Solving optimization problems with neural networks: The case of the travelling salesman problem

Workshop on Spins, Games and Networks

at

The Institute of Mathematical Sciences, Chennai

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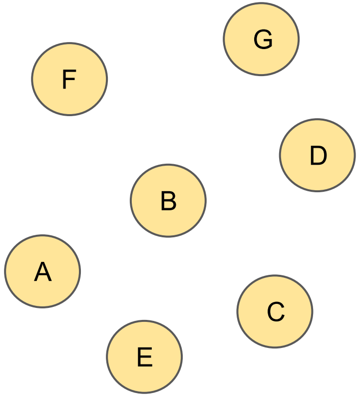
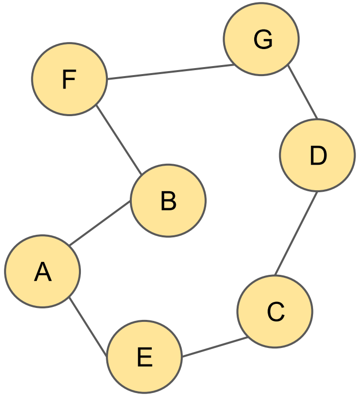
14.12.24

The travelling salesman problem



$$\begin{bmatrix} 0 & d_{AB} & d_{AC} & d_{AD} & d_{AE} & d_{AF} & d_{AG} \\ d_{BA} & 0 & d_{BC} & d_{BD} & d_{BE} & d_{BF} & d_{BG} \\ d_{CA} & d_{CB} & 0 & d_{CD} & d_{CE} & d_{CF} & d_{CG} \\ d_{DA} & d_{DB} & d_{DC} & 0 & d_{DE} & d_{DF} & d_{DG} \\ d_{EA} & d_{EB} & d_{EC} & d_{ED} & 0 & d_{EF} & d_{EG} \\ d_{FA} & d_{FB} & d_{FC} & d_{FD} & d_{FE} & 0 & d_{EG} \\ d_{GA} & d_{GB} & d_{GC} & d_{GD} & d_{GE} & d_{GF} & 0 \end{bmatrix}$$



Good path

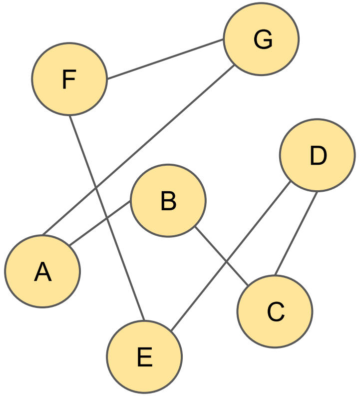
A F B G D C E A

$$d_{AB} + d_{BF} + d_{FG} + d_{GD} + d_{DC} + d_{CE} + d_{EA}$$

Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?

OR

What is the shortest Hamiltonian path/tour/route?



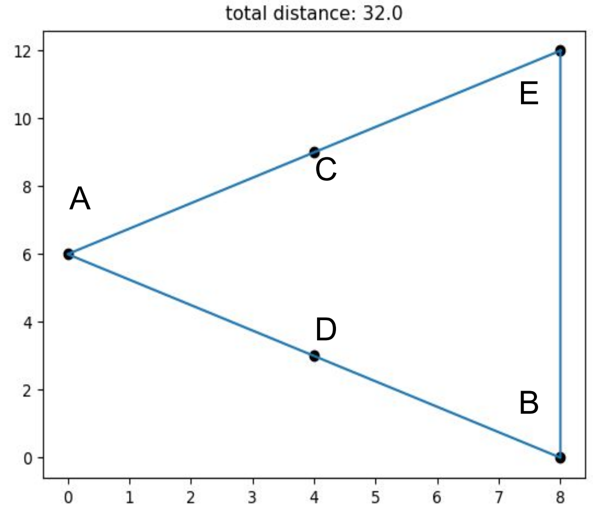
Bad path

A B C D E F G A

$$d_{AB} + d_{BC} + d_{CD} + d_{DE} + d_{EF} + d_{FG} + d_{GA}$$

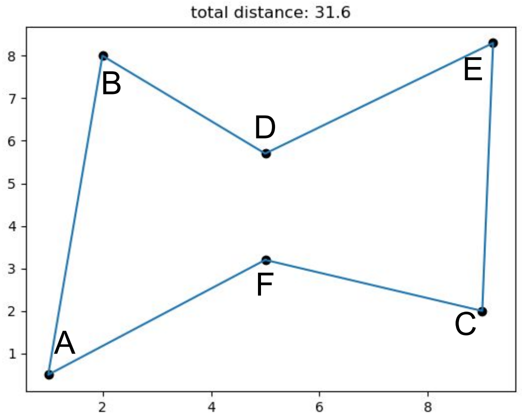
Empathy for the network

$$
\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} &
\begin{bmatrix}
0 & 10 & 5 & 5 & 10 \\
10 & 0 & 9.85 & 5 & 12 \\
5 & 9.85 & 0 & 6 & 5 \\
5 & 5 & 6 & 0 & 9.85 \\
10 & 12 & 5 & 9.85 & 0
\end{bmatrix}
\end{array}
$$



Empathy for the network hard

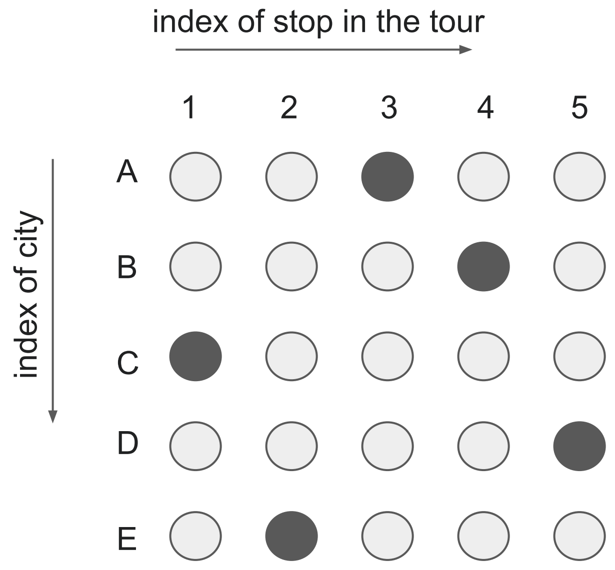
$$\begin{array}{c c} & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} & \begin{bmatrix} 0 & 7.57 & 8.14 & 6.56 & 11.32 & 4.83 \\ 7.57 & 0 & 9.22 & 3.78 & 7.21 & 5.66 \\ 8.14 & 9.22 & 0 & 5.45 & 6.3 & 4.18 \\ 6.56 & 3.78 & 5.45 & 0 & 4.94 & 2.5 \\ 11.32 & 7.21 & 6.3 & 4.94 & 0 & 6.61 \\ 4.83 & 5.66 & 4.18 & 2.5 & 6.61 & 0 \end{bmatrix} \end{array}$$



Setting up the neural network

A salesman has to travel through n cities.
The salesman has to make n stops such that a new city appears at a every stop and the total distance travelled is minimum.

$V_{X,i}$- 1 if stop i of the tour is at city X; 0 otherwise



The tour is C E A B D

Constraints for a valid tour of the salesman:

1. A city is visited exactly once
$$\sum_i V_{X,i} = 1, \quad \forall X$$

2. Two cities cannot be visited simultaneously at the same stop
$$\sum_X V_{X,i} = 1, \quad \forall i$$

Cost incurred for different paths –
$$\mathrm{d_{tot}} = \sum_{X,Y,i} d_{XY} V_{X,i}(V_{Y,i+1} + V_{Y,i-1})$$

The total energy function becomes

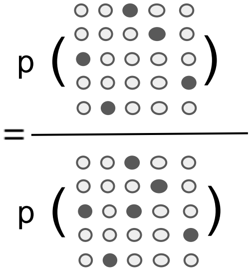
$$E = \frac{A}{2}\sum_X \sum_i \sum_{j \neq i} V_{X,i} V_{X,j} + \frac{B}{2}\sum_i \sum_X \sum_{Y \neq X} V_{X,i} V_{Y,i} + \frac{C}{2}\sum_{X,Y,i,j} (V_{X,i}V_{Y,j} - N)^2 + \frac{D}{2} d_{tot}$$

Hopfield, Tank, 1985, Biol. Cybern., 52

Discrete dynamics

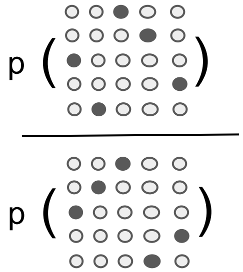
The dynamics takes place in the corners of a N^2 dimensional hypercube for an N-city problem

$$\frac{p(V_a)}{p(V_b)} = e^{-\frac{1}{T}(E(V_a)-E(V_b))} =$$

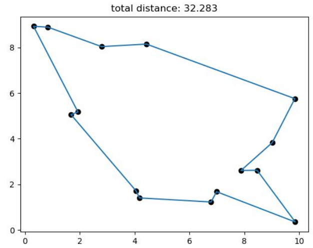


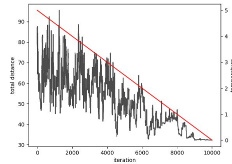
$$p(V_{X,i} = V^a_{X,i}) = \frac{1}{1 + e^{\frac{1}{T}(E(V_a)-E(V_b))}}$$

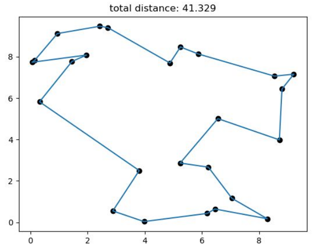
Instead, swap columns of an initial suboptimal valid tour

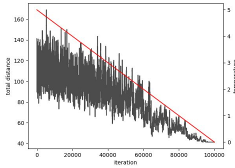


$$= \frac{1}{1 + e^{\frac{1}{T}(d_a - d_b)}}$$

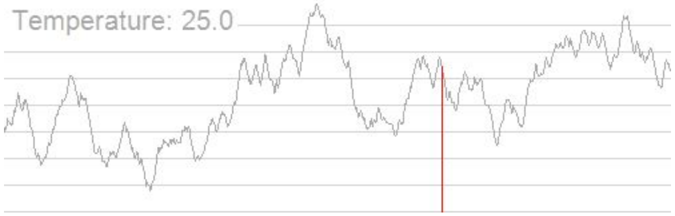


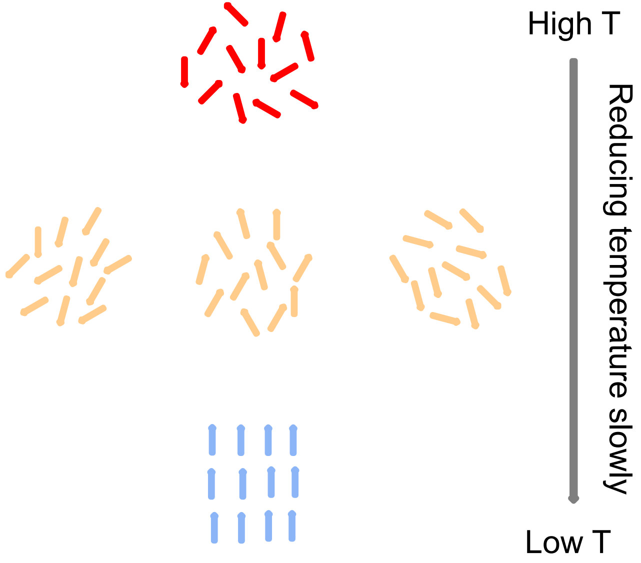






Simulated Annealing





Kirkpatrick et. al. 1983, 220 (4598)

Continuous dynamics

The dynamics takes place in $\mathbb{R}^{N^2}$
dimension for an N-city problem

The initial condition is kept at the centre
to ensure non-convergence to bad solutions
and with a tiny noise to break symmetry.

$$E = \frac{\gamma}{2}\left(\sum_i\left(\sum_X V_{X,i} - 1\right)^2 + \sum_X\left(\sum_i V_{X,i} - 1\right)^2\right) + \frac{D}{2} d_{tot}$$

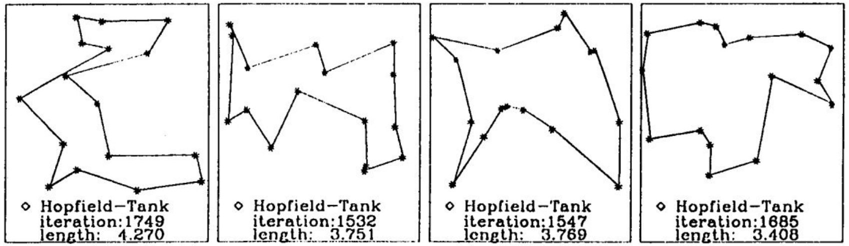
Continuous version of Hopfield model

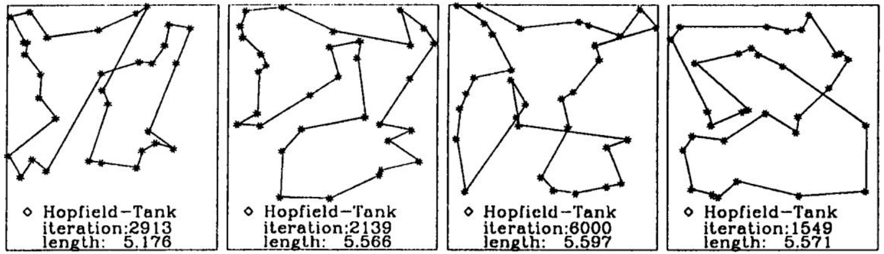
$$\frac{dU_i}{dt} = -\tau U_i - \frac{\partial E}{\partial V_i}$$

$$\frac{\partial E}{\partial V_i} = -\sum_{j \neq i} W_{ij} V_j + \theta_i$$

$$V_i = \sigma_T(U_i)$$

Hopfield, 1984, PNAS 81





Brandt et. al., 1988, IEEE,(333-340)

Thank you

Questions?