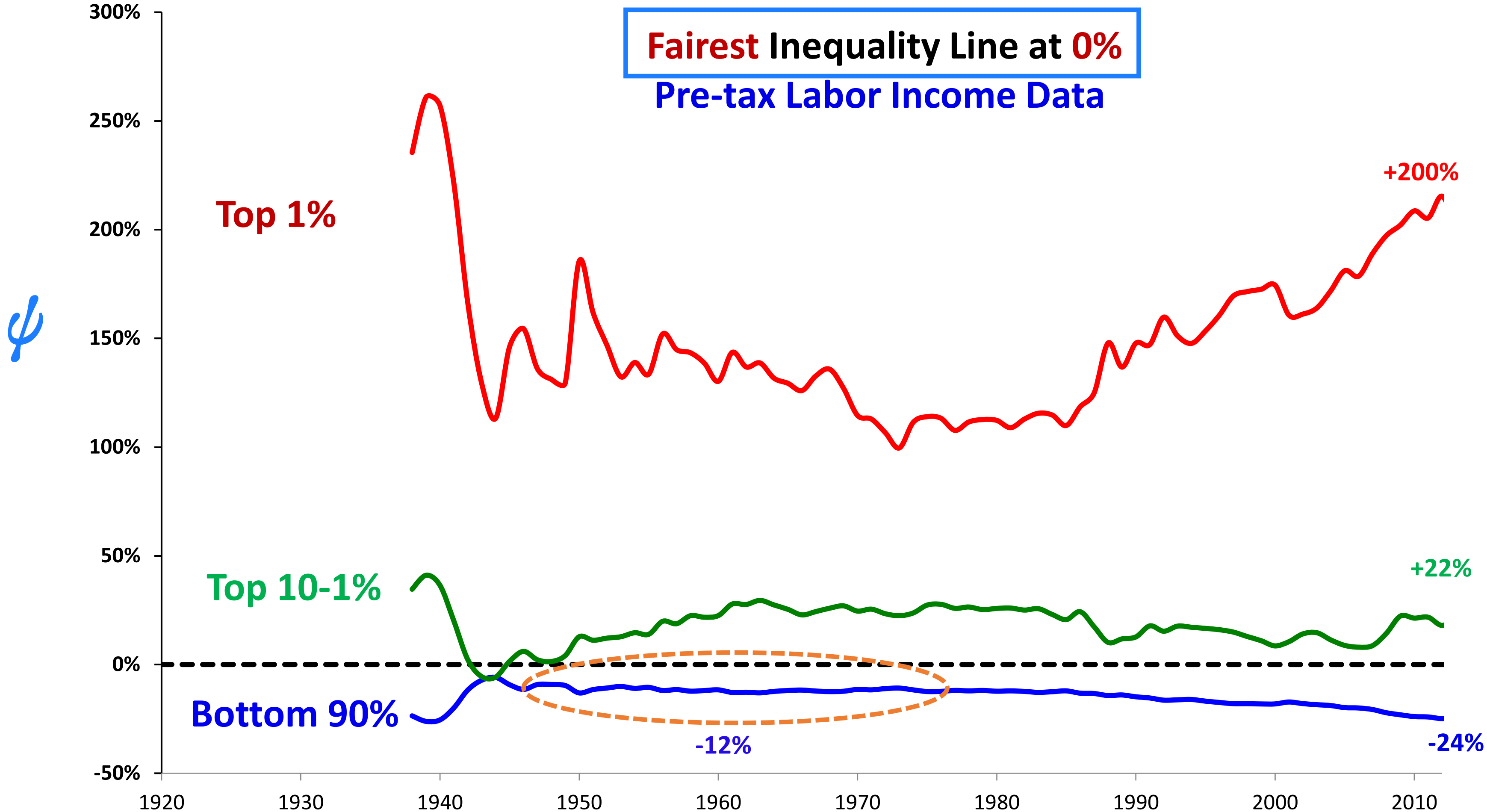
Norway: Non-ideal Inequality ψ



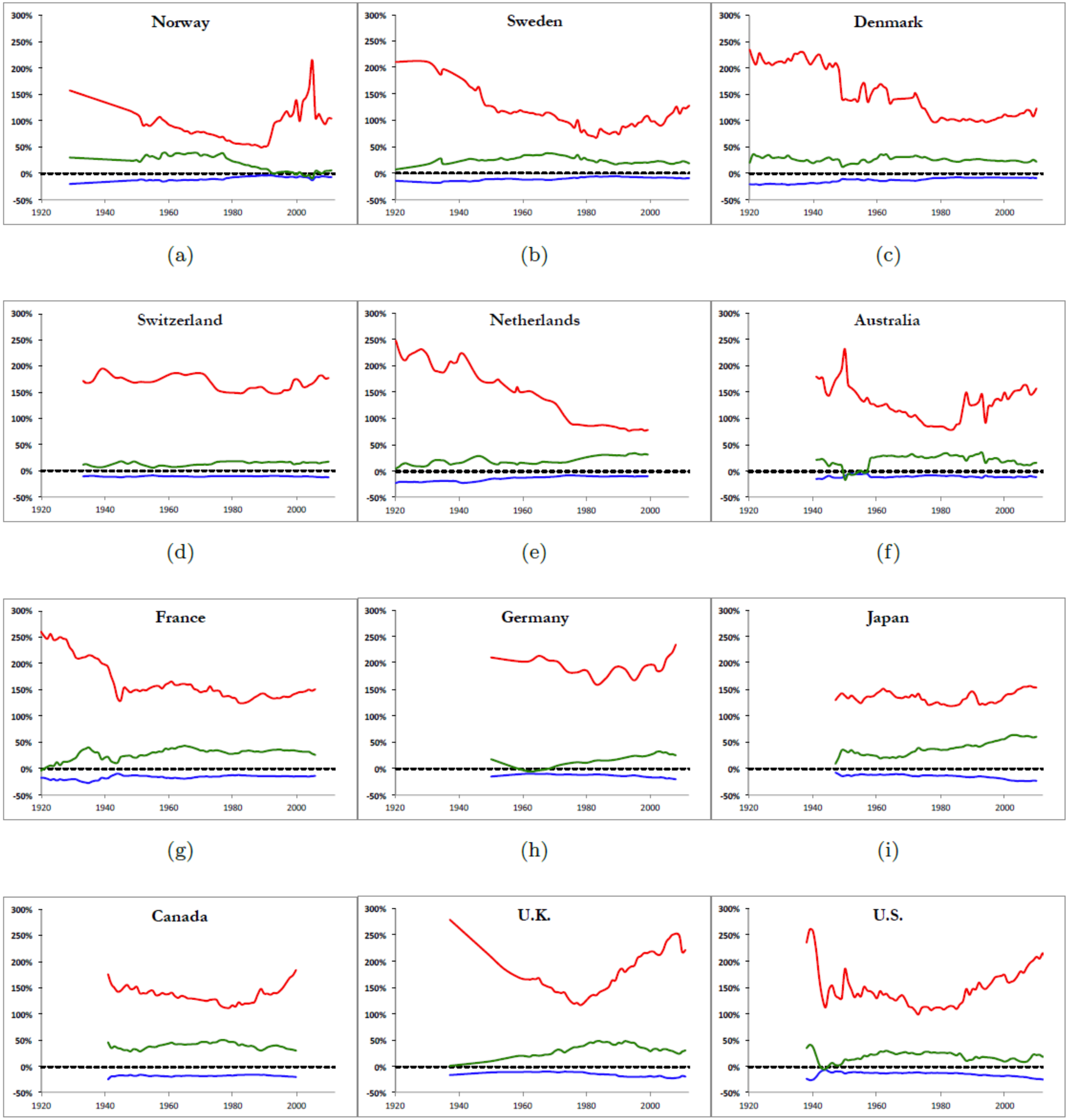
Norway: Non-ideal Inequality ψ



USA: Non-ideal Inequality ψ



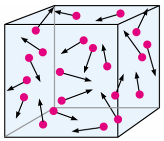
Data: Piketty's World Top Income Database



Ideal Inequality Line at 0%

Pre-tax Labor Income Data

Statistical Teleodynamics of Biological Active Matter



Agency Spectrum

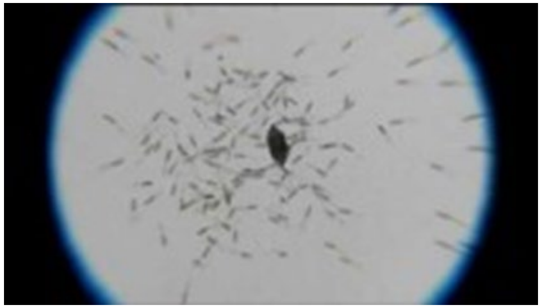
0%
Goal-free
e.g., Gas Molecules
Statistical
Thermodynamics
Statistical Equilibrium

x%
Somewhat
Purposeful Agents
Active Matter
Arbitrage Equilibrium

100%
Goal-driven
Rational Agents
Statistical
Teleodynamics
Arbitrage Equilibrium

Birds Flocking

Bacterial Chemotaxis



Ant Crater Formation

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Venkatasubramanian, Sivaram, and Das, “A unified theory of emergent equilibrium phenomena in active and passive matter”, Comp. & Chem. Engg., **2022**.

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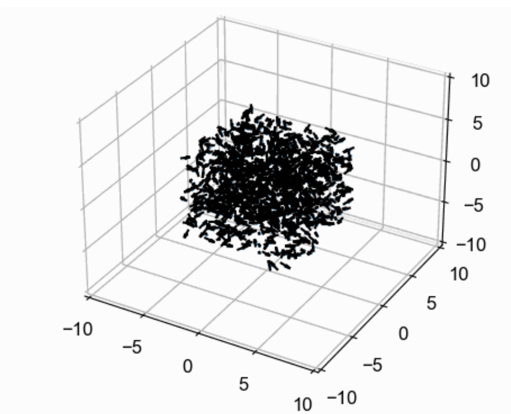
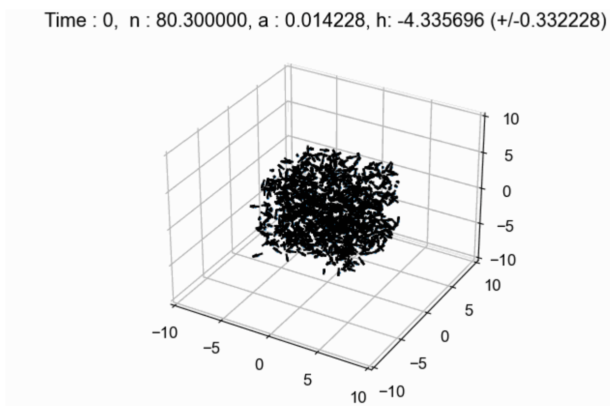
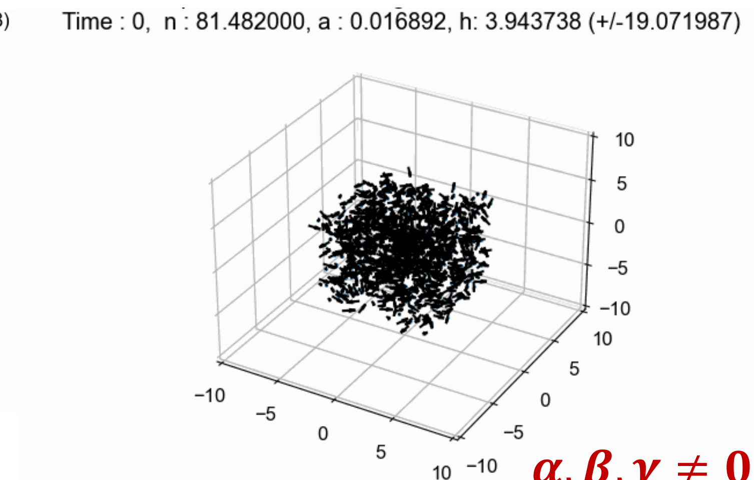
Birds Flocking:

Agents-based Simulation





$\boldsymbol{\alpha, \beta, \gamma = 0}$



$\boldsymbol{\alpha, \beta, \gamma \neq 0}$

- **Darwinian survival instinct**
- **Effective Utility**

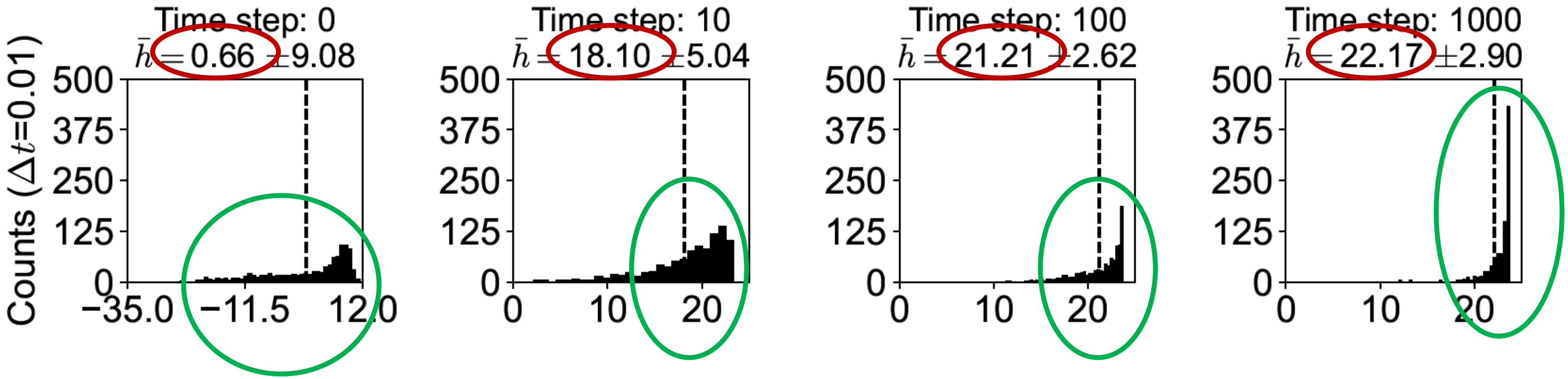
$$h_i = \alpha n_i - \beta n_i^2 + \gamma \sum_j n_{ij} \mathbf{s}_i \cdot \mathbf{s}_j - \delta \ln n_i$$

Company **Congestion** **Coherence** **Competition**

Ideal gas model
Ising model

- **VV**, Sivaram, and Das, "A unified theory of emergent equilibrium phenomena in active and passive matter", *Comp. & Chem. Engg.*, **2022**.
- Sivaram and **VV**, "Arbitrage Equilibrium, Invariance, and the Emergence of Spontaneous Order in the Dynamics of Birds Flocking", *Arxiv,* **2022**.

Arbitrage Equilibrium – *h**



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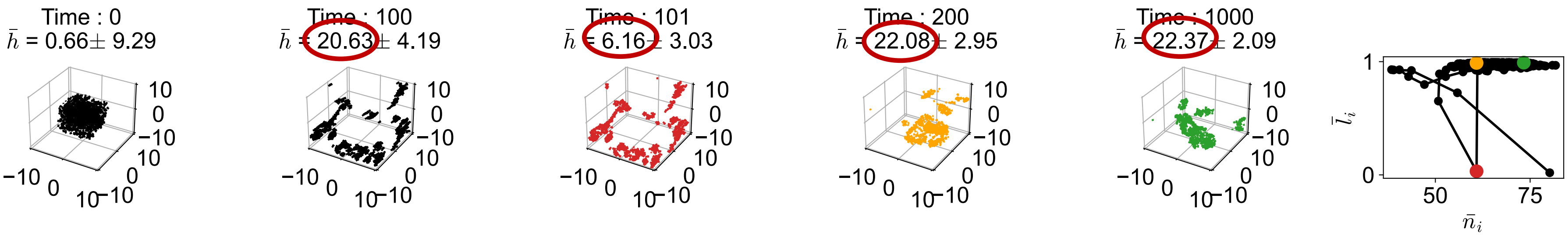
- Sivaram and **VV**, “Arbitrage Equilibrium, Invariance, and the Emergence of Spontaneous Order in the Dynamics of Birds Flocking”, *Entropy,* **2022**.

Arbitrage Equilibrium is Asymptotically Stable



- Randomly reset the velocities at time step 101
- System self-heals automatically!
- Could this be a core mechanism of self-healing and resilience in other biological systems?

Lyapunov function (*V*)

$$V(\mathbf{x}) = \phi^*(\mathbf{x}) - \phi(\mathbf{x})$$

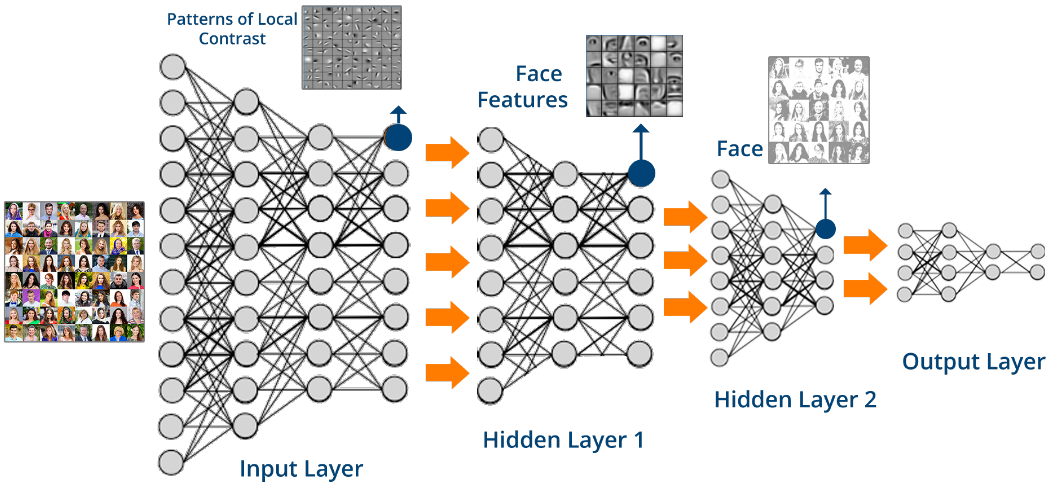
$$\dot{V} \text{ is } \textit{negative definite}.$$

Asymptotically Stable

- **VV**, Sivaram, and Das, "A unified theory of emergent equilibrium phenomena in active and passive matter", *Comp. & Chem. Engg.*, **2022**.
- Sivaram and **VV**, "Arbitrage Equilibrium, Invariance, and the Emergence of Spontaneous Order in the Dynamics of Birds Flocking", *Entropy*, **2022**.

Theory of Deep Neural Networks

- A gas container has ~10^{23} molecules interacting dynamically.
- The theory of predicting the macroscopic properties of a gas given the microscopic properties of its molecules is called statistical mechanics.
- Similarly, what is such a theory of deep neural networks with millions of neurons and billions of connections?
- Theory should answer questions such as
  - What is the distribution of weights in a trained network?
  - What is the distribution of neuronal iota?



Theory of Deep Neural Networks

- Hopfield model and Boltzmann machine cannot answer these questions!
- Backprop algorithm cannot answer either.
- These are not theories of deep neural networks.
- They are algorithmic recipes for training networks.
- They are like having molecular dynamics algorithms, periodic boundary conditions, and other such tricks to simulate the system without knowing the laws of thermodynamics.

Theory of Deep Neural Networks

- So, what are the laws of neural networks?
- This is the question I had asked myself in 1982 before hearing Hopfield's talk at Cornell.
- After his talk, I felt intuitively that he was on the right track, but his formulation was not quite correct.
- So, I went on to develop the correct formulation.
- It took me 42 years and I finally found the correct formulation this year!
- I call the new formulation the Jaynes Machine.

Edwin T. Jaynes
1922-1998

Jaynes Machine

- Hopfield and Hinton looked to physics for inspiration
  - Ising model
  - Minimize Energy
  - Statistical Thermodynamics
- I looked to game theory and economics
  - Potential games
  - Maximize Utility
  - Statistical Teleodynamics
- Neurons and connections compete during training

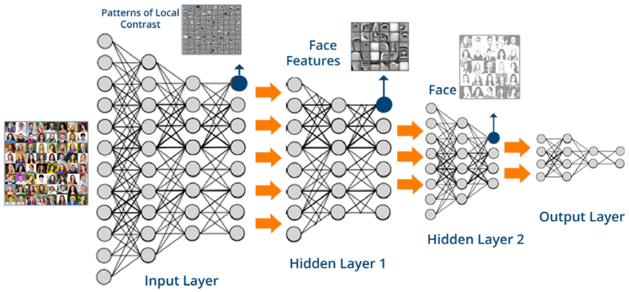
Theory of Deep Neural Networks:
Statistical Teleodynamics

- Each connection contributes an effective utility towards error minimization

$$h_{ijk}^{l} = \alpha^{l} \ln \mid w_{ijk}^{l} \mid - \beta^{l} (\ln \mid w_{ijk}^{l} \mid)^{2} - \ln M_{k}^{l}$$

- Arbitrage equilibrium:

$$h_{ijk}^{l} = h^{*}$$



- Lognormal distribution of weights

$$x_{k}^{l} = \frac{1}{\mid w_{ijk}^{l} \mid \sigma^{l} \sqrt{2\pi}} \exp \left[- \frac{(\ln \mid w_{ijk}^{l} \mid - \mu^{l})^{2}}{2\sigma^{l2}} \right]$$

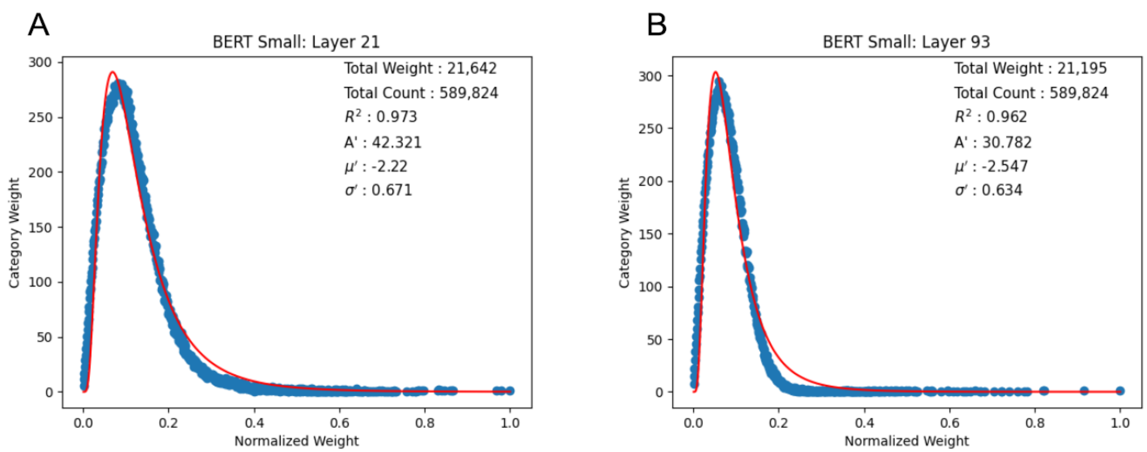
$$\mu^{l} = \frac{\alpha^{l} + 1}{2\beta^{l}} \text{ and } \sigma^{l} = \sqrt{\frac{1}{2\beta^{l}}}.$$

VV., Sanjeevrajan, N., Khandekar, M., Sivaram, A., and Szczepanski, C., "Jaynes machine: The universal microstructure of deep neural networks," *Comp. & Chem. Engg.*, 192, 108908, **2025**.

VV, N. Sanjeevrajan, M. Khandekar, "Jaynes Machine: The universal microstructure of deep neural networks", #5144789, *ArXIV*, **2023**

Deep Neural Networks:
6 Case Studies – 798 layers

Model	Architecture	Parameters size	Application
BlazePose	Convolution	2.8×10^6	Computer Vision
Xception	Convolution	20×10^6	Computer Vision
BERT Small	Transformer	109×10^6	Natural Language Processing
BERT Large	Transformer	325×10^6	Natural Language Processing
LLAMA-2 (7B)	Transformer	7×10^9	Natural Language Processing
LLAMA-2 (13B)	Transformer	13×10^9	Natural Language Processing



Lognormal!

Same σ!

Model	Layers	R^2	A'	μ'	σ'
BlazePose	39	0.93 ± 0.02	3.75 ± 2.09	-1.74 ± 0.52	1.49 ± 0.60
Xception	32	0.98 ± 0.01	6.53 ± 3.64	-2.87 ± 0.18	0.70 ± 0.05
BERT Small	75	0.96 ± 0.01	66.15 ± 46.46	-2.47 ± 0.95	0.65 ± 0.02
BERT Large	144	0.96 ± 0.01	44.84 ± 143.6	-2.37 ± 0.98	0.64 ± 0.01
LLAMA-2 (7B)	226	0.97 ± 0.01	11464 ± 8170	-2.96 ± 0.54	0.66 ± 0.05
LLAMA-2 (13B)	282	0.94 ± 0.03	1513 ± 1116	-3.02 ± 0.53	0.67 ± 0.06

Derive Hopfield Network and Boltzmann Machine as a special case of Jaynes Machine

- Neuronal lotum $$Z_i^l = z_i^l y_i^l = \left(\sum_{j=1}^{N^{(l-1)}} w_{ij}^l y_j^{l-1} + b_i^l \right) y_i^l$$
- Jaynes neuronal utility $$H^{l*} = \eta \ln Z_q^l - \zeta (\ln Z_q^l)^2 - \ln N_q^{l*}$$
- Jaynes neuronal potential Max $$\Phi_N = \sum_{l=1}^{L} \phi_N^l = \sum_{l=1}^{L} \sum_{q=1}^{n} \left[\eta x_q^l \ln Z_q^l - \zeta x_q^l (\ln Z_q^l)^2 \right] + \mathscr{S}_N$$
- Hopfield utility $$H^{l*} = B^l - \zeta Z_q^l - \ln N_q^{l*}$$
- Hopfield potential Max $$\Phi_N = -\zeta \sum_{l=1}^{L} \sum_{i=1}^{N^l} Z_i^l + \mathscr{S}_N$$
- Hopfield "energy" $$\mathrm{Min} \sum_{l=1}^{L} \sum_{i=1}^{N^l} Z_i^l = \mathrm{Min} \sum_{l=1}^{L} \sum_{i=1}^{N^l} \left(\sum_{j=1}^{N^{(l-1)}} w_{ij}^l y_j^{l-1} y_i^l + b_i^l y_i^l \right)$$

Laws of Statistical Teleodynamics

First Law: A large system of competing goal-driven agents will dynamically evolve such that all agents will continuously strive to achieve maximum effective utility allowed by the constraints imposed by their operating environment.

Second Law: The system will reach an arbitrage equilibrium when the system's potential ϕ is maximized.

These two laws are generalizations of the thermodynamic laws for goal-driven agents.

This is what I was seeking in 1982! Found it in 2024!

Max $$\Phi_N = \sum_{l=1}^{L} \sum_{q=1}^{n} \left[\eta x_q^l \ln Z_q^l - \zeta x_q^l (\ln Z_q^l)^2 \right] + \mathscr{S}_N$$

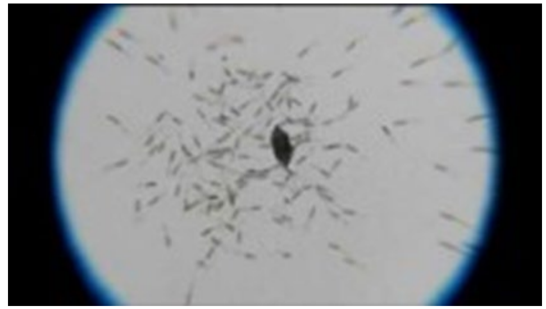
"Negative Gibbs Free Energy"
- G = - E + TS

Hopfield &
Boltzmann Machine

$$\text{Min} \sum_{l=1}^{L} \sum_{i=1}^{N^l} Z_i^l$$

Ising model

Equilibrium in Active Matter



- Passive matter
  - Forces are equal → Mechanical Equilibrium
  - Temperatures are equal → Thermal Equilibrium
  - Chemical potentials are equal → Phase Equilibrium
- Active matter has a new equilibrium
  - Effective utilities are equal → Arbitrage Equilibrium

Is there a Unifying Self-organizing Principle?

Nature uses only the longest threads to weave her patterns, so that each small piece of her fabric reveals the organization of the entire tapestry.

Richard P. Feynman

Wikipedia

- Maximum utility-driven **Arbitrage Equilibrium** could be such a thread.
- Could be the unifying principle in the self-organization and emergent behavior of active matter.
- This is what Adam Smith called the Invisible Hand!

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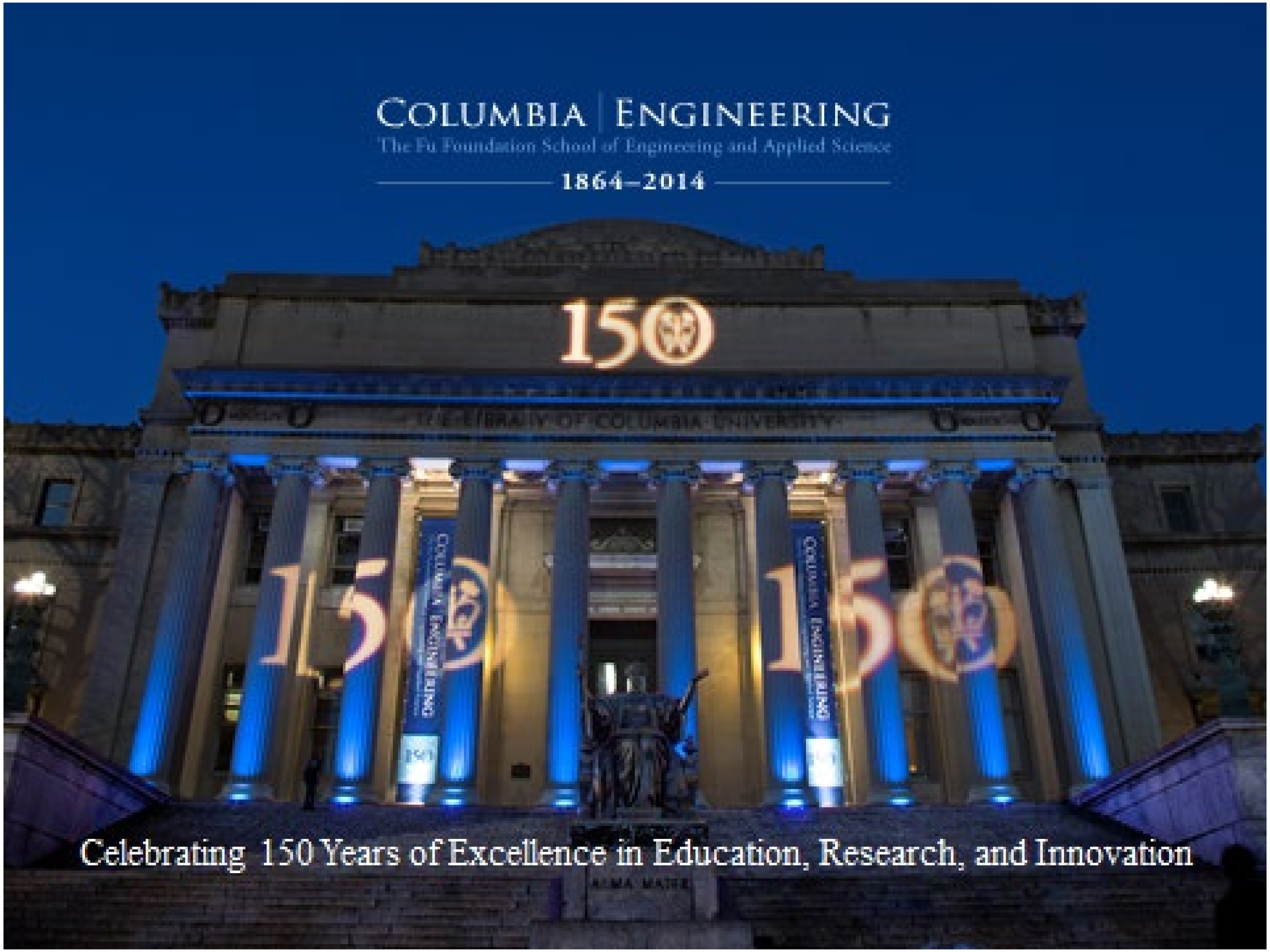
Thank You for Your Attention!



Questions?