Spins, Games and Networks

Congestion and traffic routing

Sushmita Gupta

23rd December, 2024

Games in the wild

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So what is a (congestion) game ?

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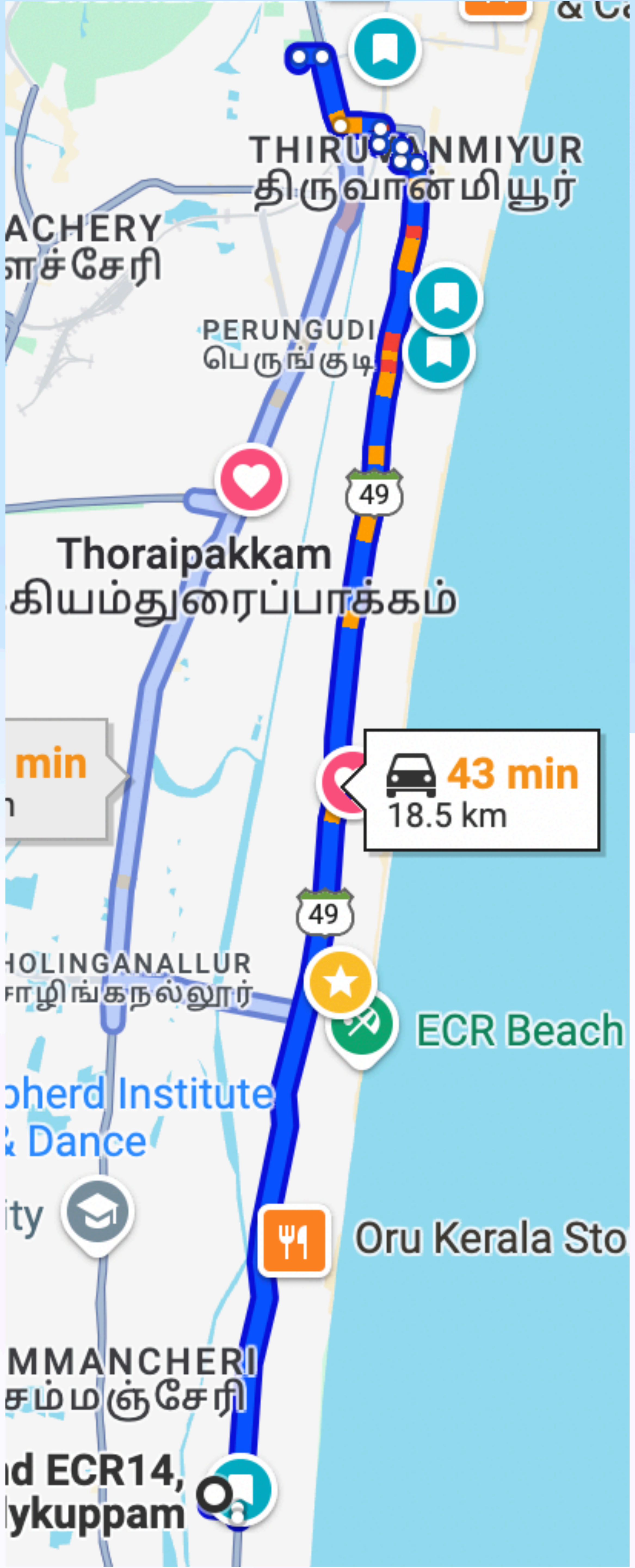
This map will give us the answer



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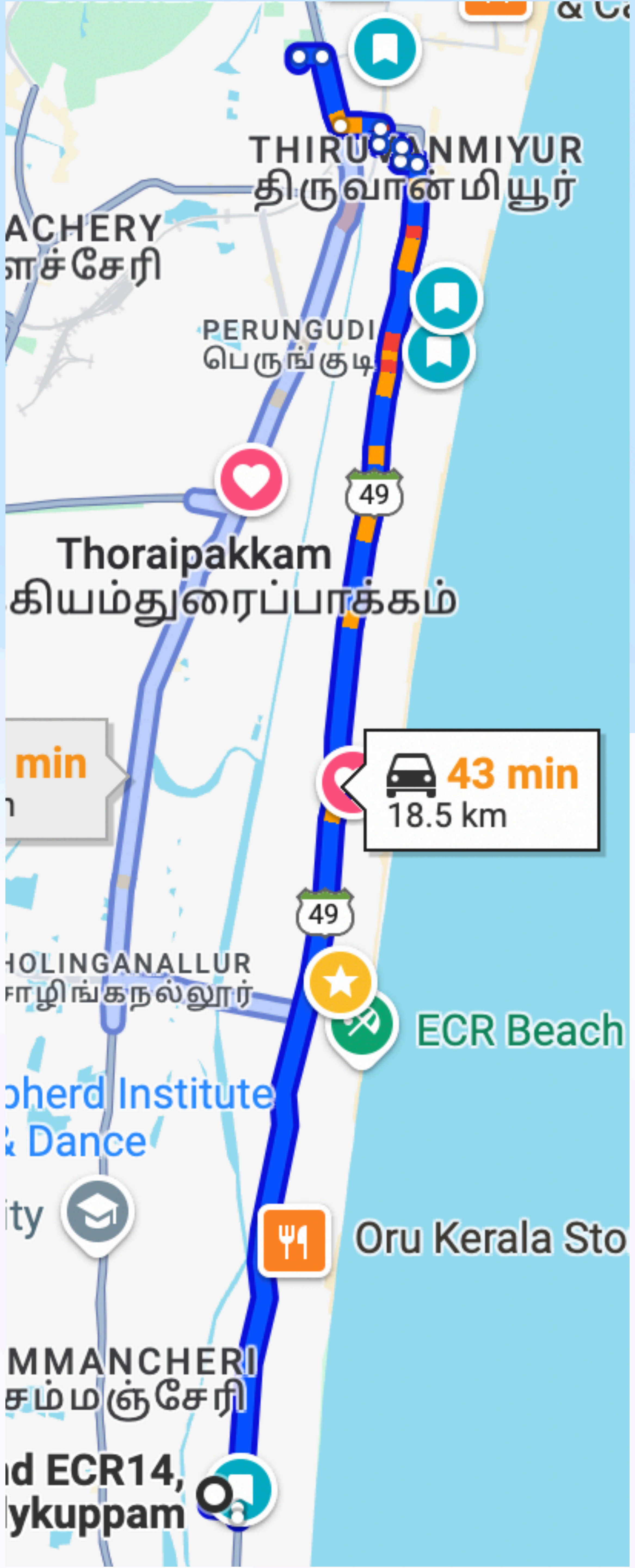
When a situation consists of <u>entities</u>:



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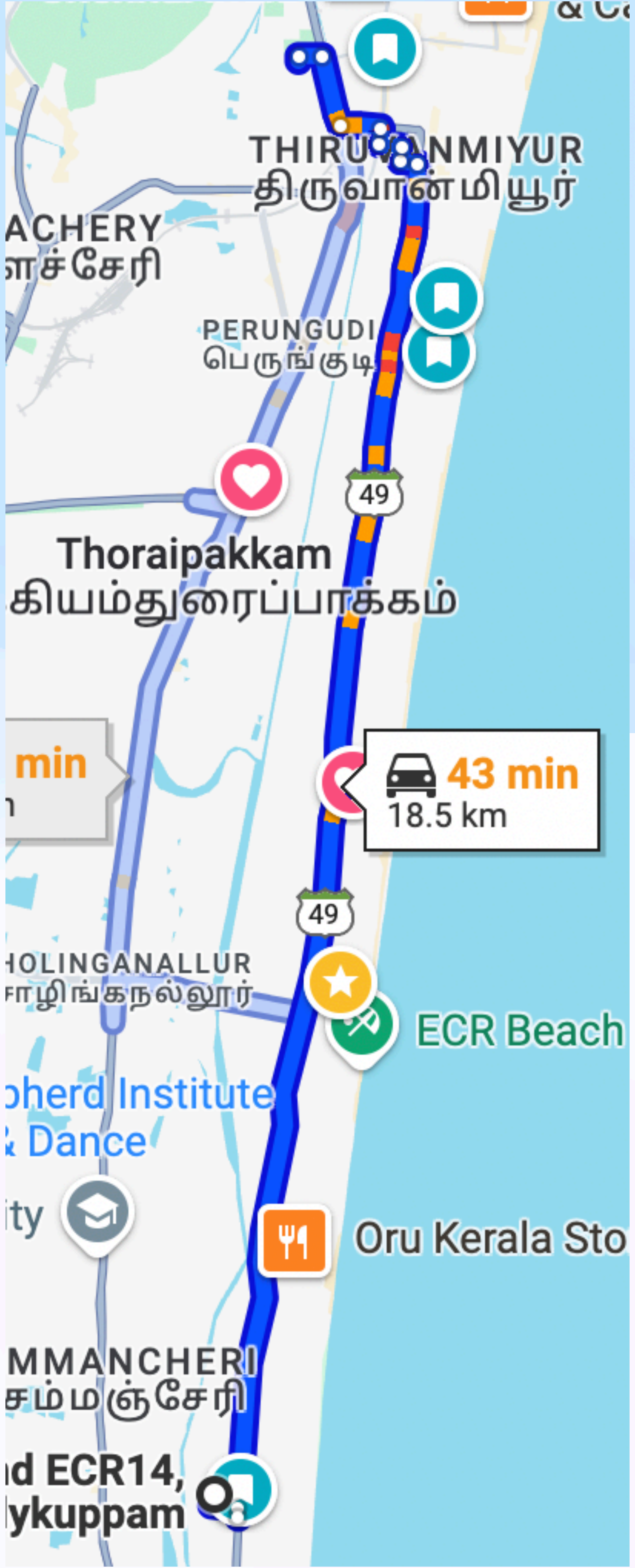

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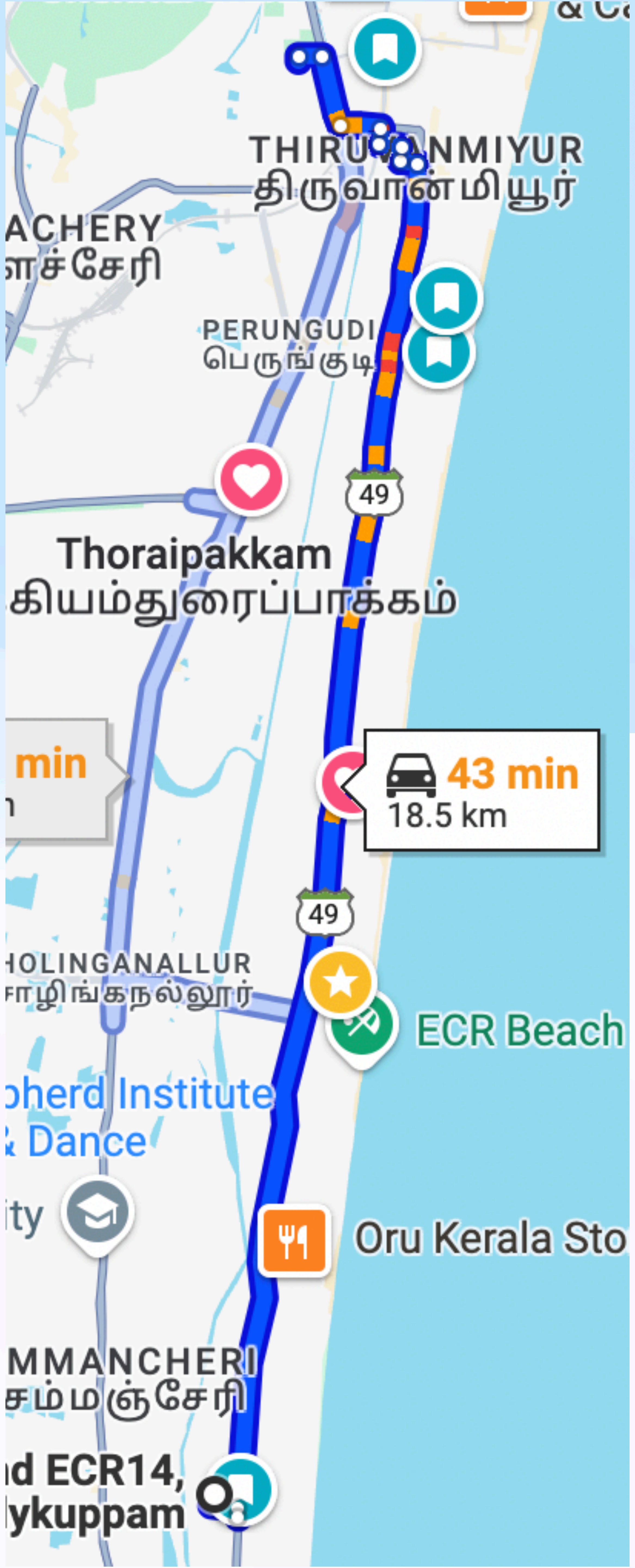
- whose outcome depends on the behaviour of the other entires



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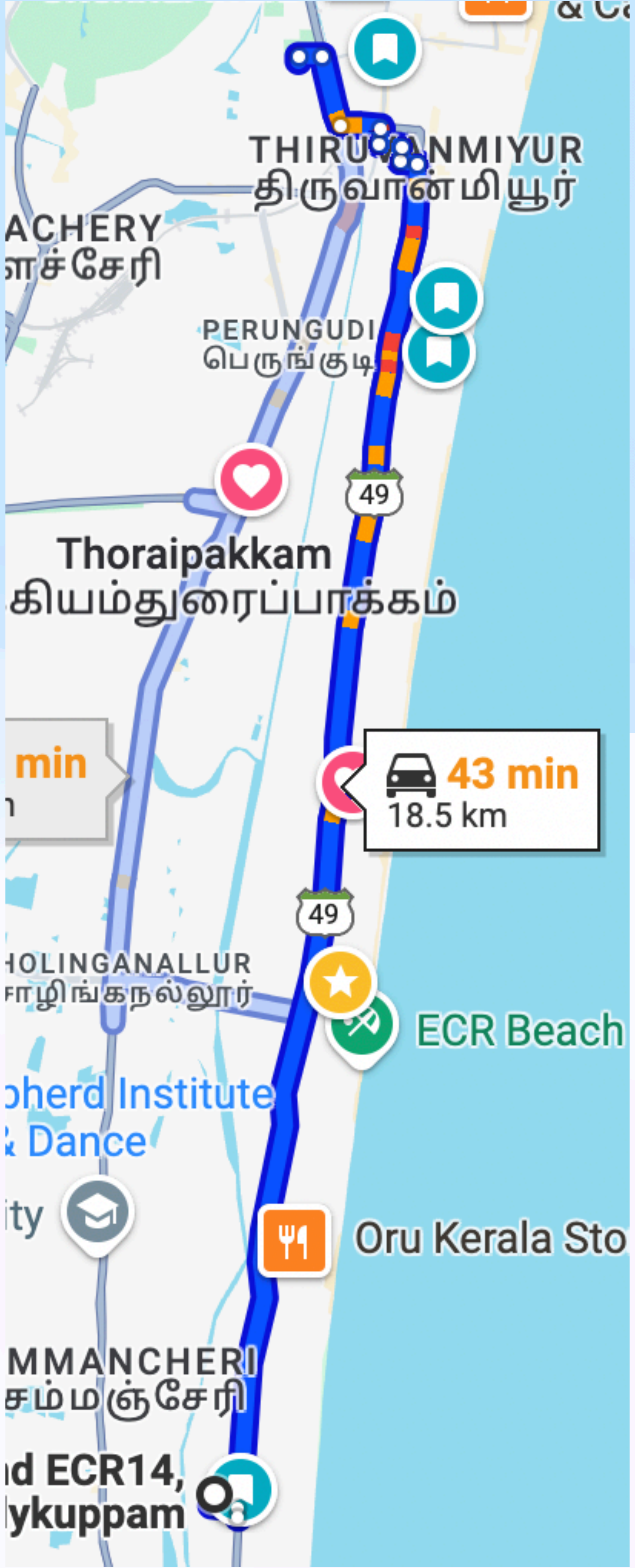

When a situation consists of entities:

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- And its behaviour can differ based on (knowledge) of other people's behavior

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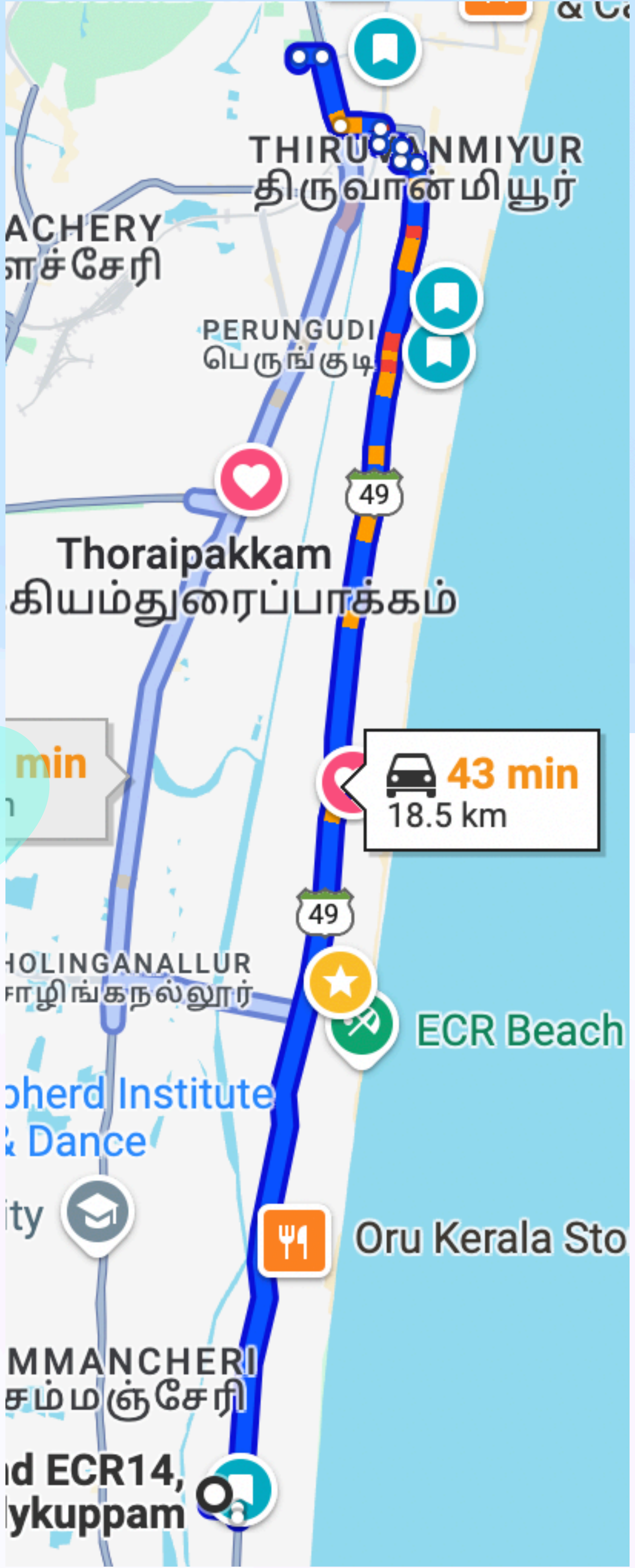
Strategic Behavior



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When a situation consists of <u>entities</u>:

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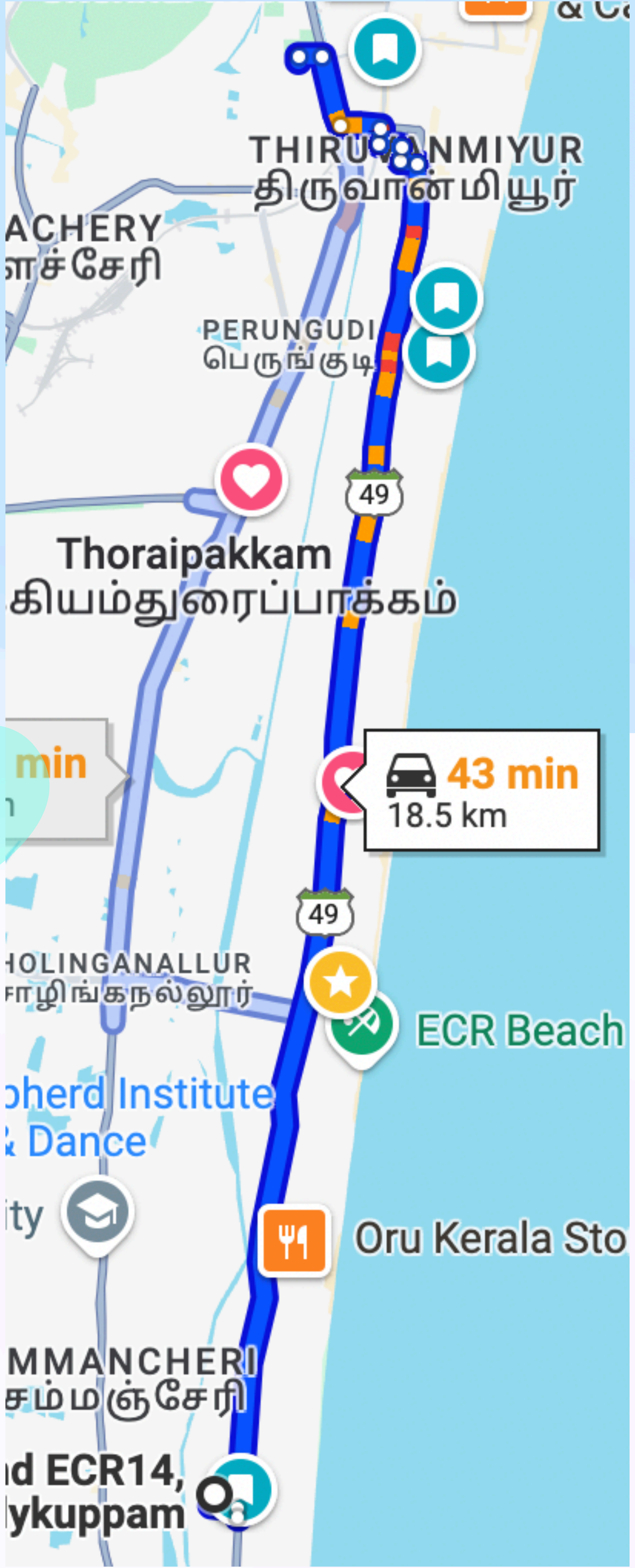
Strategic Behavior

When the outcome of one depends on the actions of other...we have a game!



Let's try to understand this phenomenon

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Pigou's Network 1920

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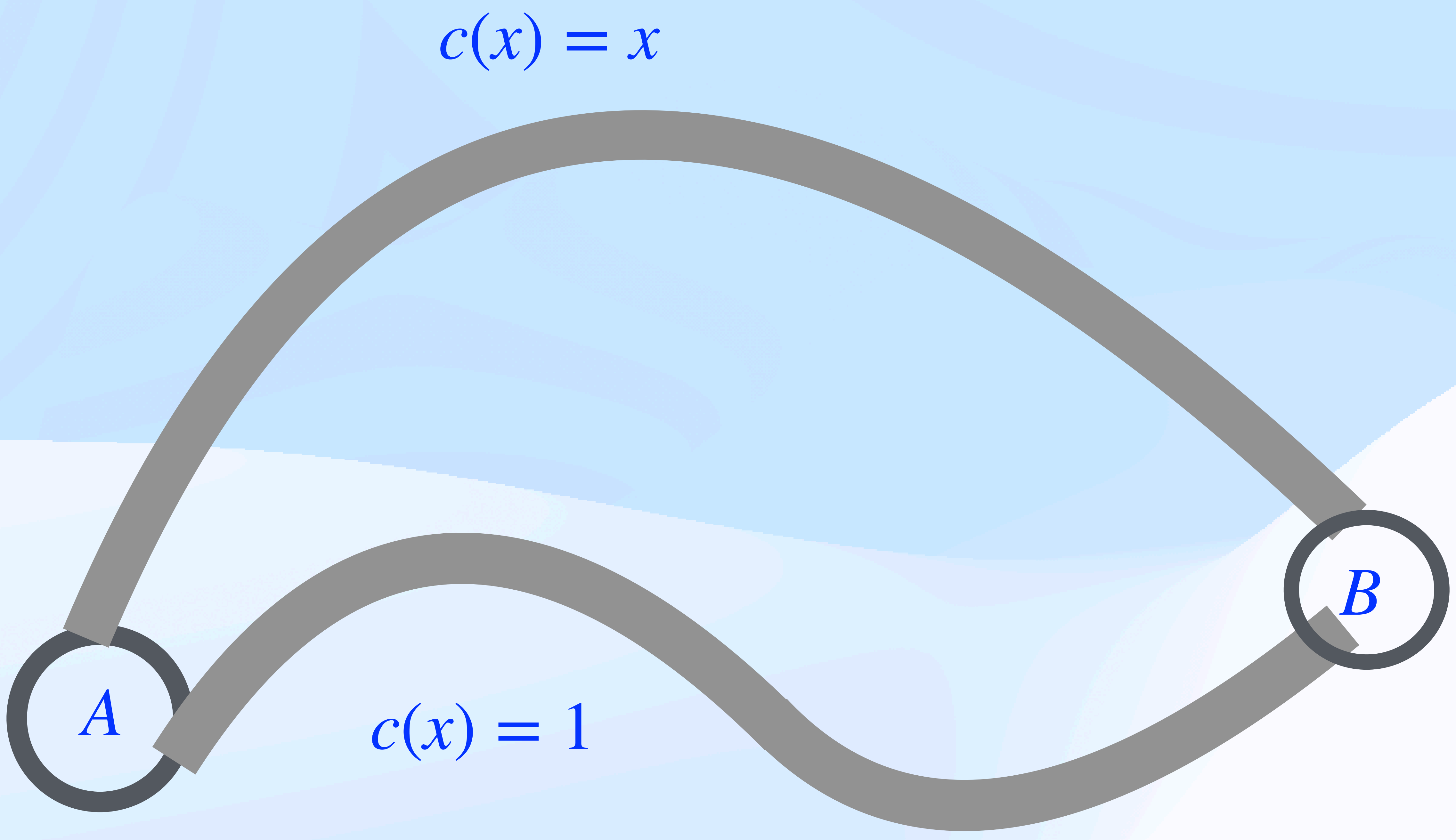

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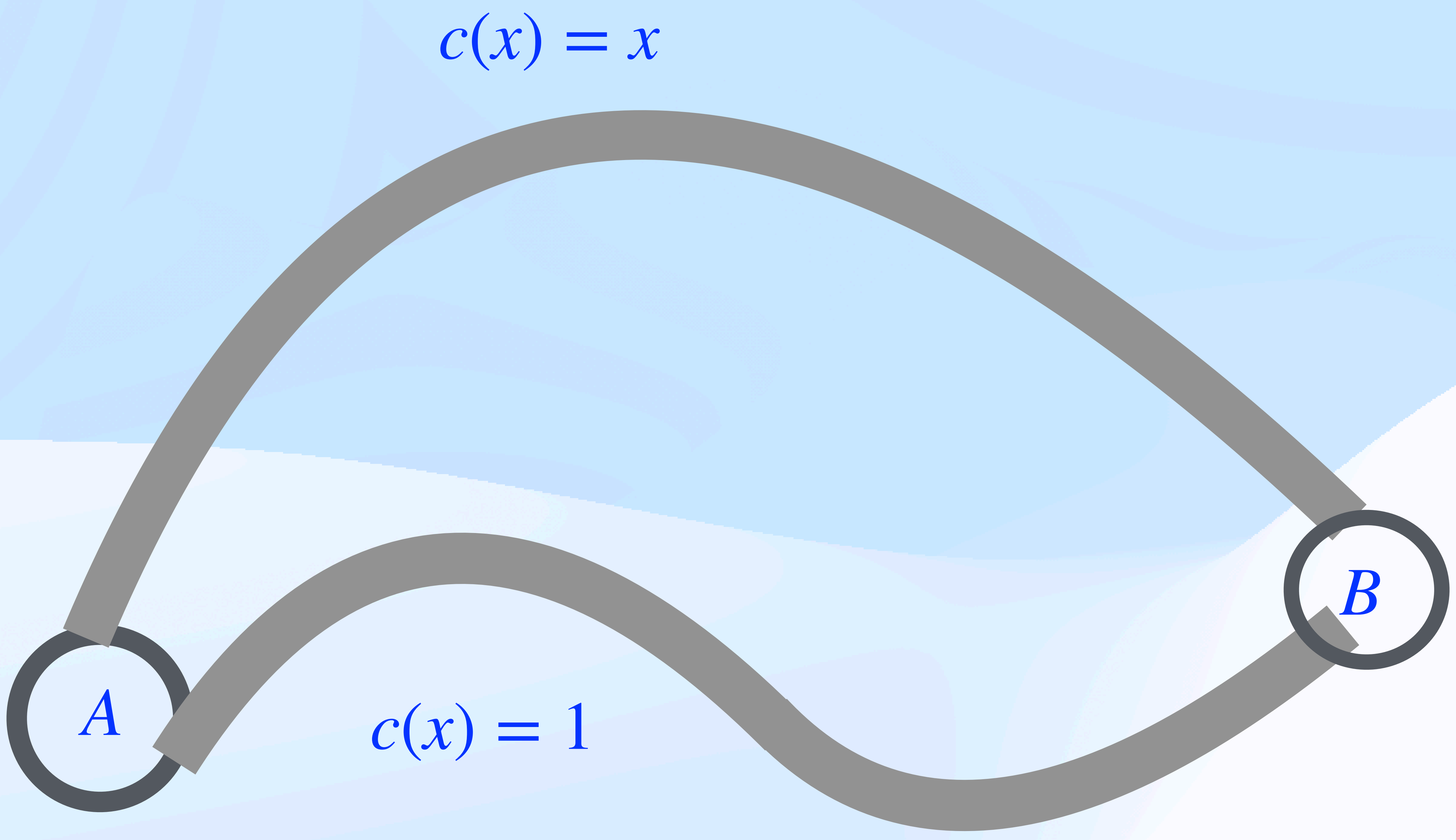
Pigou's Network 1920

1 unit of traffic



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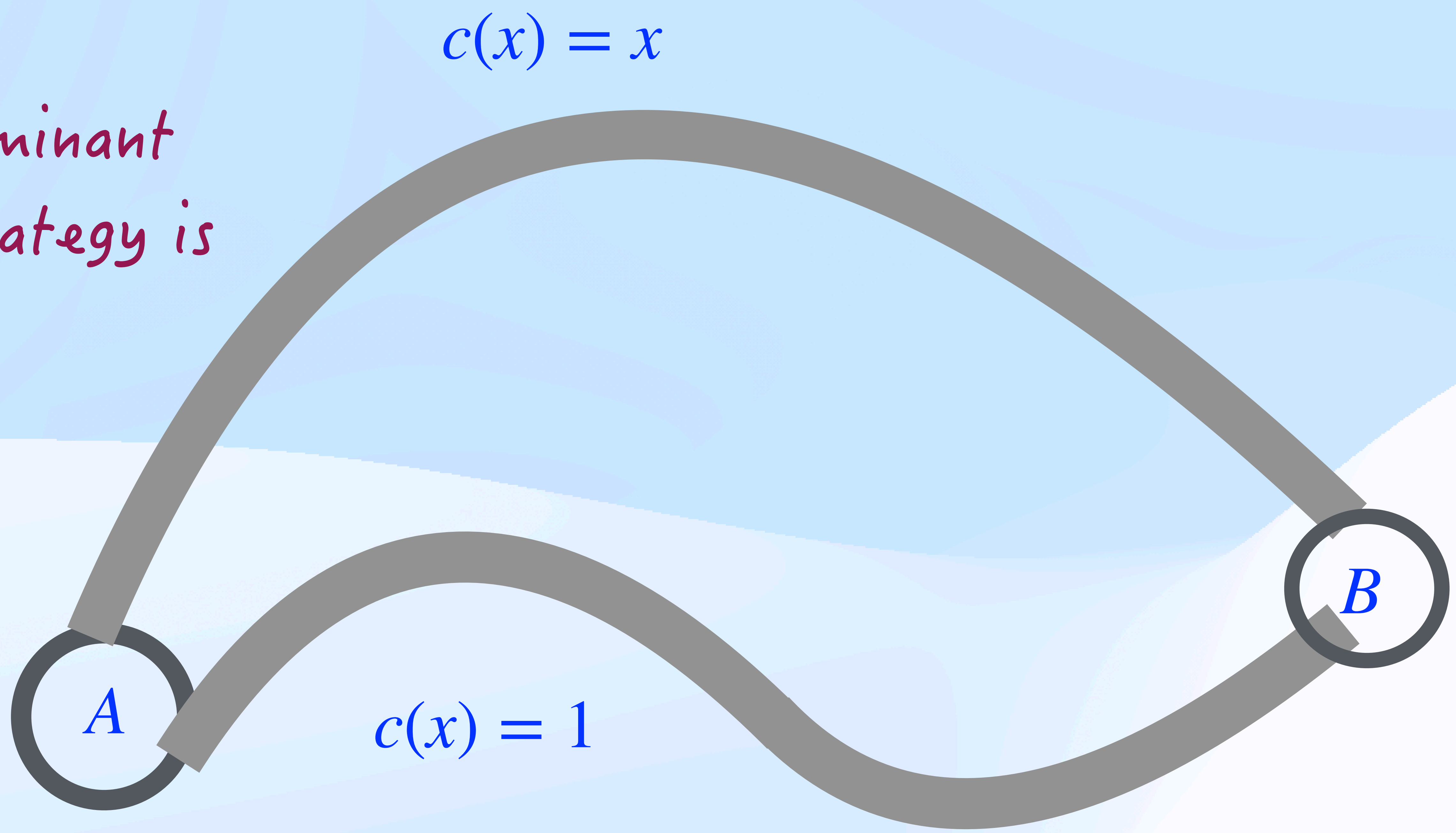

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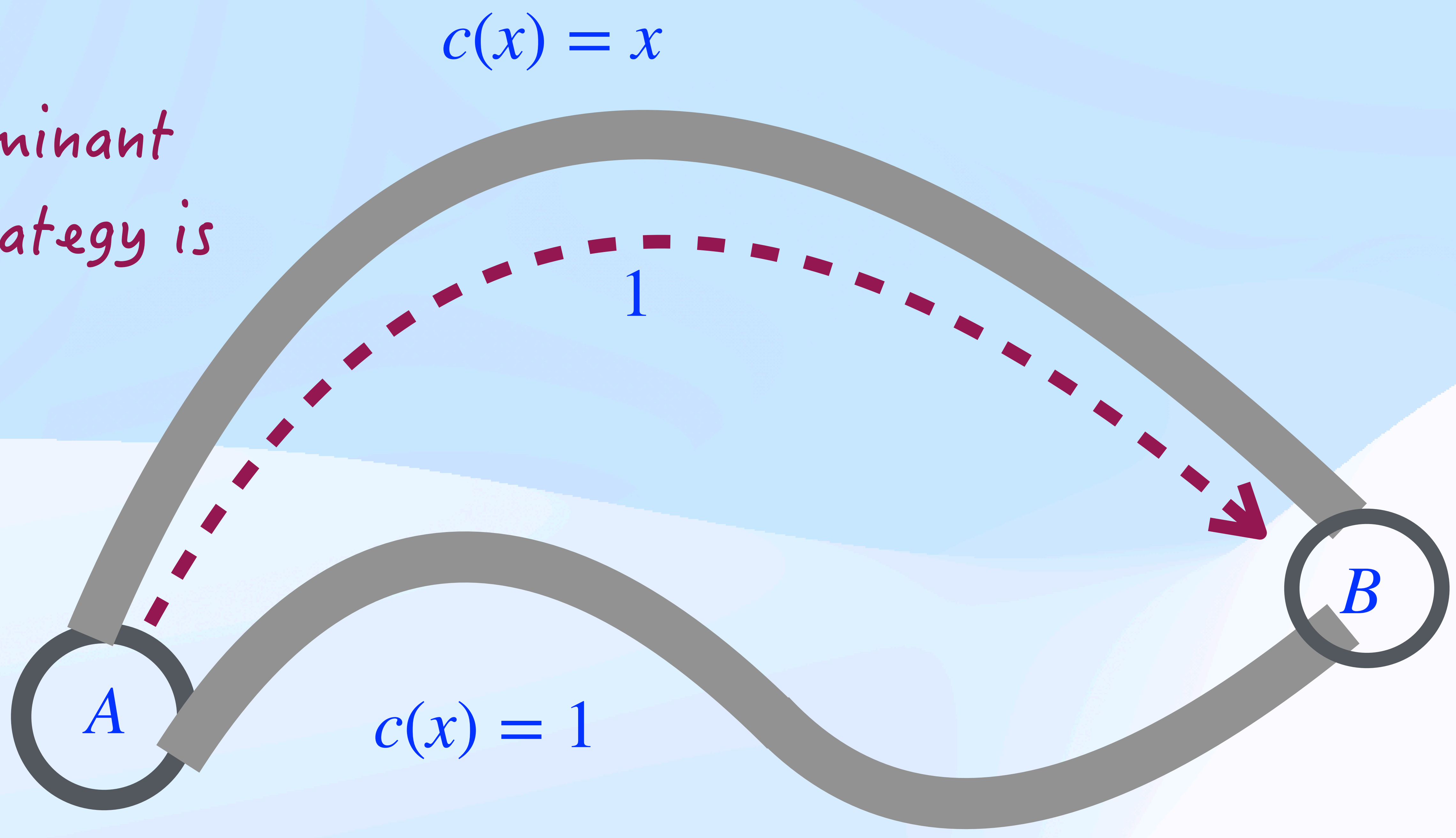

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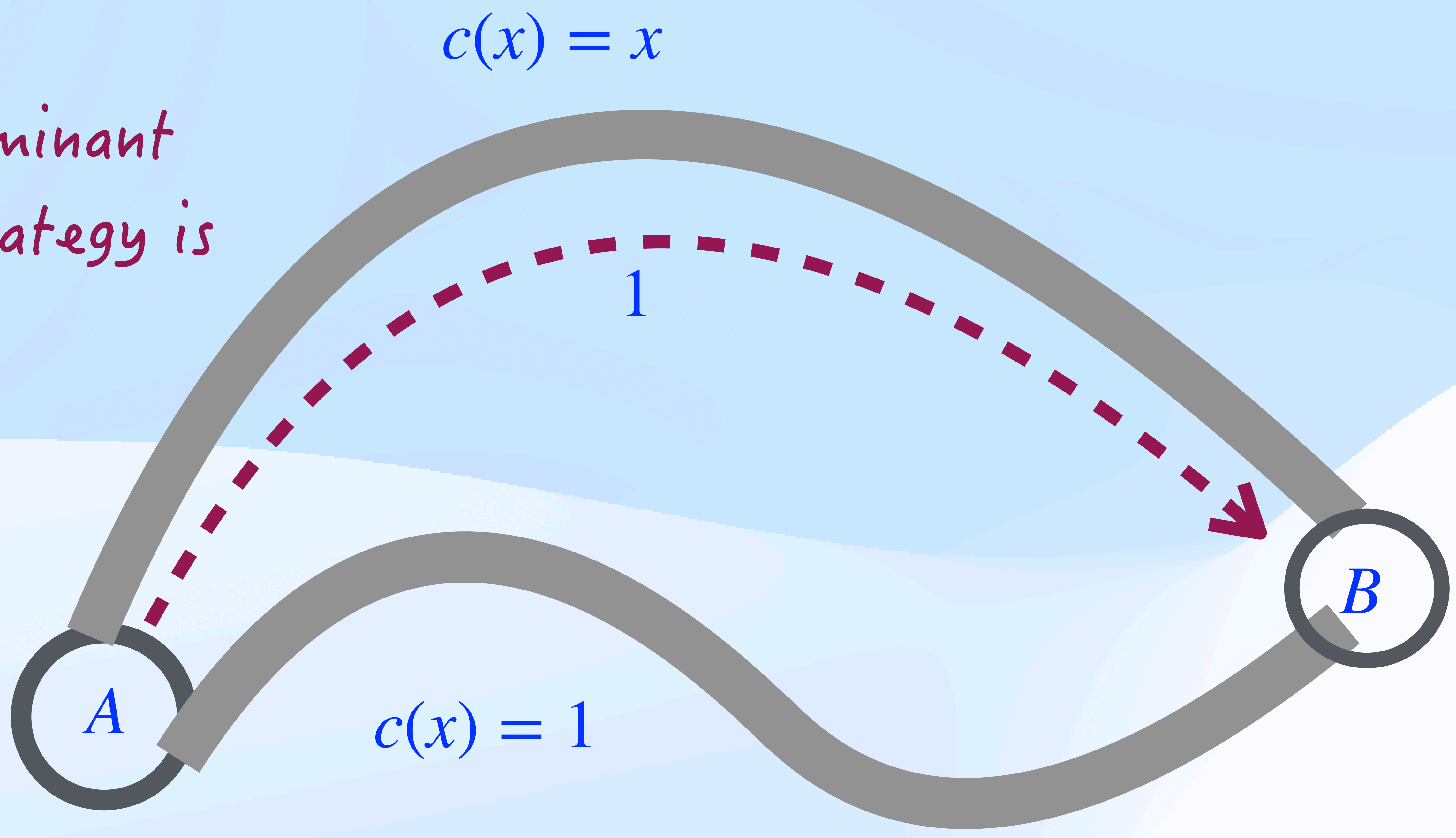

Expected travel time in DS equilibria $= 1 \cdot 1 = 1$

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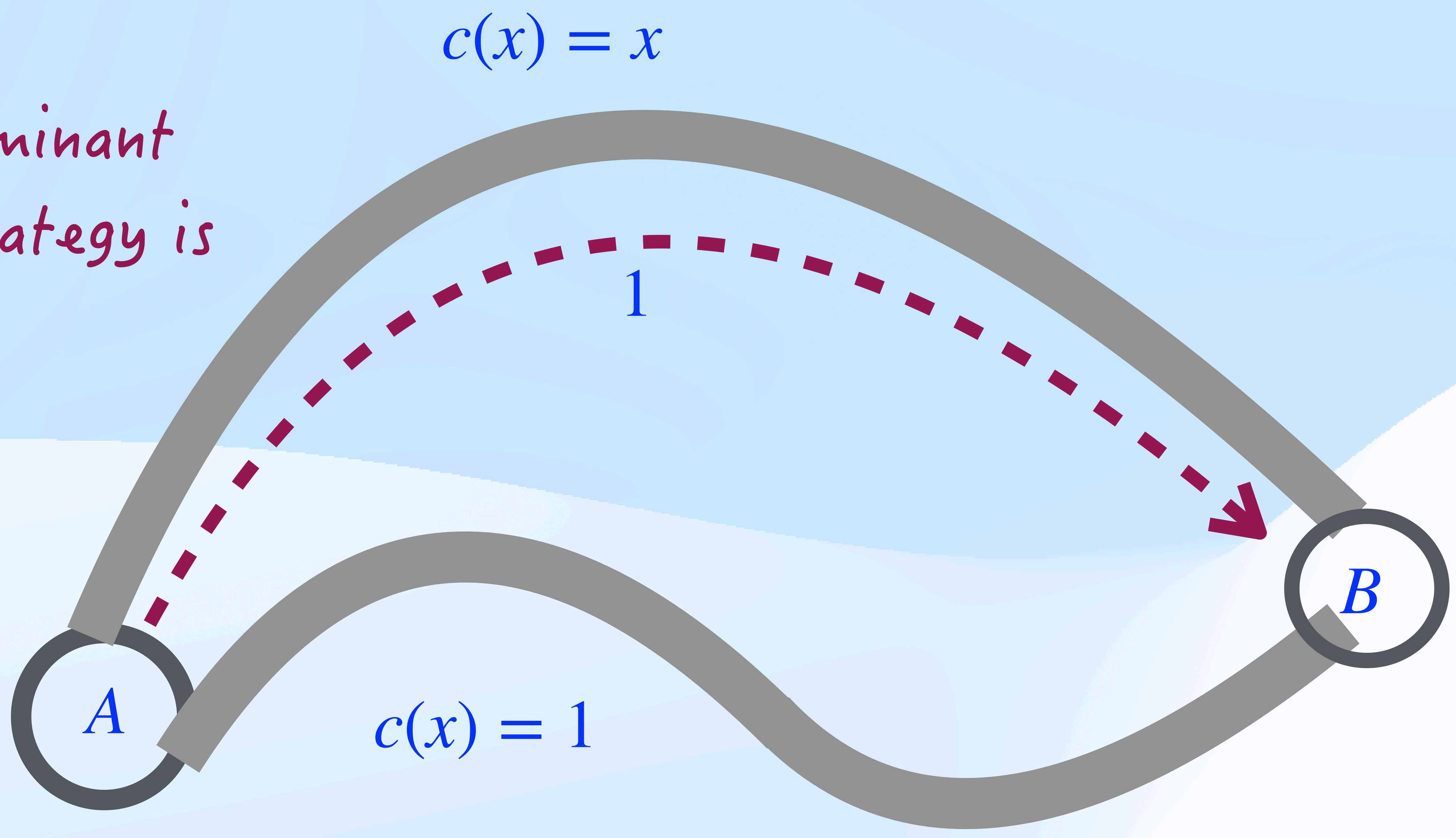

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Q: Is this the ideal outcome ?

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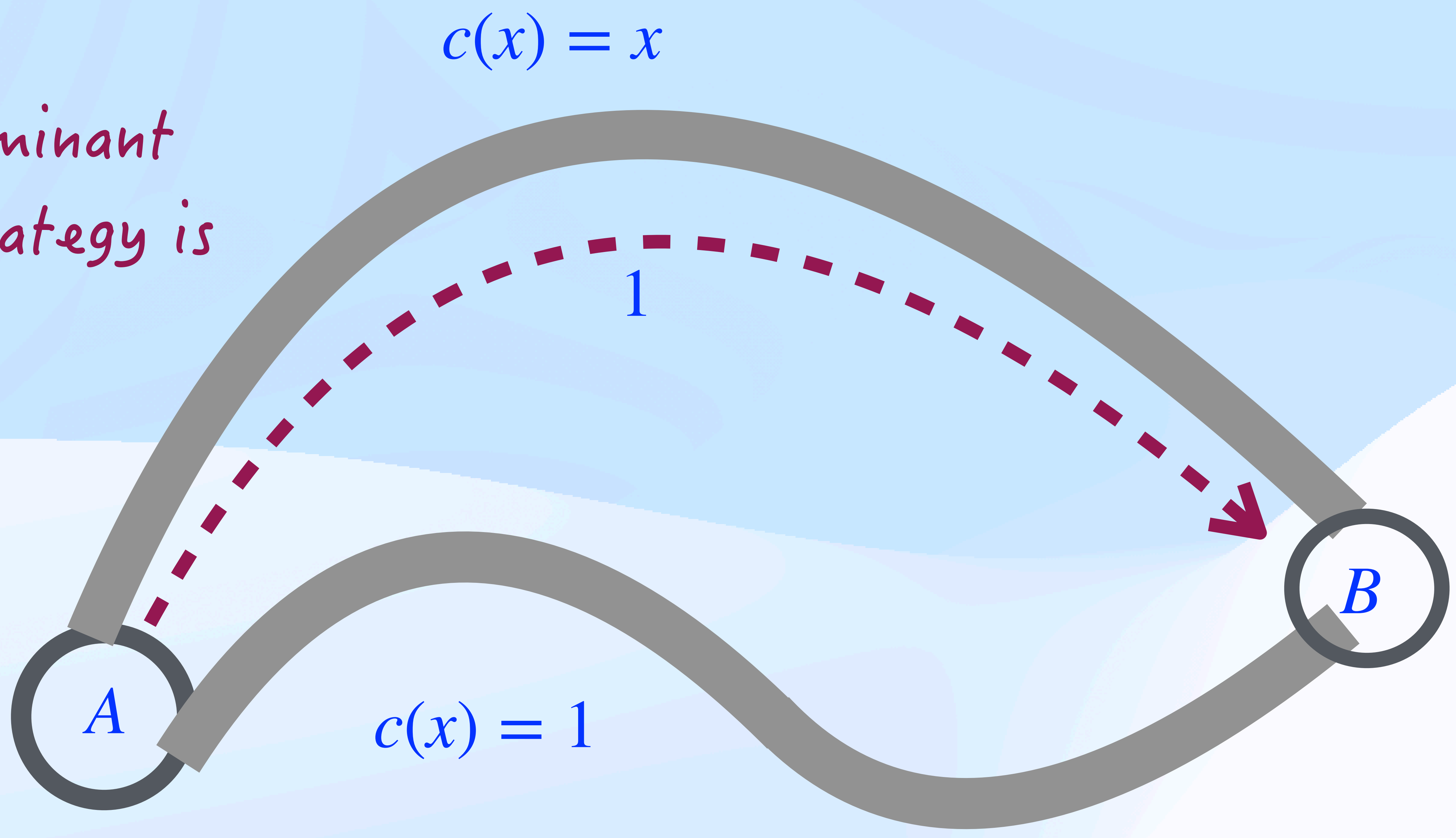

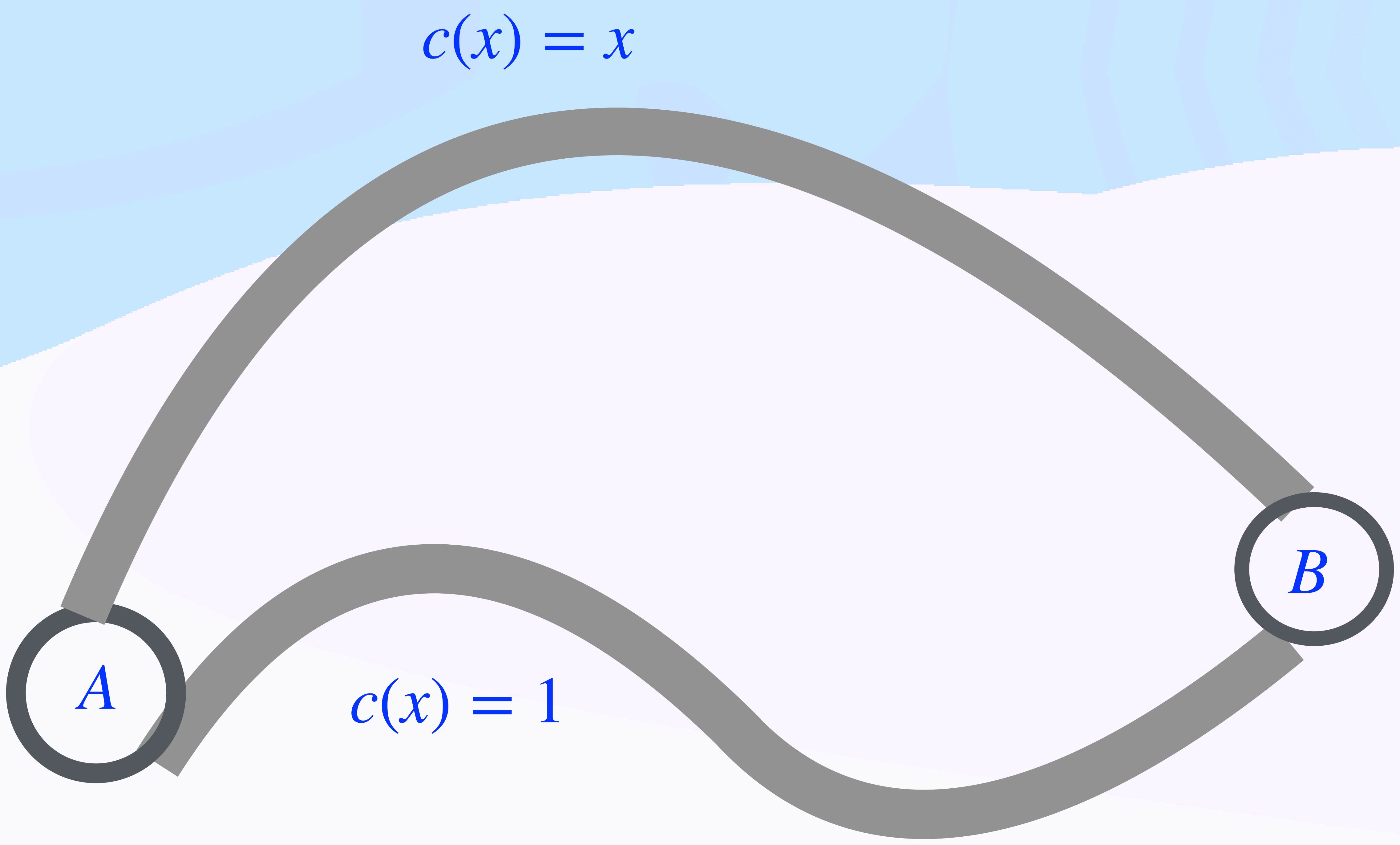

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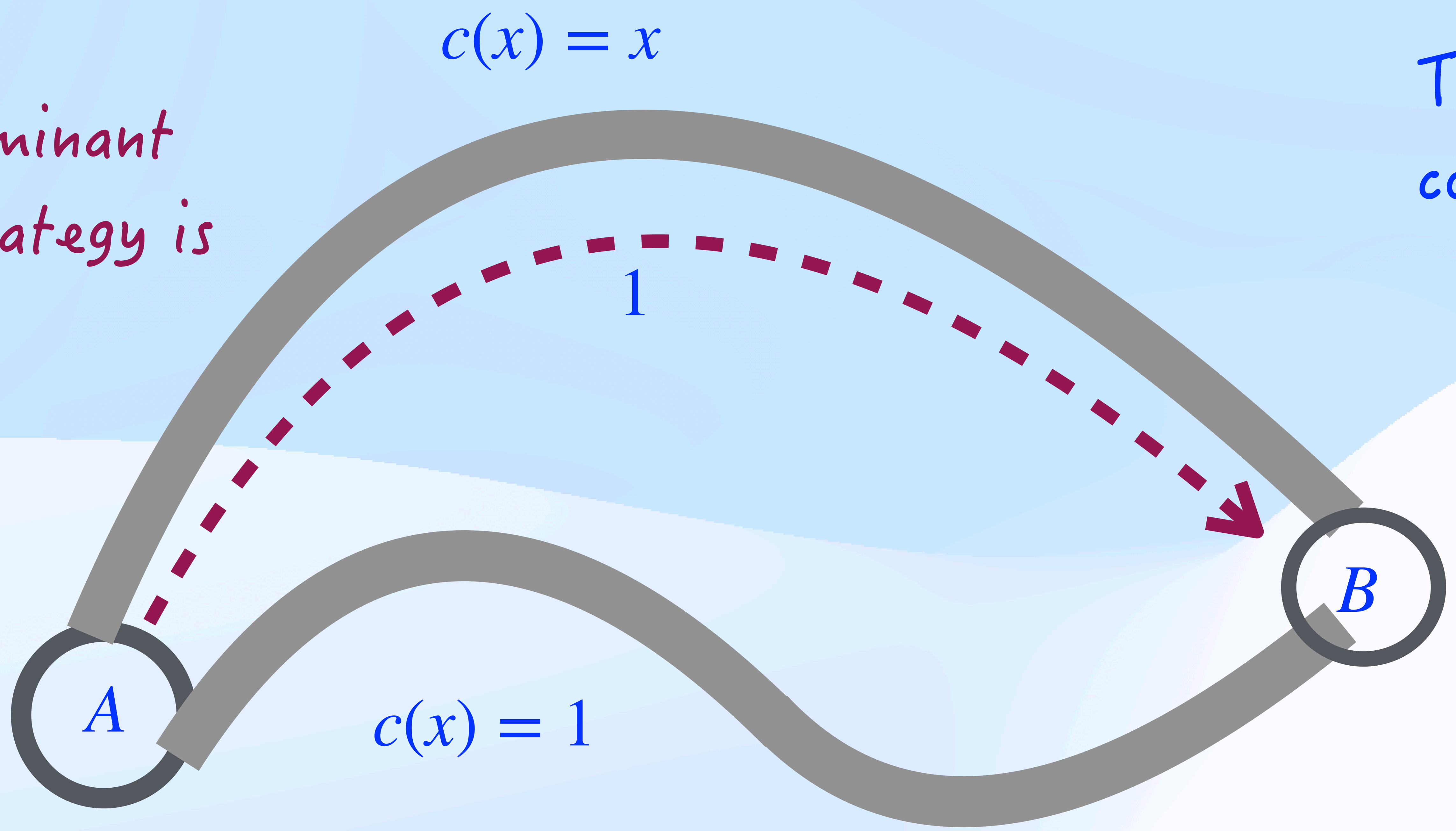
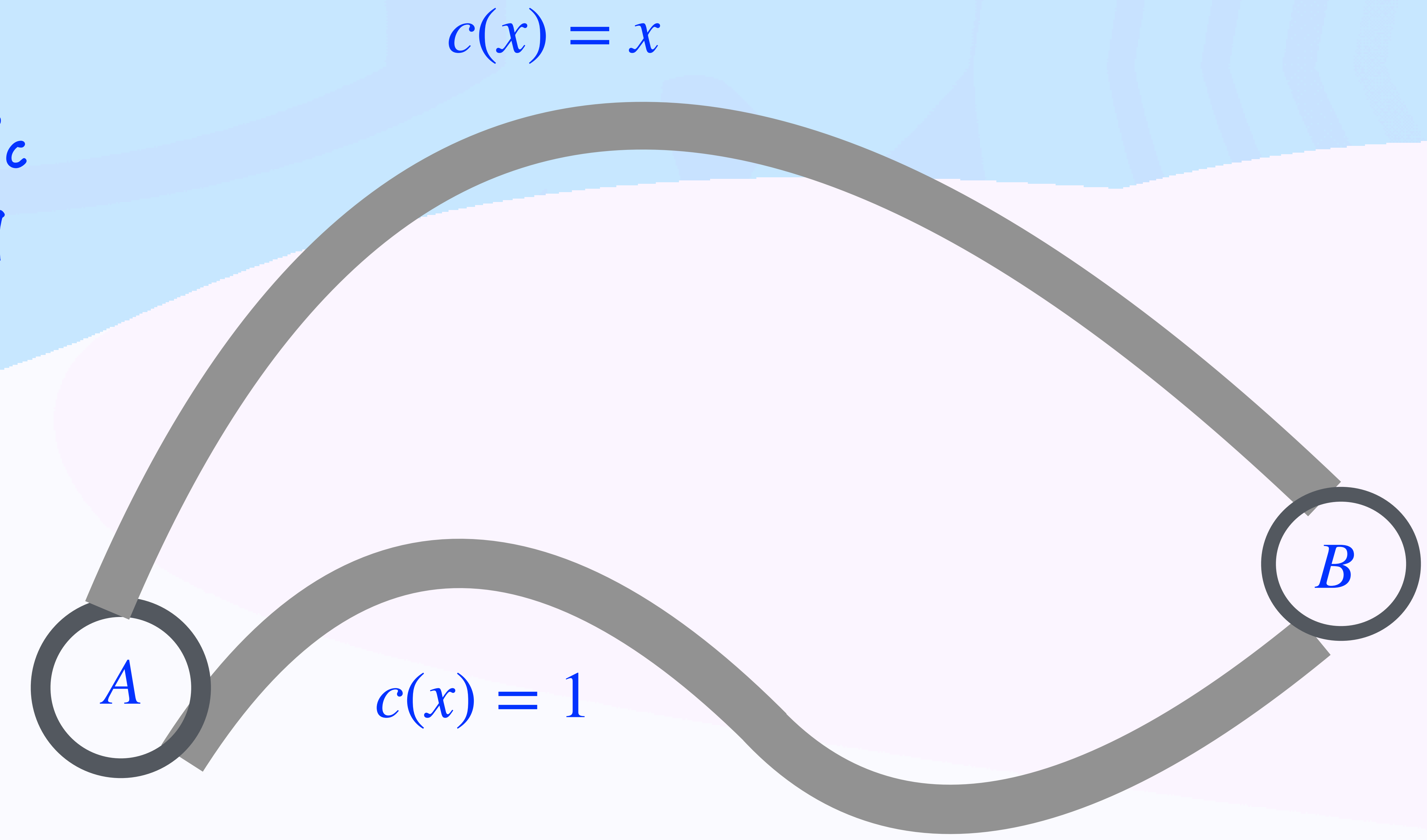


Traffic control



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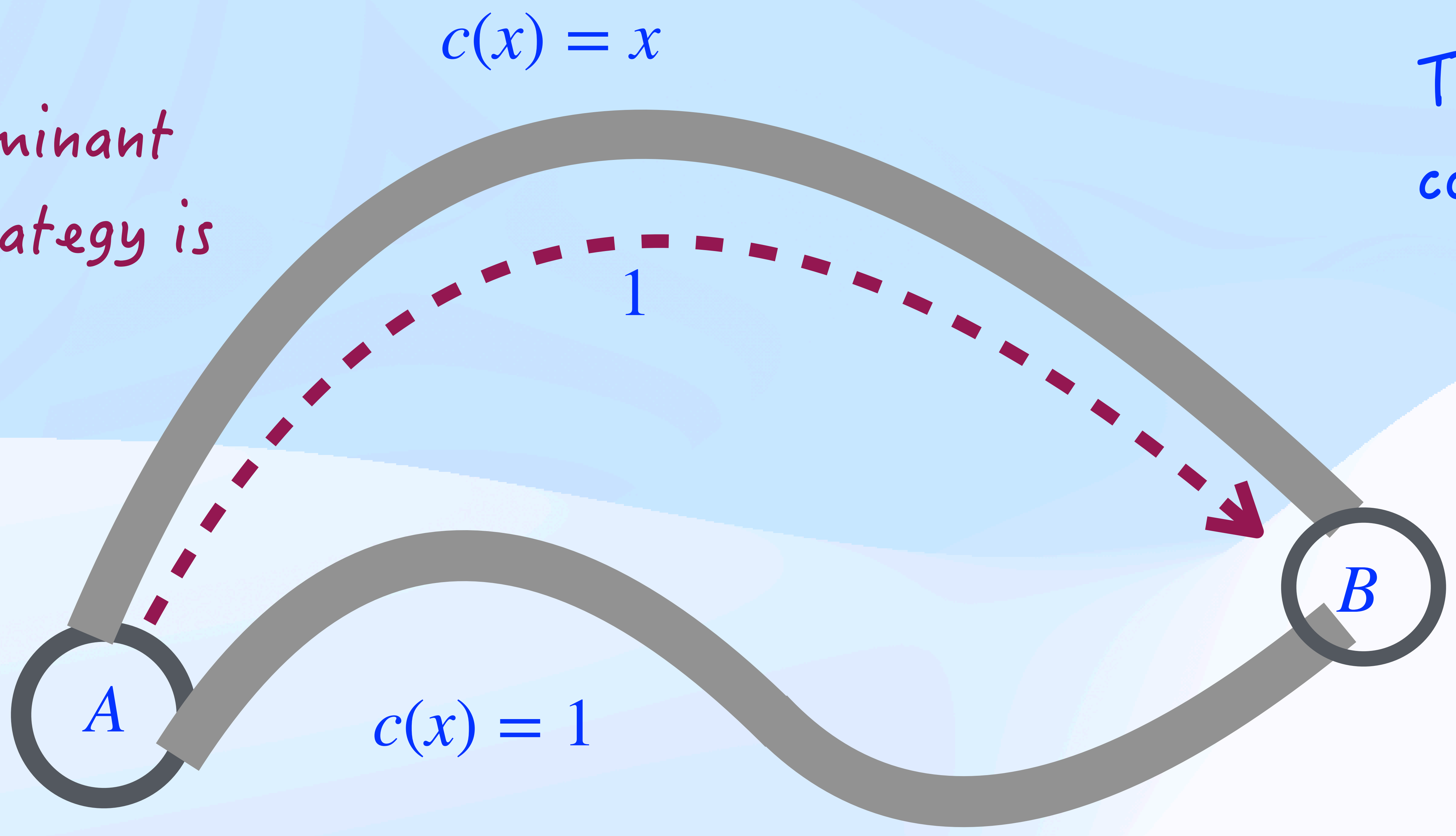

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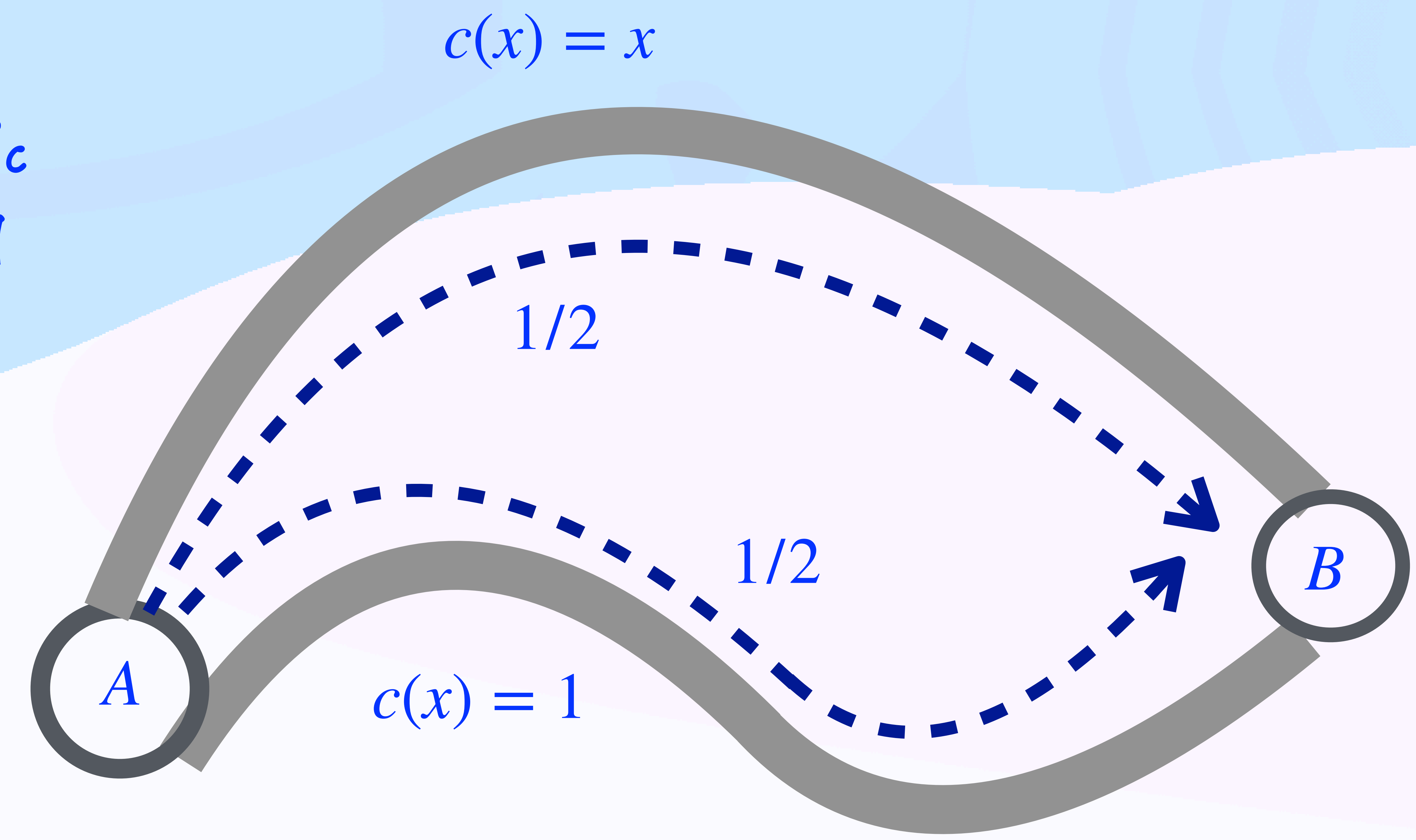

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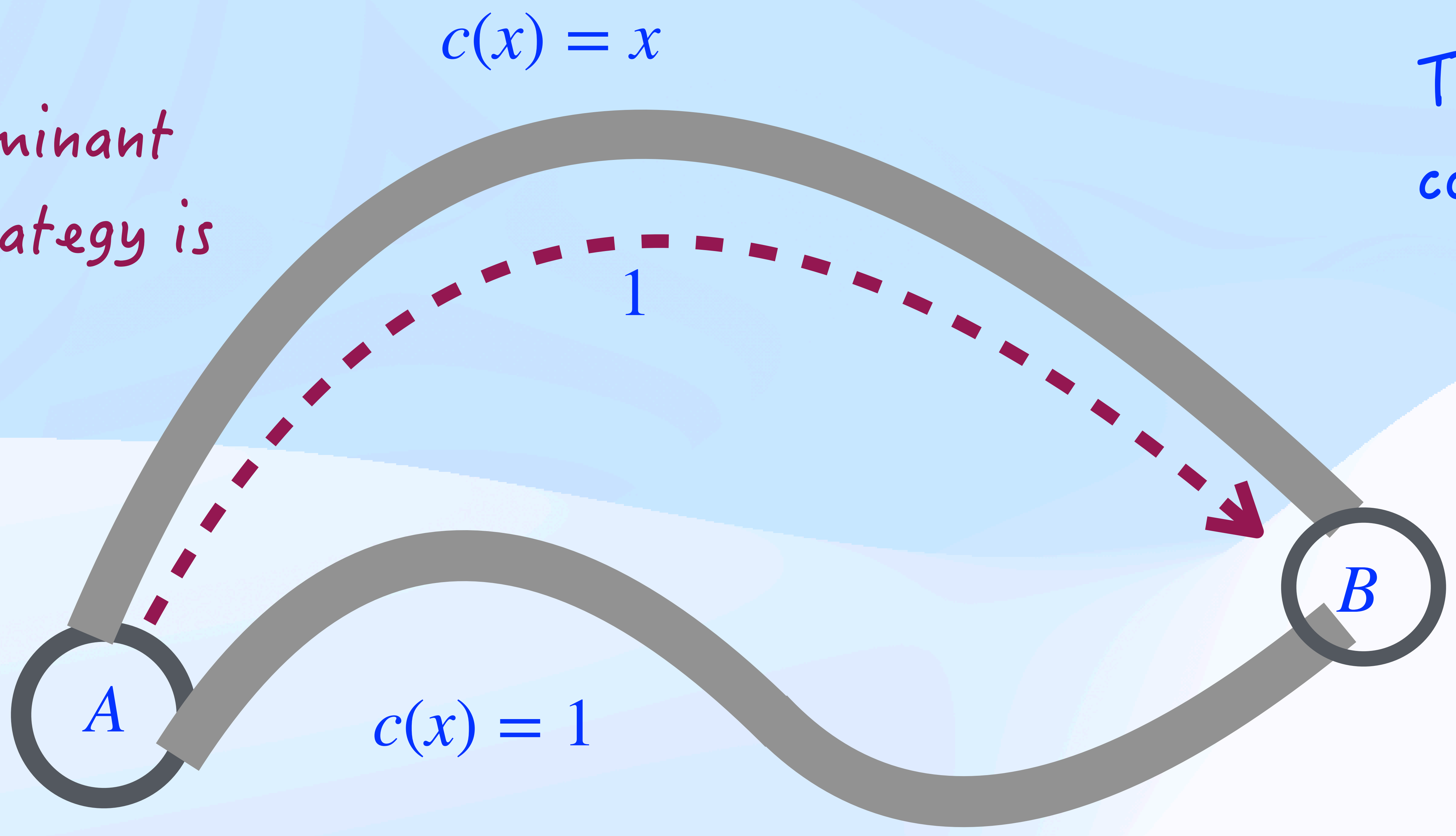

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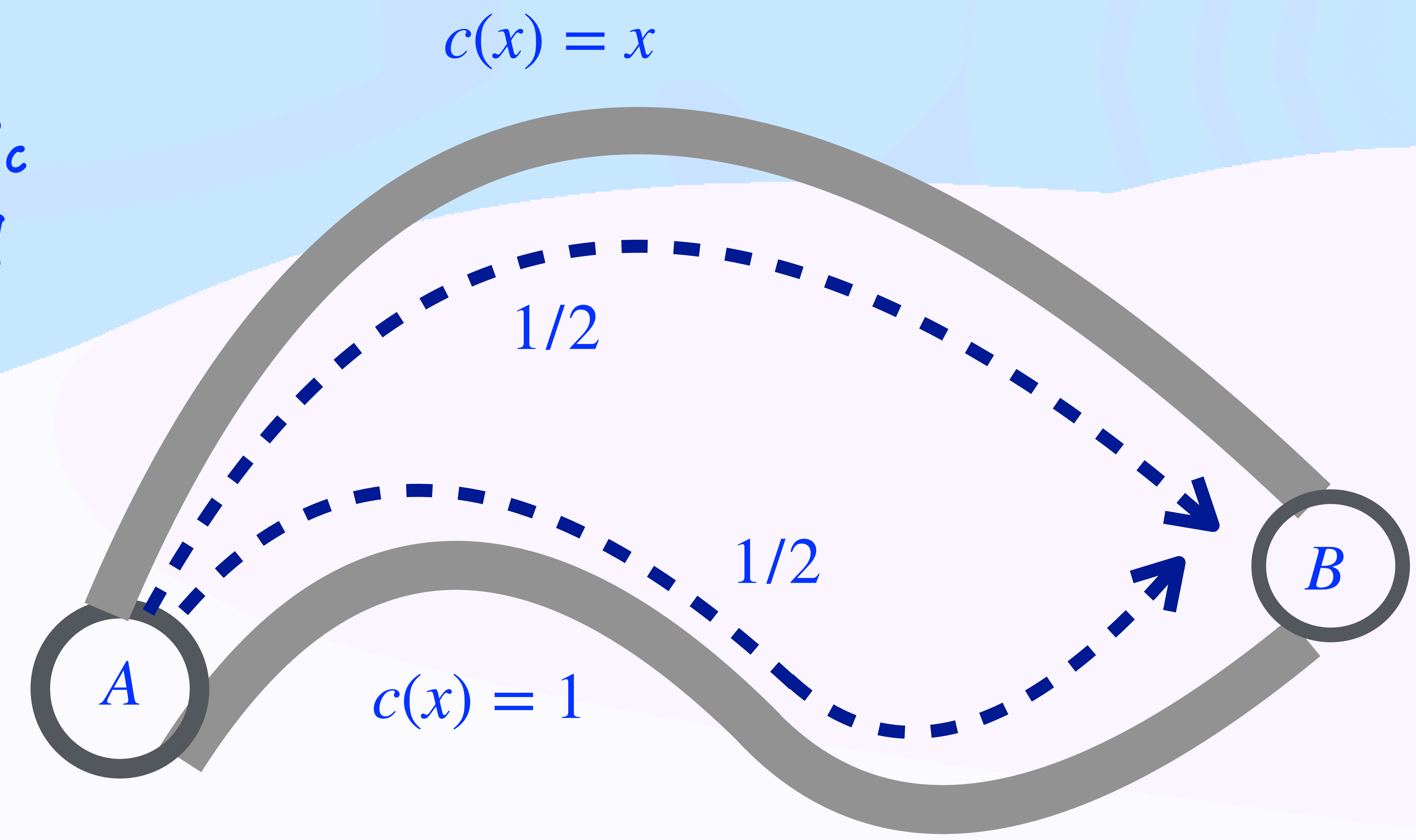

Expected travel time in DS equilibria $= 1 \cdot 1 = 1$

Expected travel time $= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Q: Is this the ideal outcome ?

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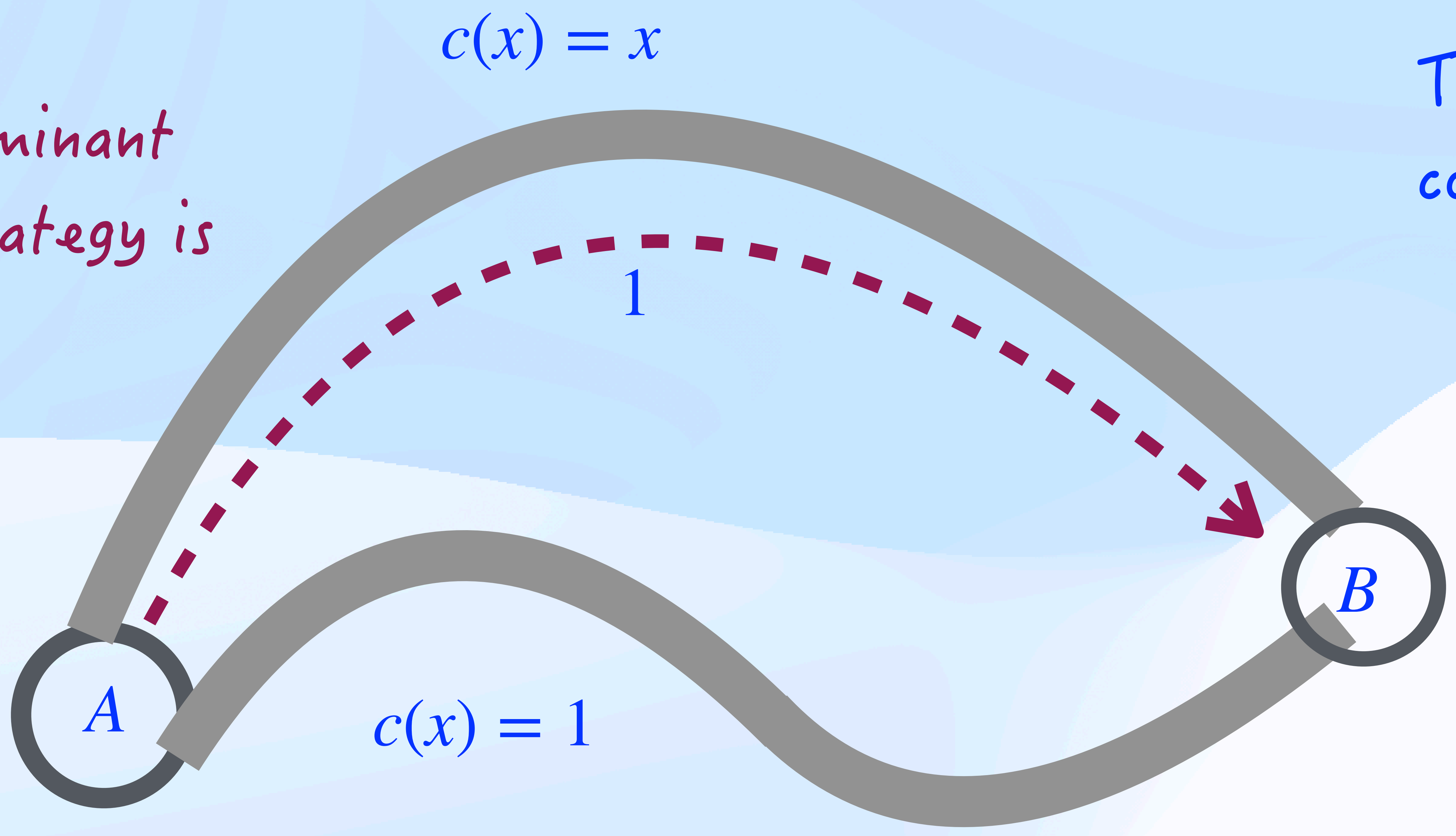

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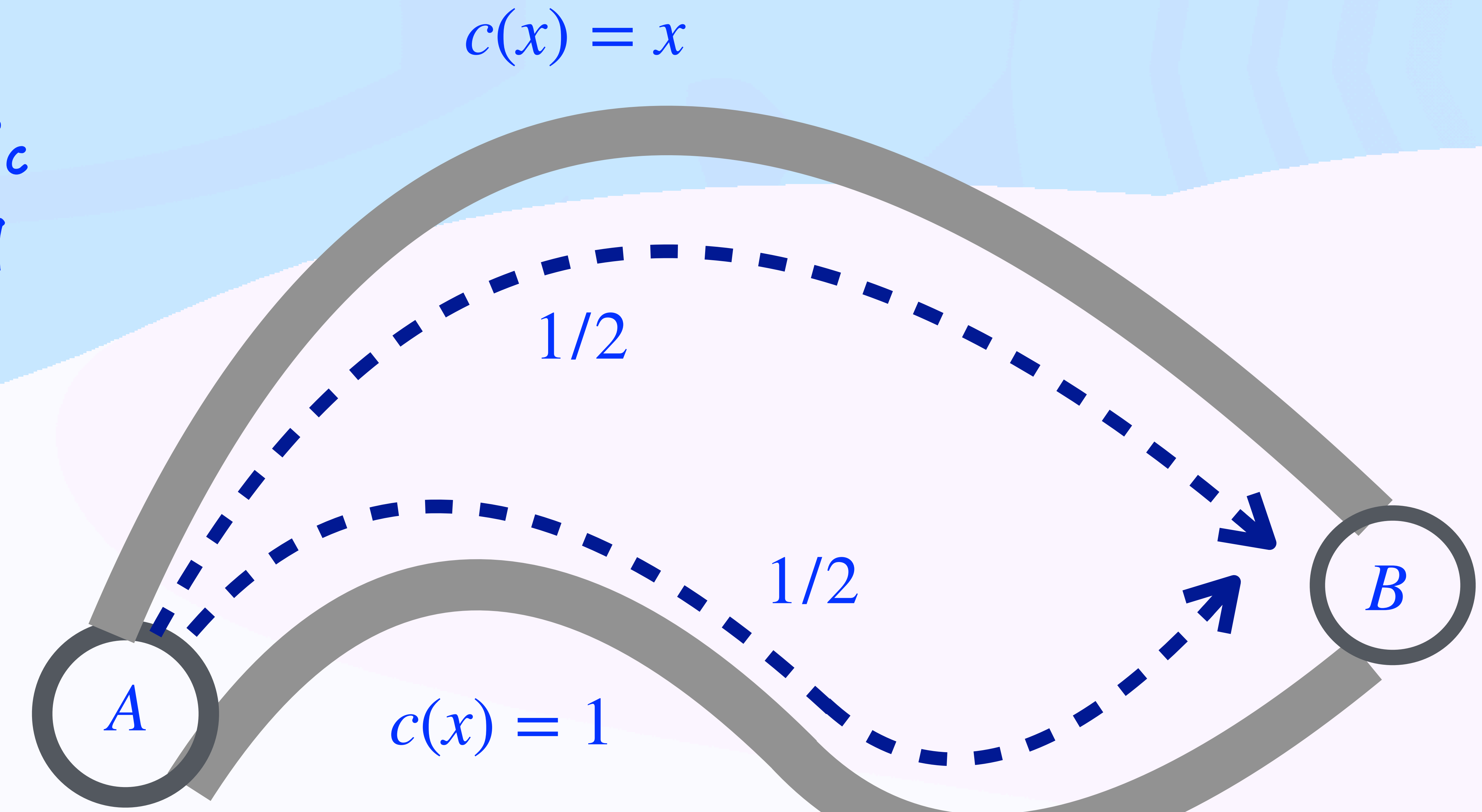

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Q: Is this the ideal outcome ?

$$\frac{\text{Travel time in DS equilibrium}}{\text{Min average travel time}} = \frac{1}{3/4} = \frac{4}{3}$$

Q: What should be our social objective ?

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Ans: Minimize average travel time

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Price of Anarchy (POA)

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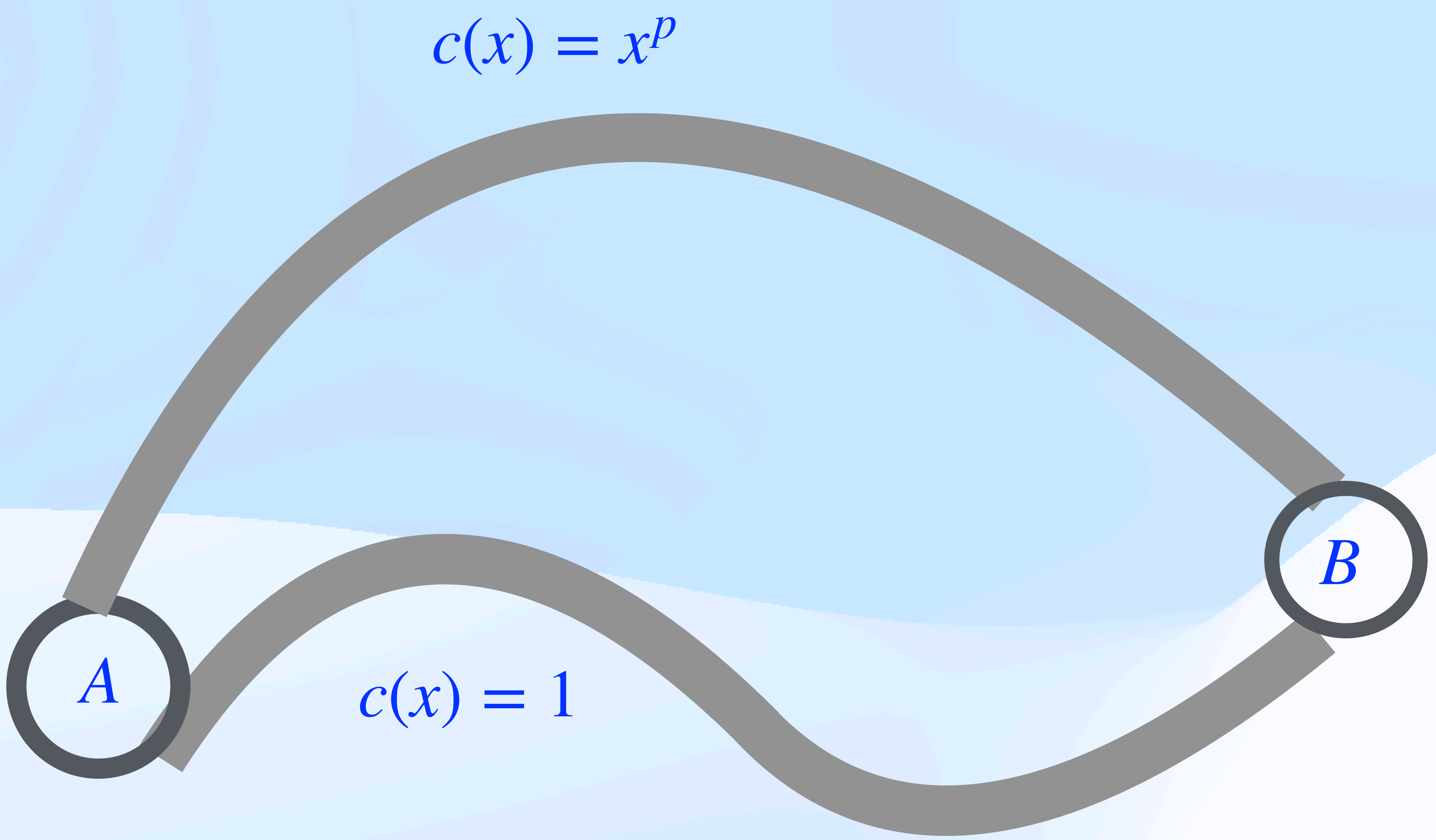
Ans: Minimize average travel time

Q: How far can we be from this objective ?

$$\text{Price of Anarchy (POA)} = \frac{\text{Travel time in DS equilibrium}}{\text{Min possible average travel time}}$$

What if the cost is non-linear ?



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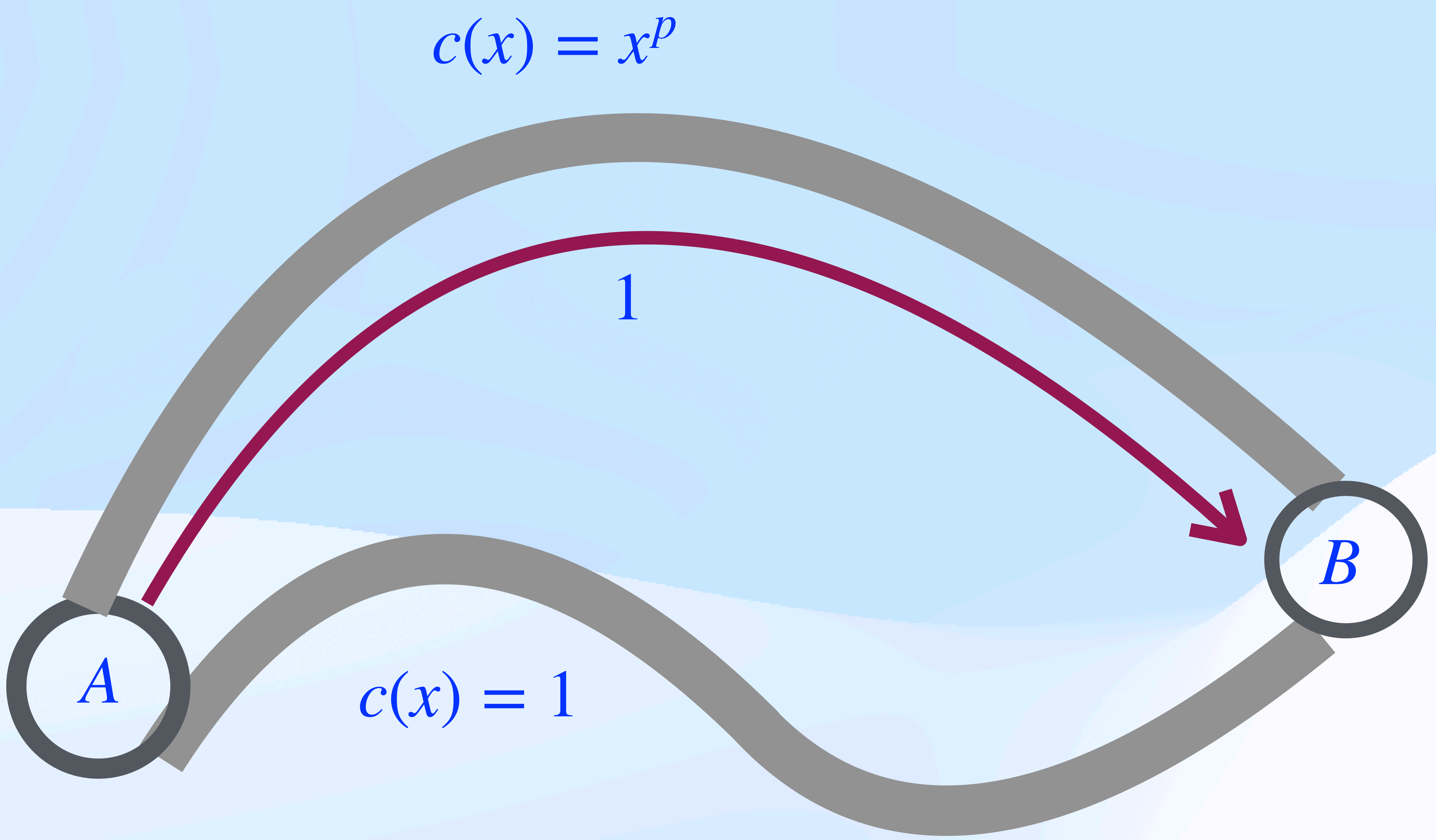

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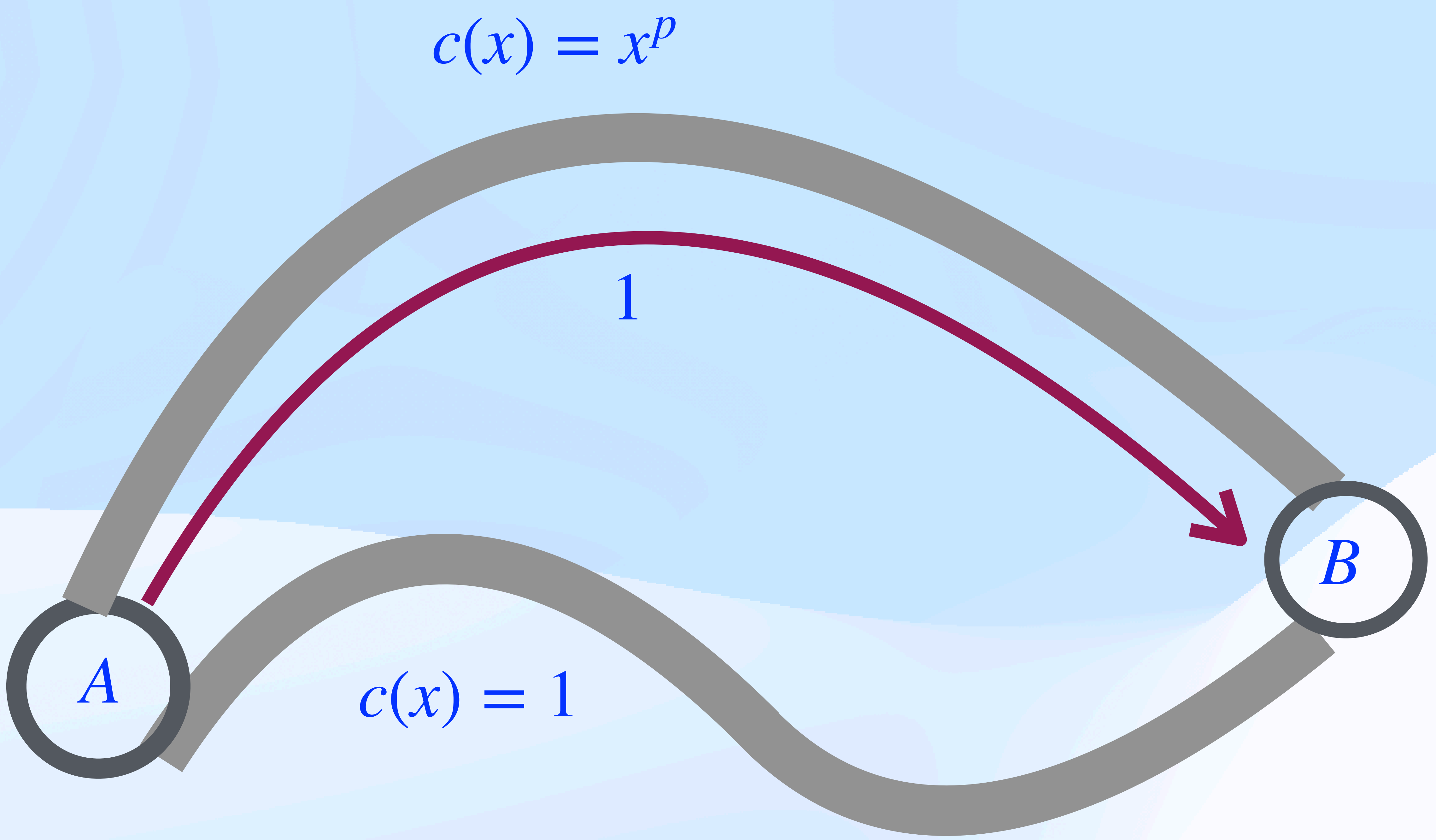

Travel time in DS equilibria $= 1$

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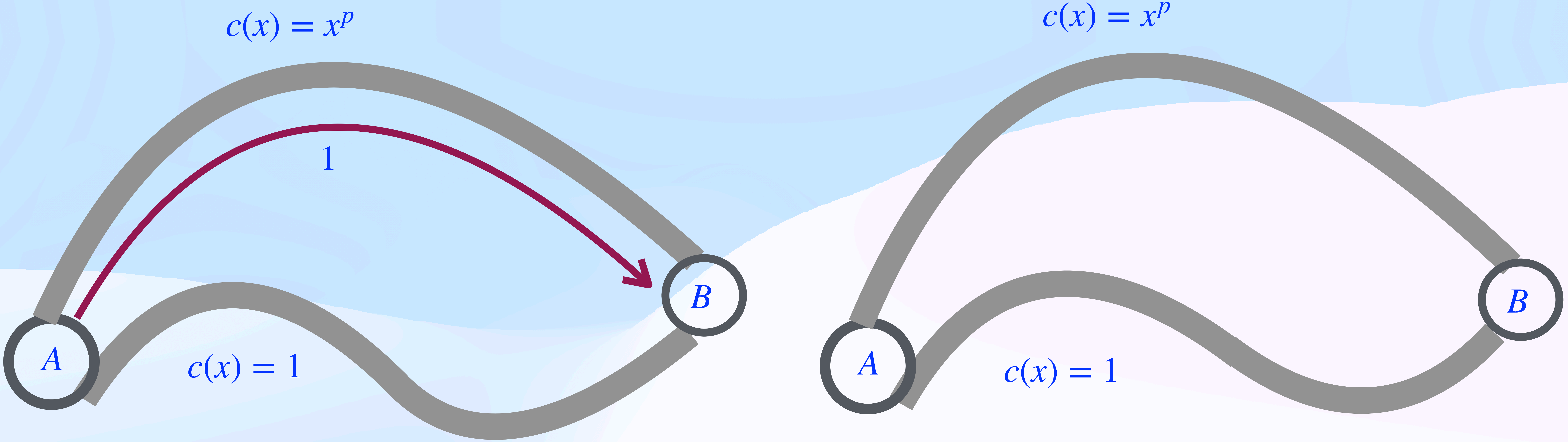

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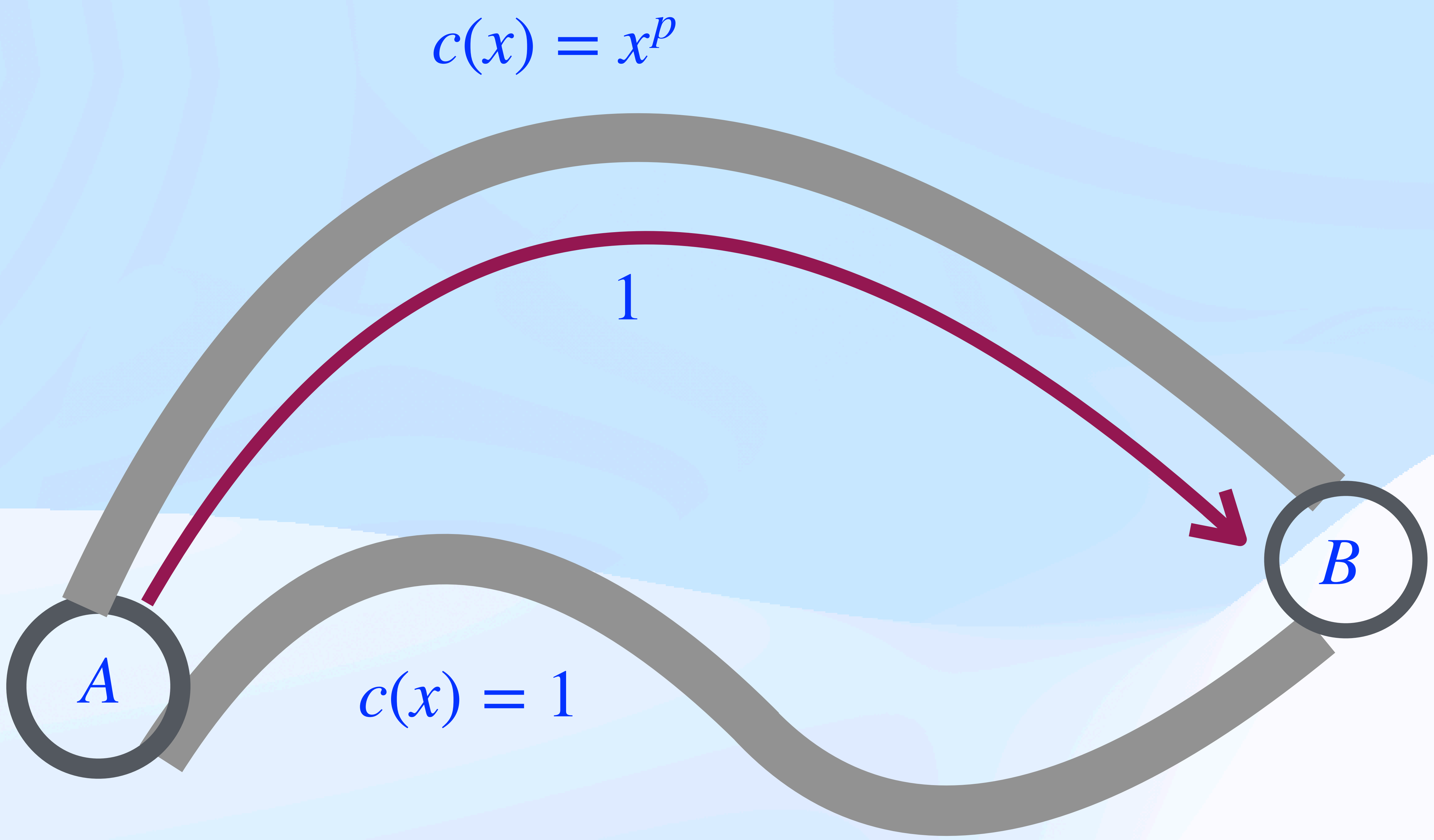

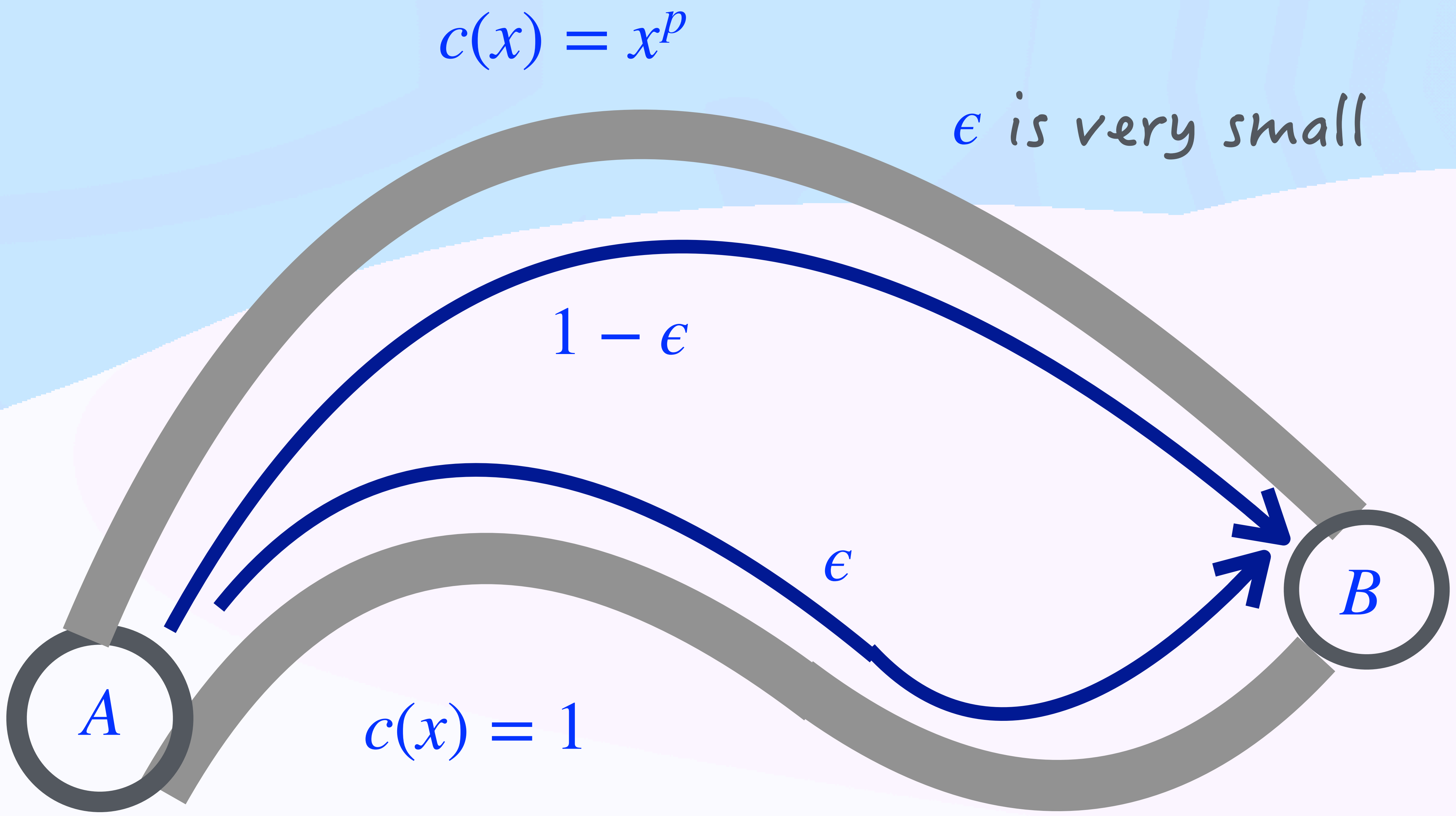

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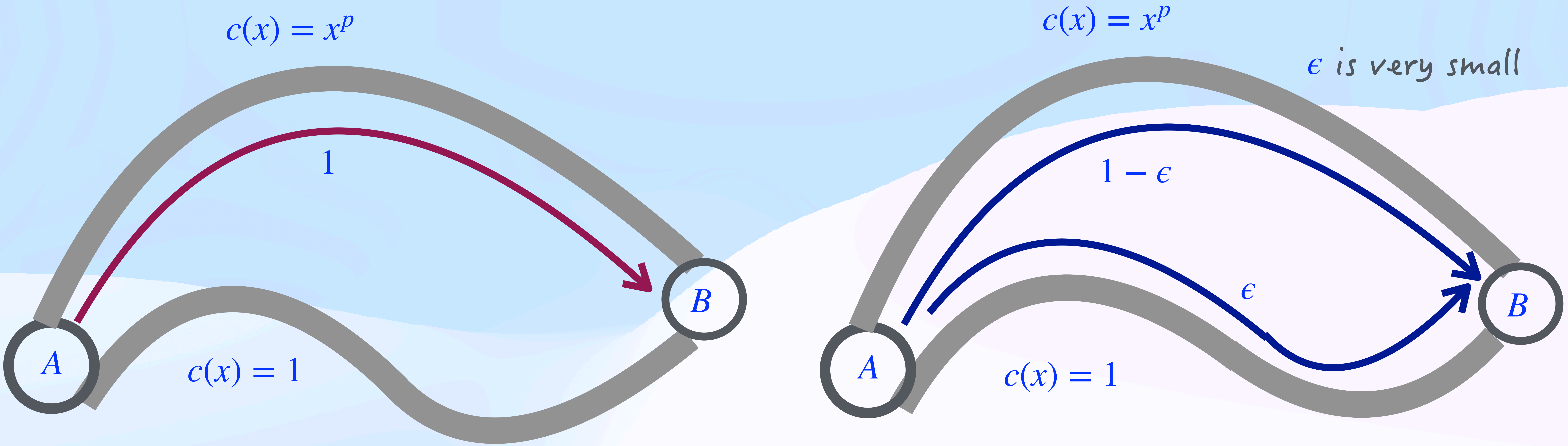

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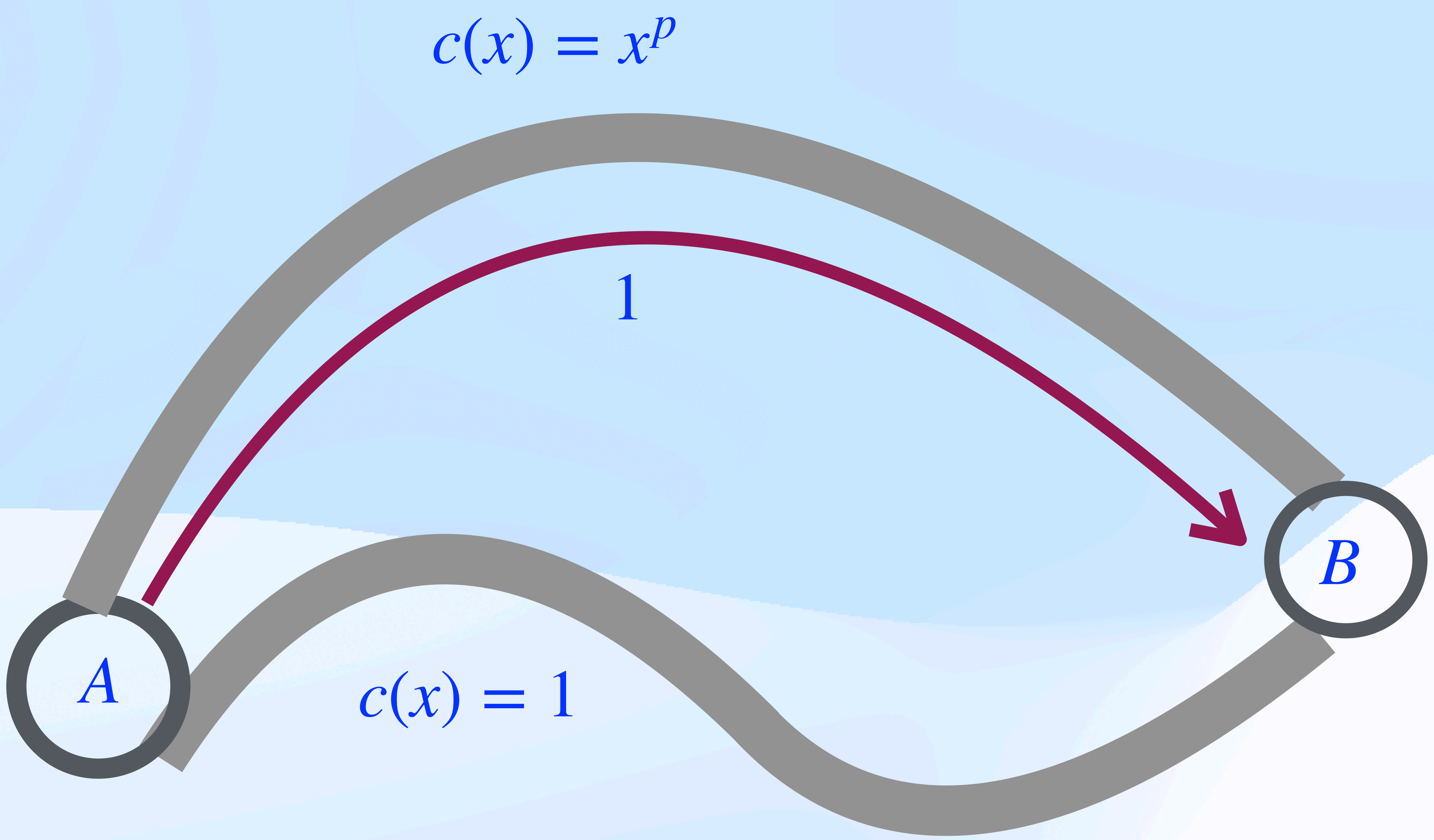

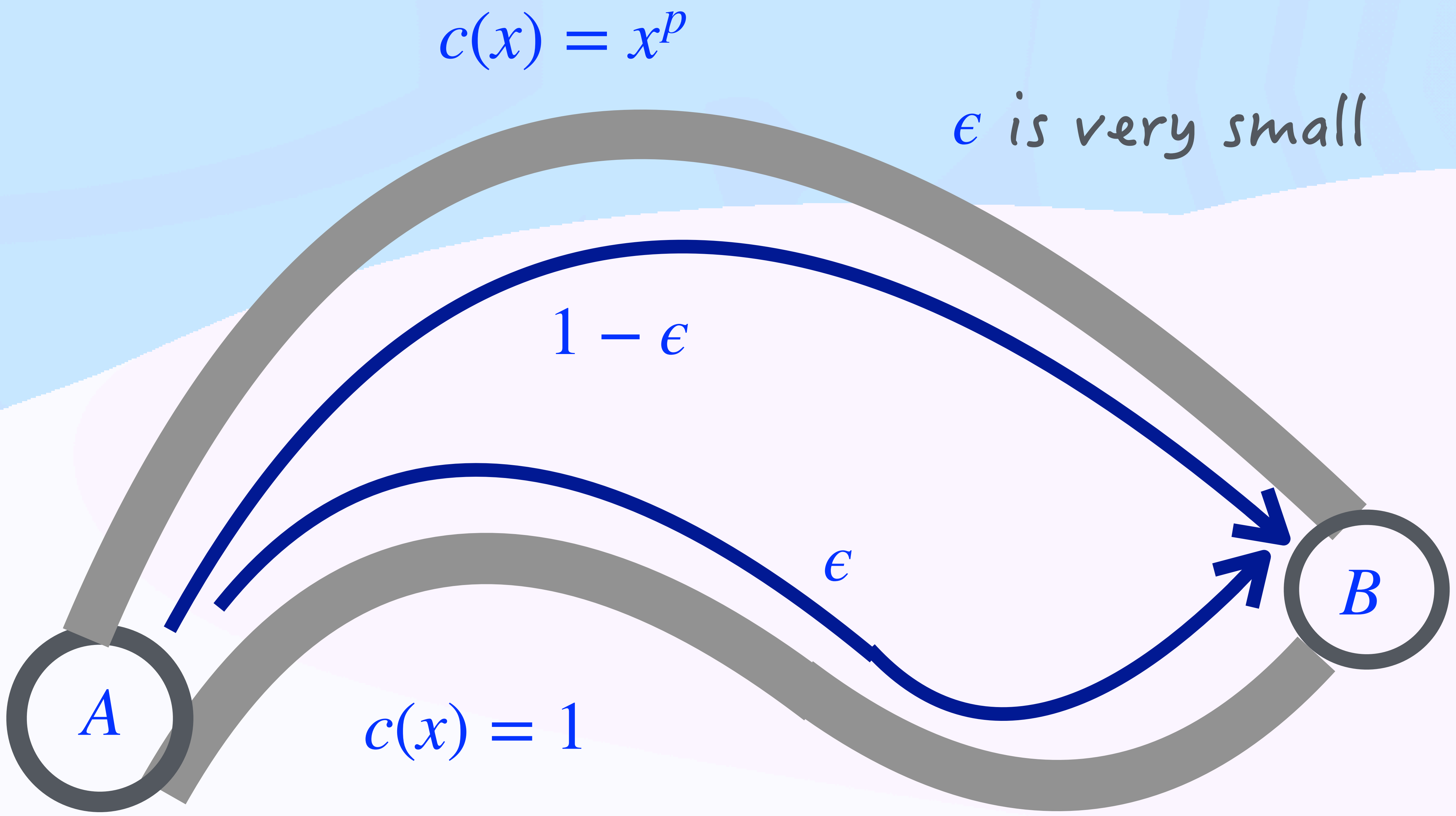

Travel time in DS equilibria $= 1$

Travel time

$$= \epsilon \cdot 1 + (1-\epsilon)\cdot(1-\epsilon)^p = \epsilon + (1-\epsilon)^{p+1}$$

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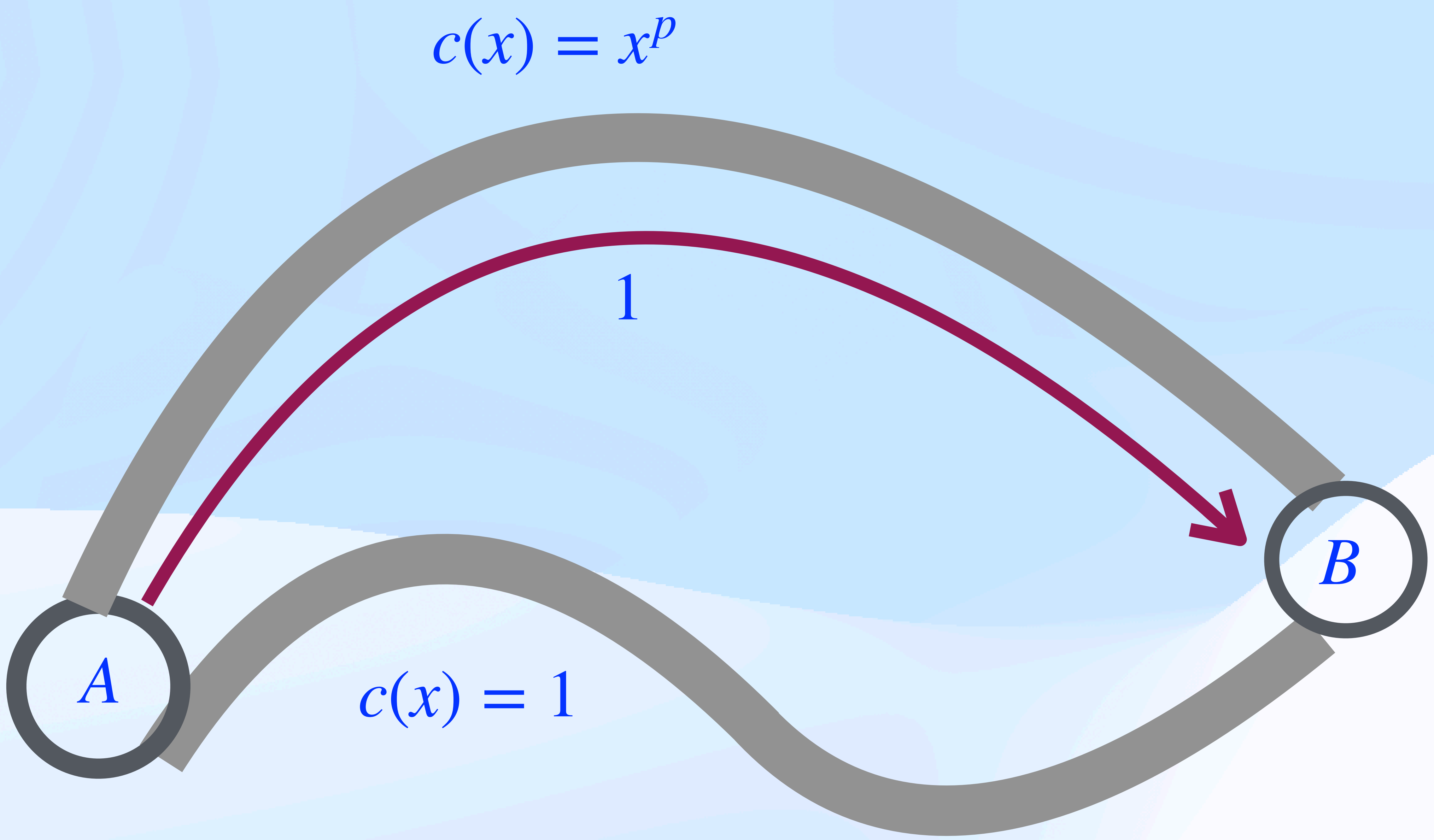

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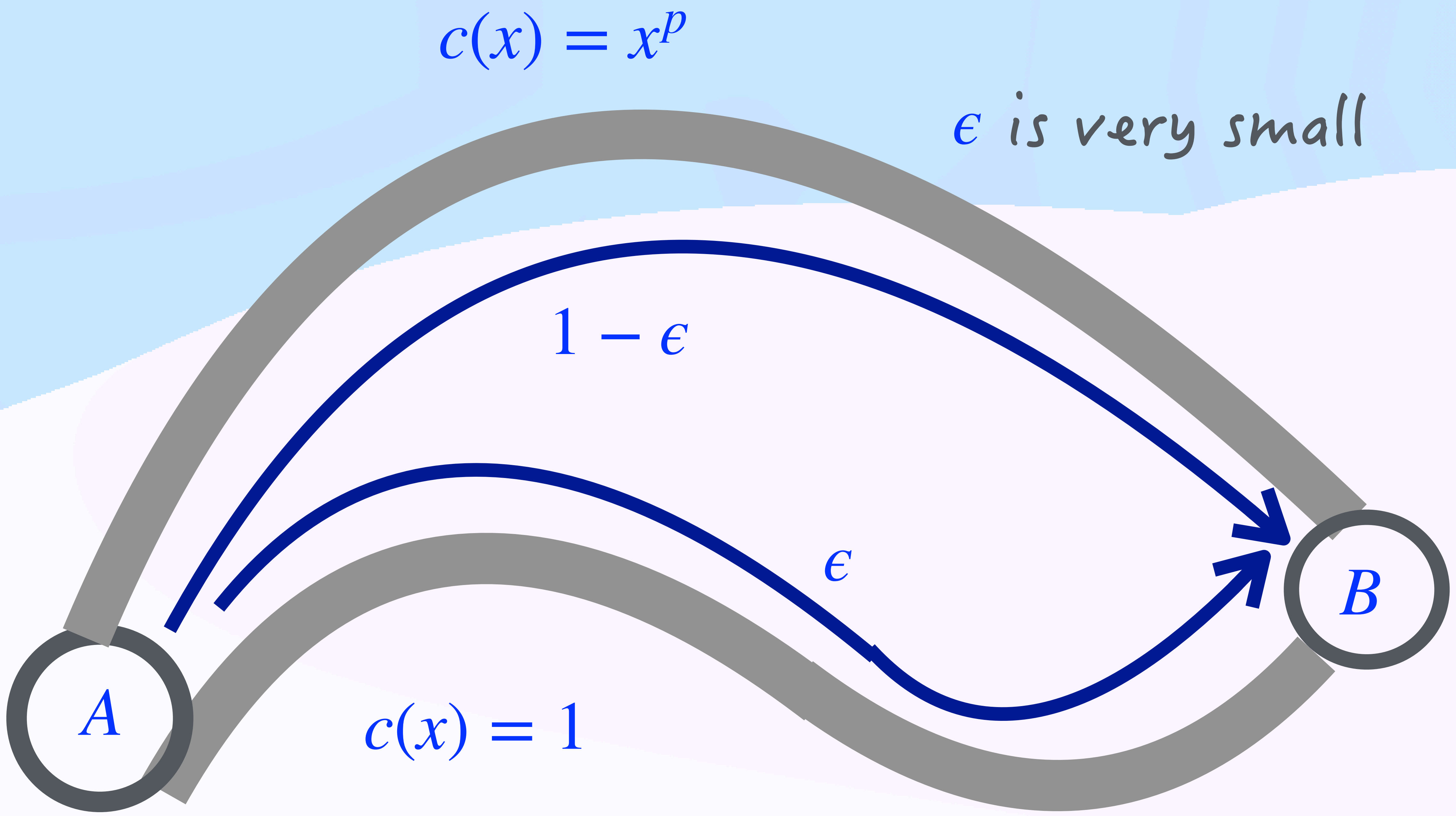

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$\to 0$ when p is large enough

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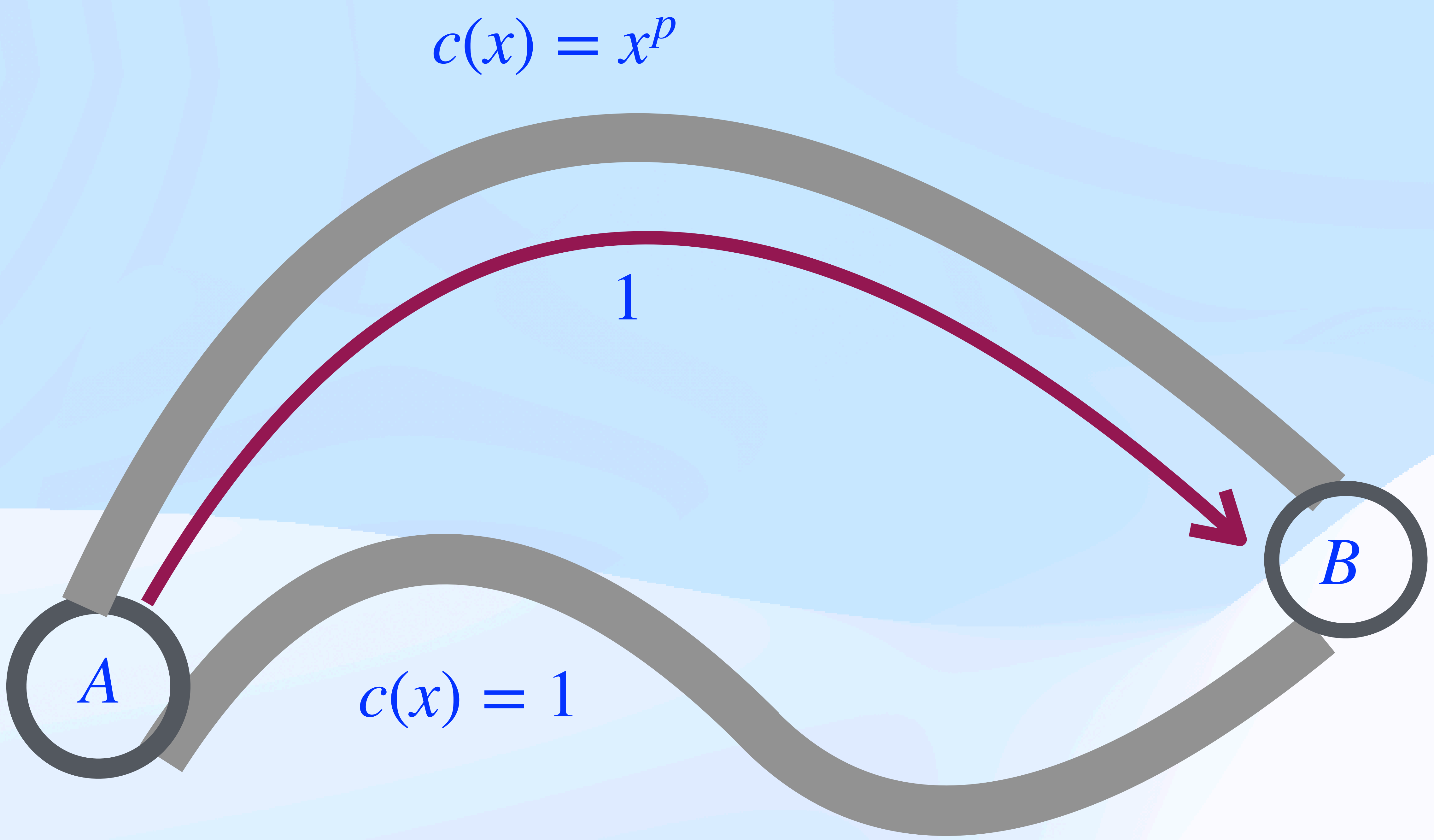

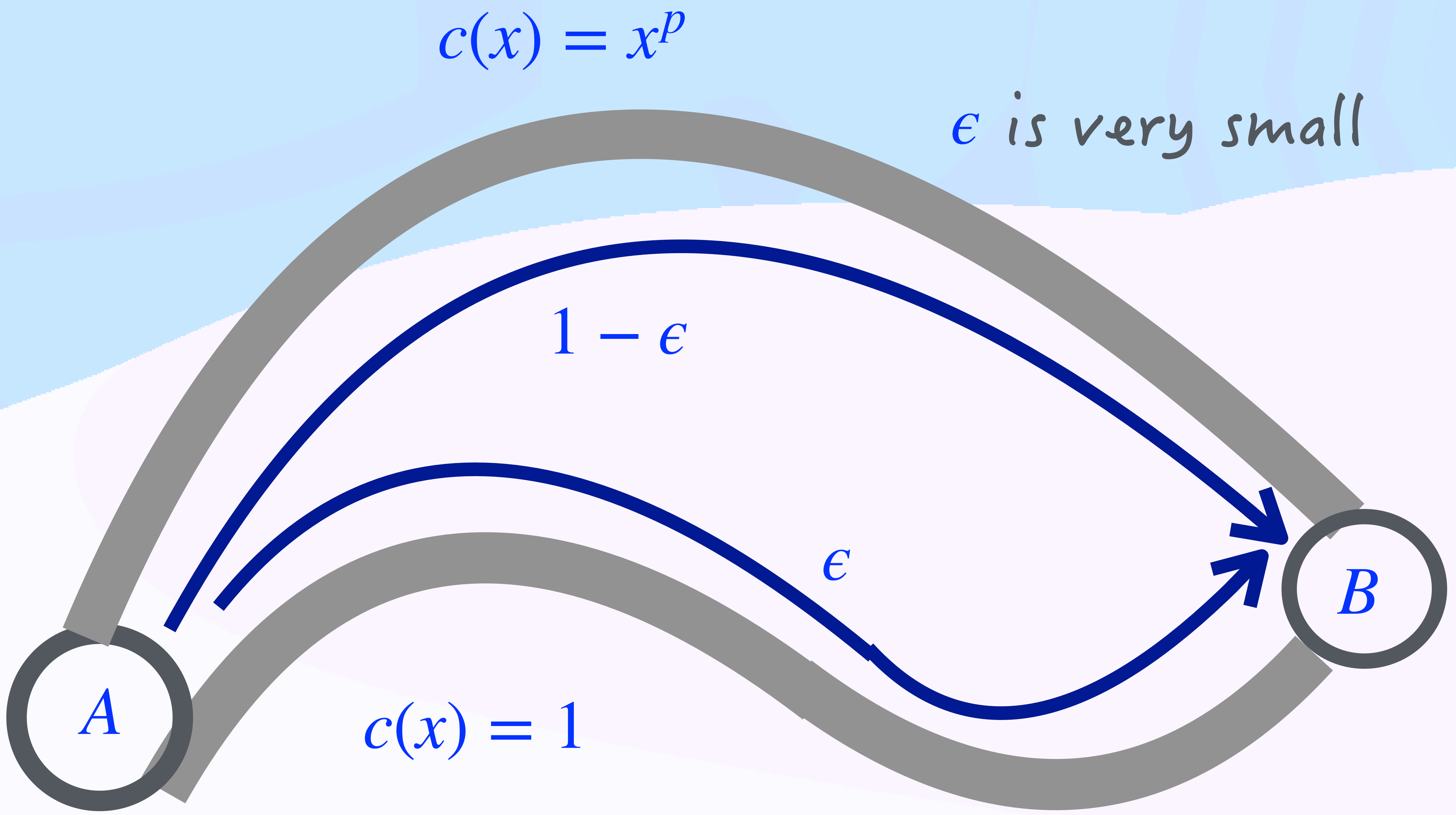

ϵ is very small

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$$\text{POA} = \frac{\text{Travel time in DSE}}{\text{Min possible travel time}}$$

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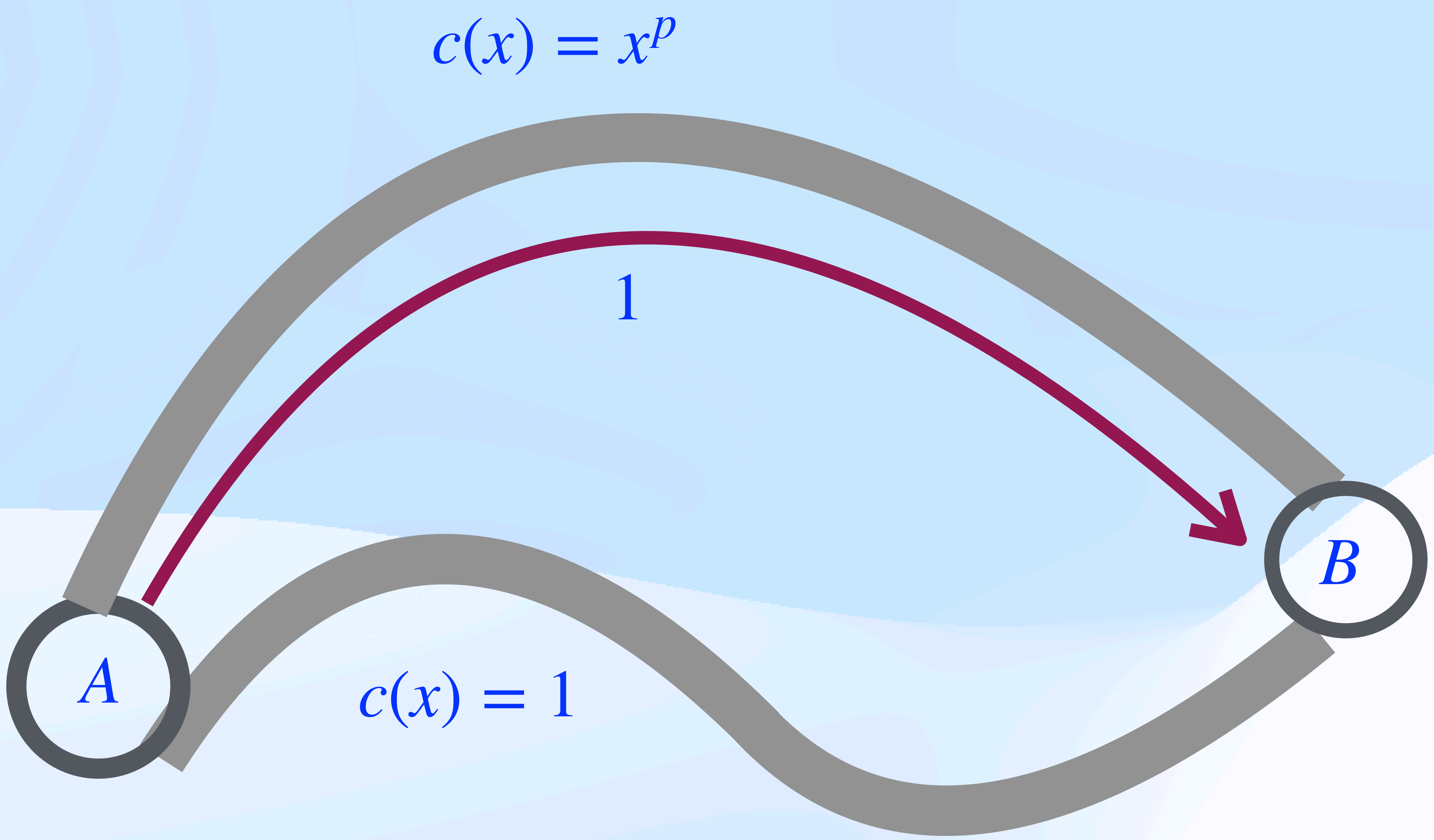

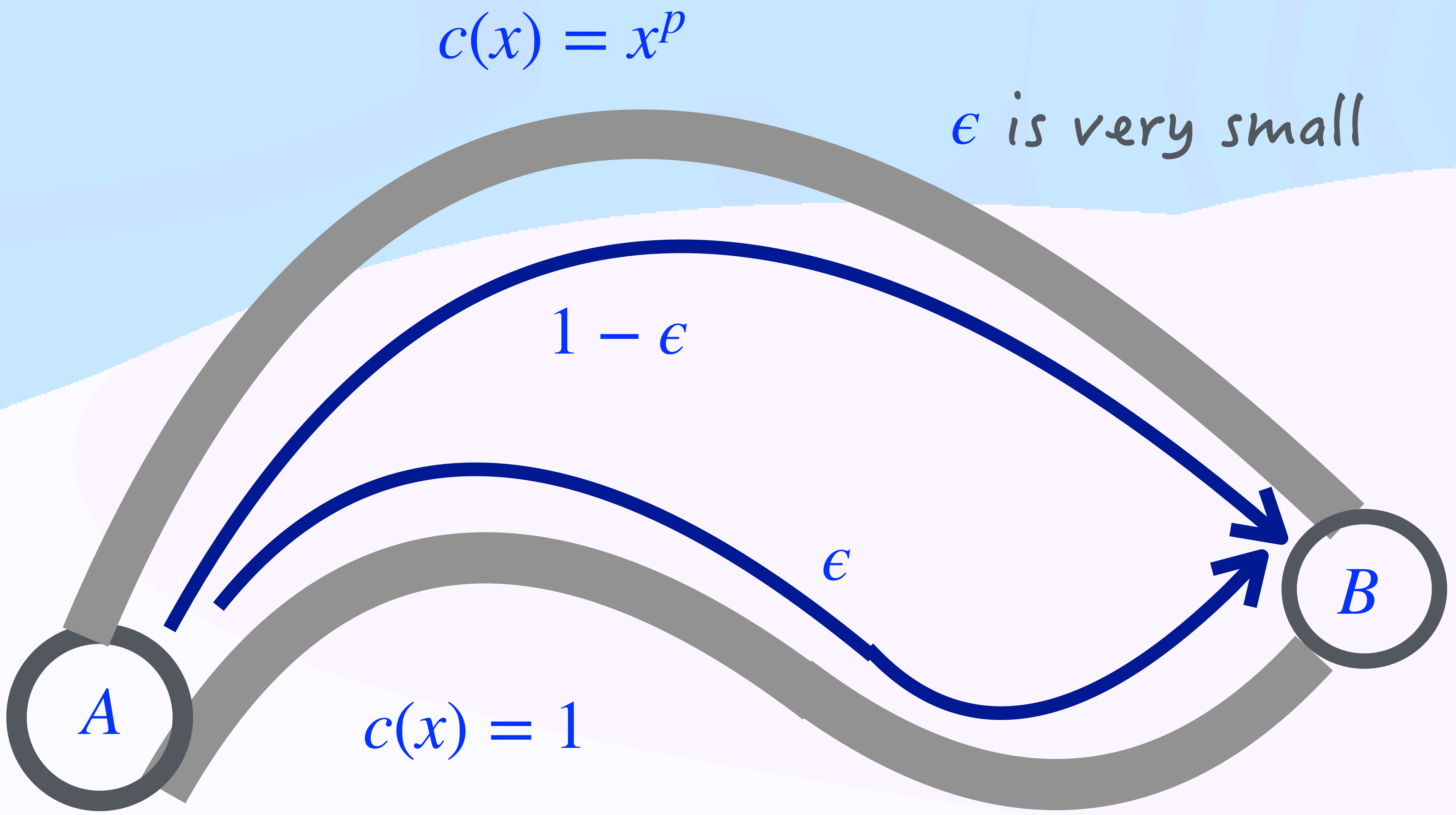

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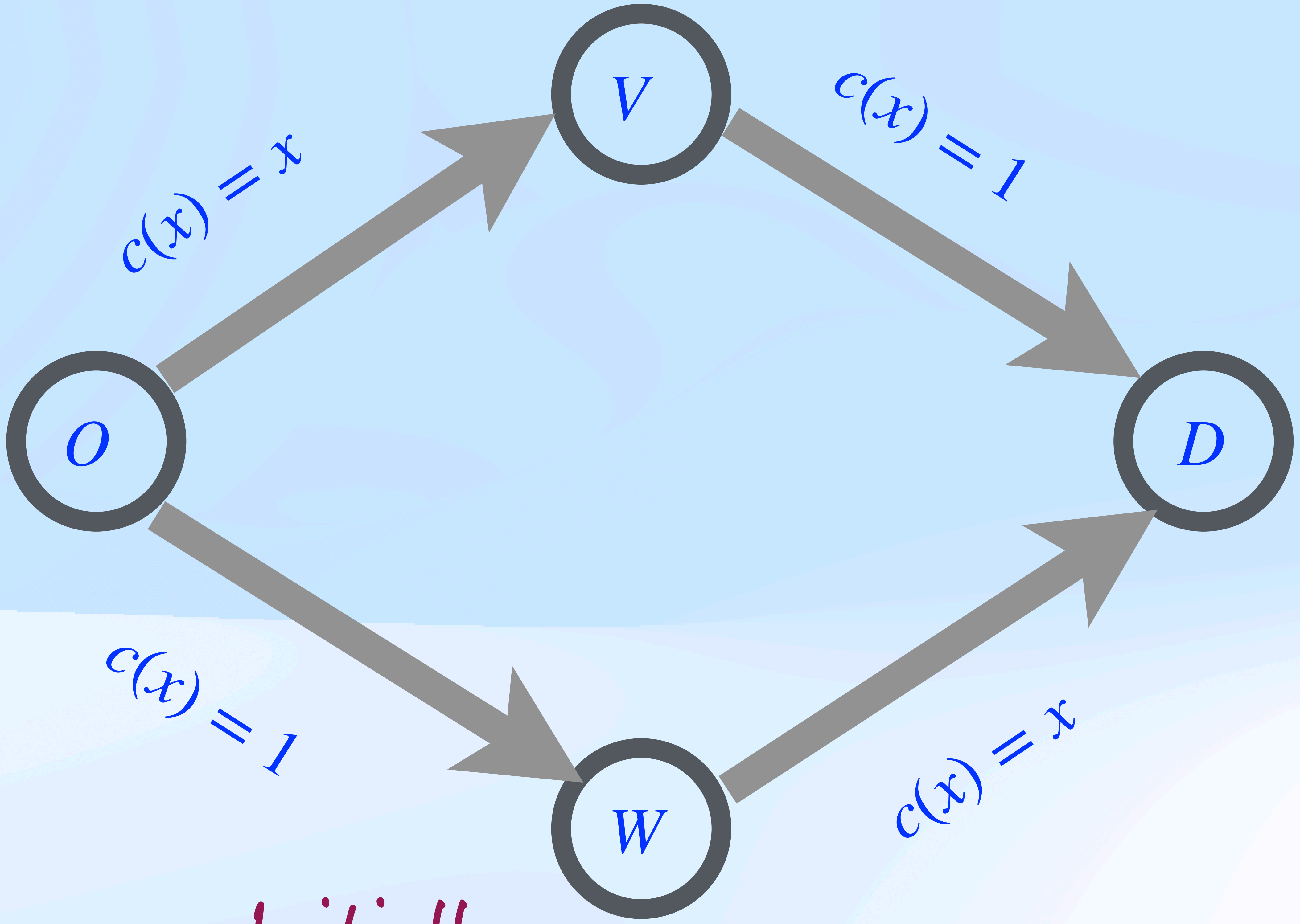
Is unbounded!

Braess Network, 1965

Initially

Braess Network, 1965

1 unit of traffic



Initially

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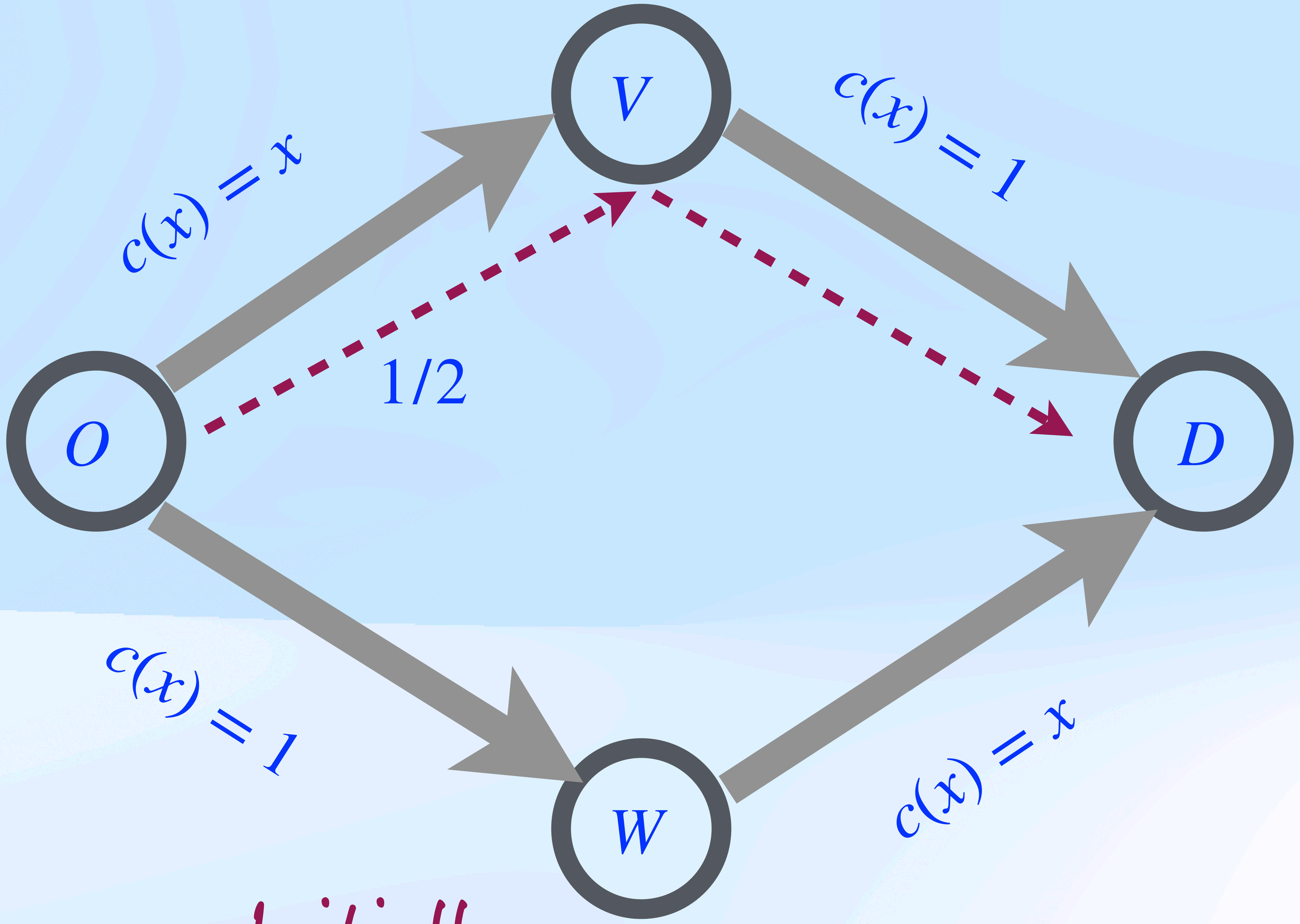

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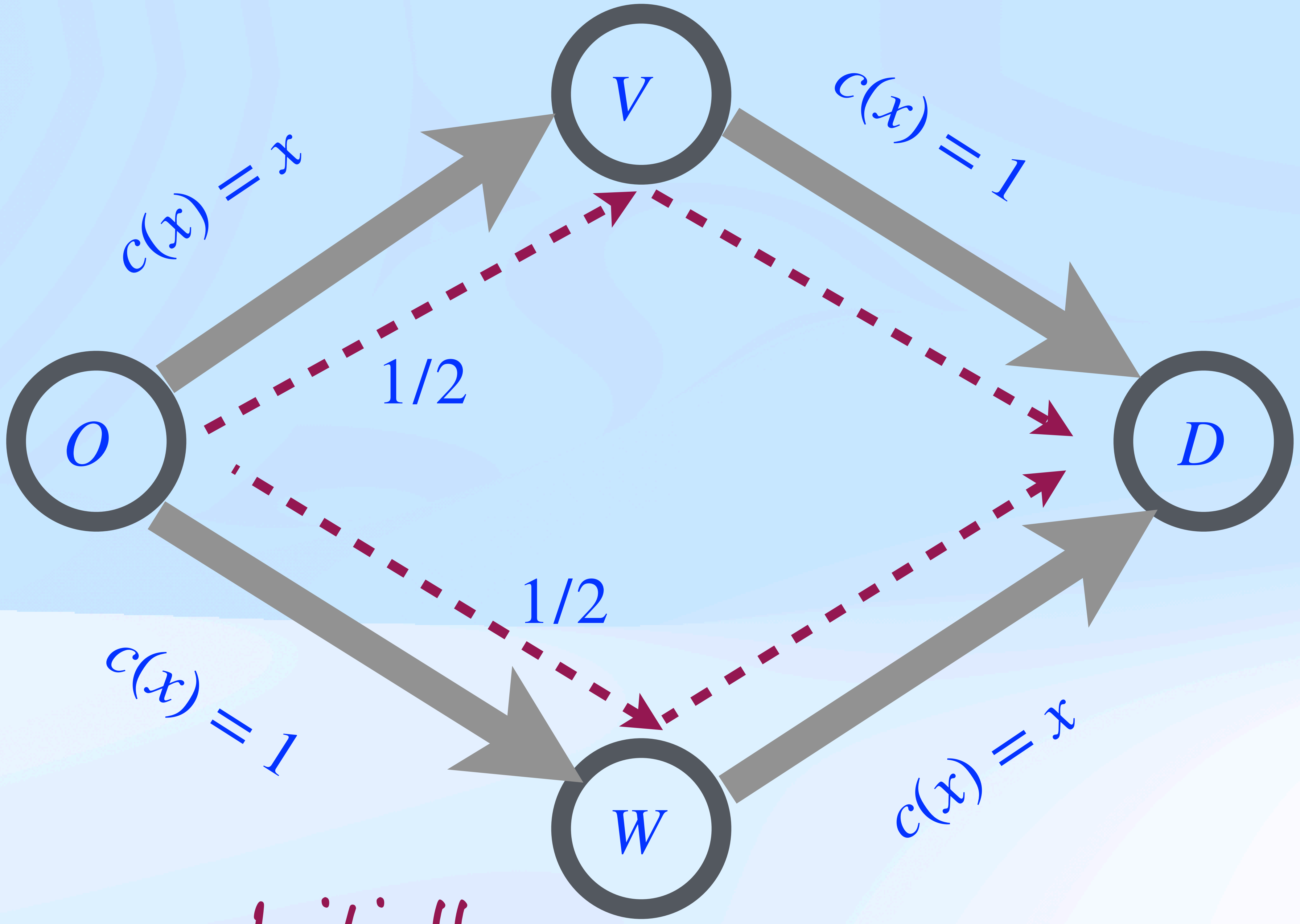

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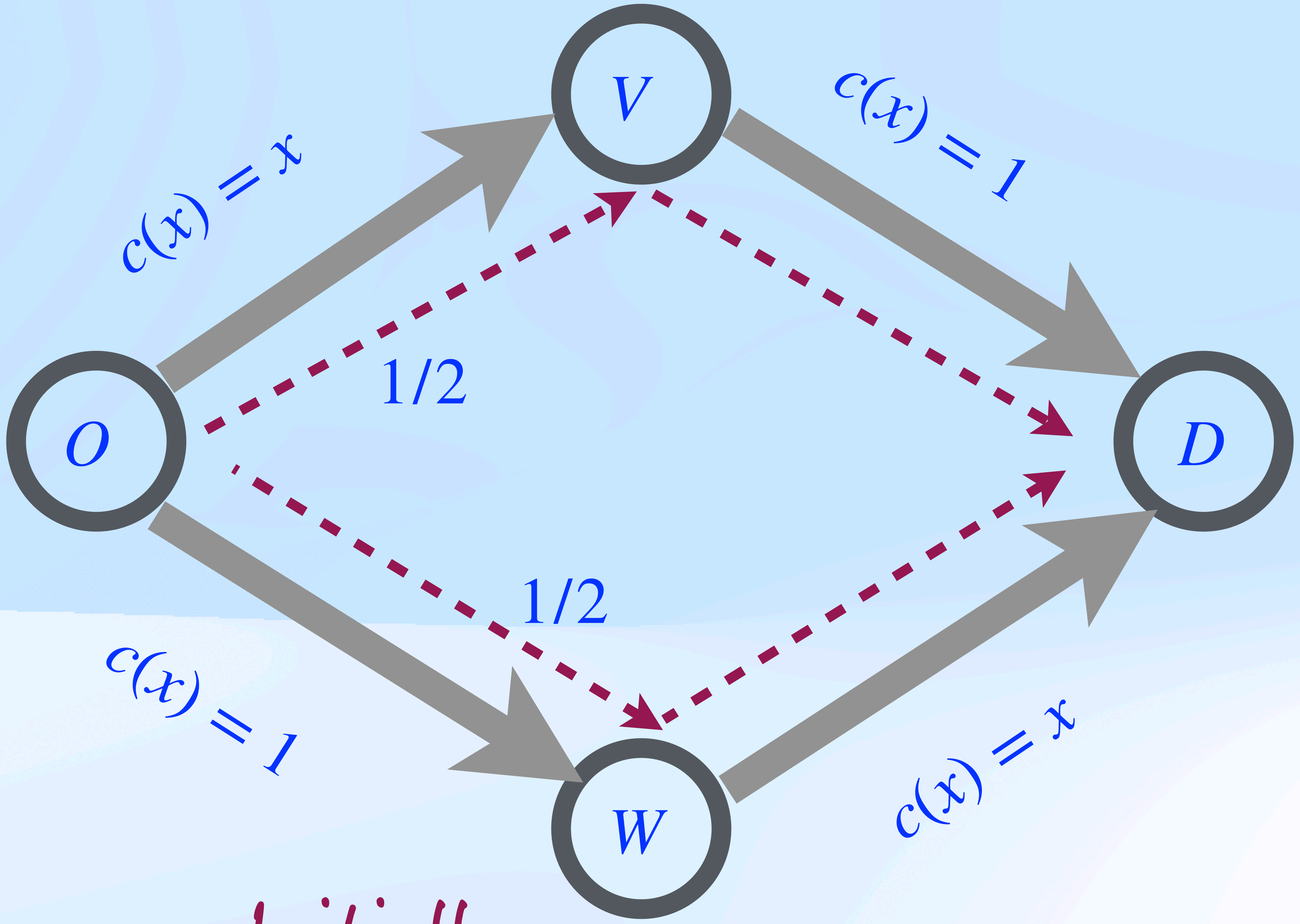

Initially

Travel time

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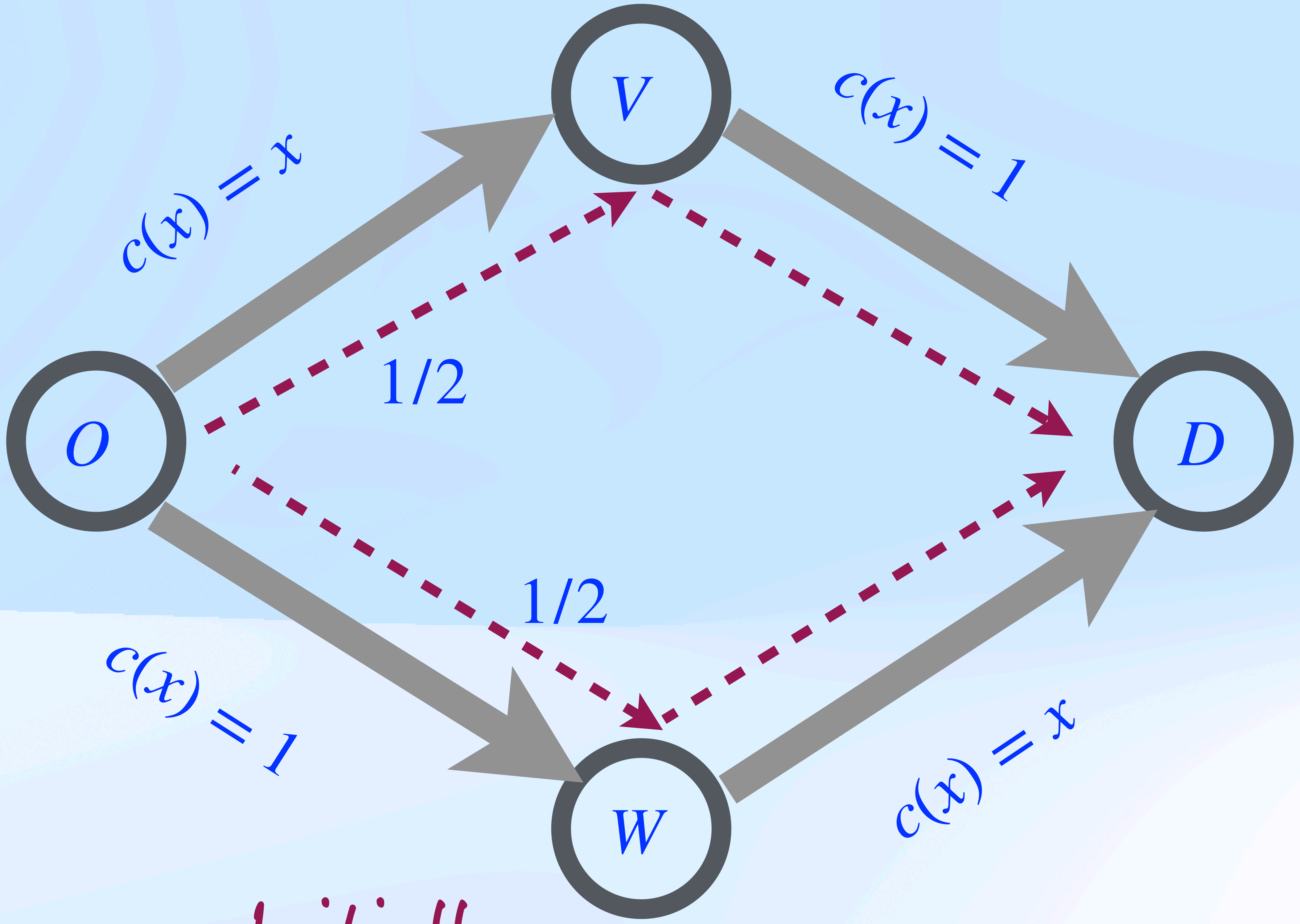

Initially

Travel time $= 2\left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1\right) = \frac{3}{2}$

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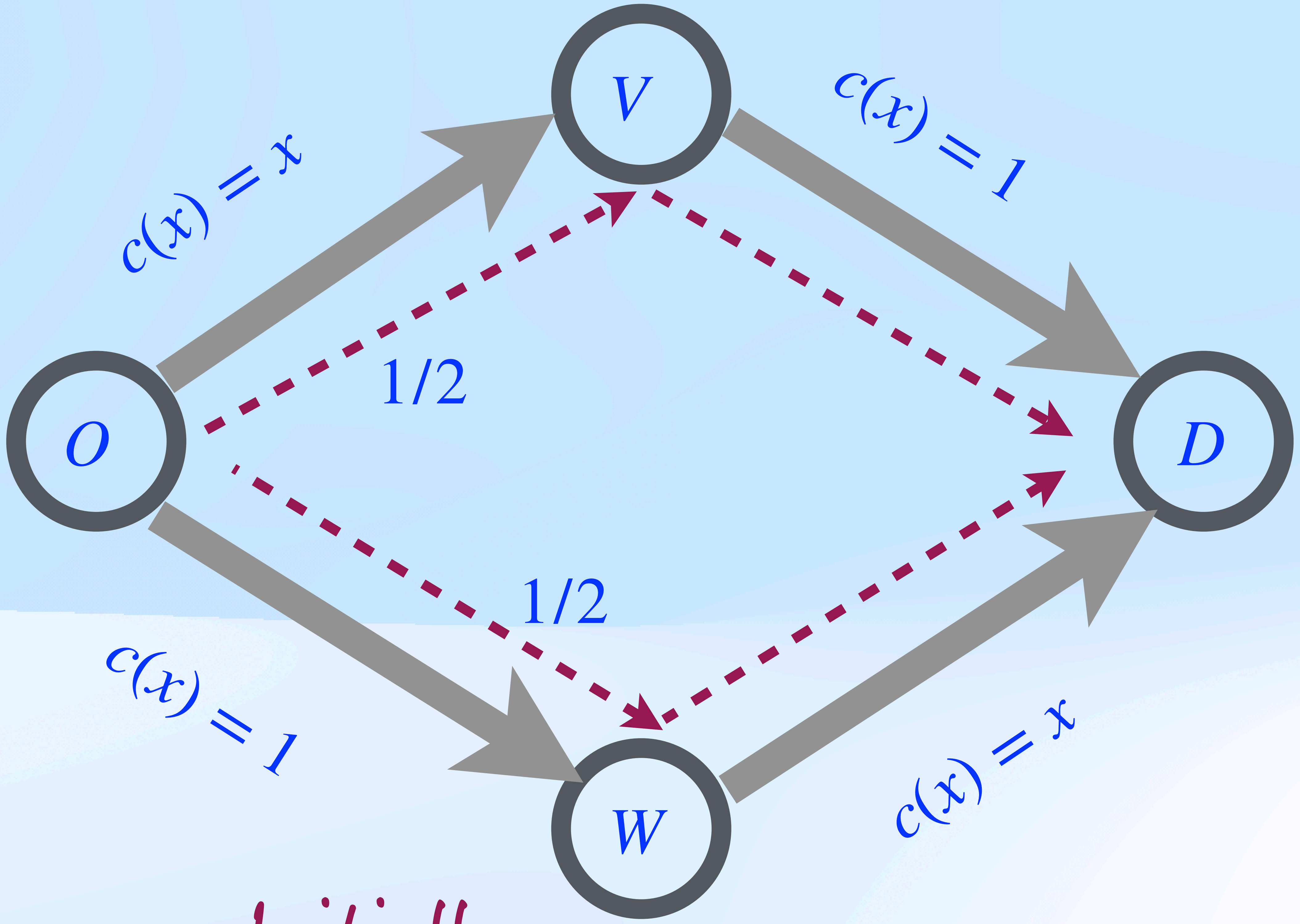

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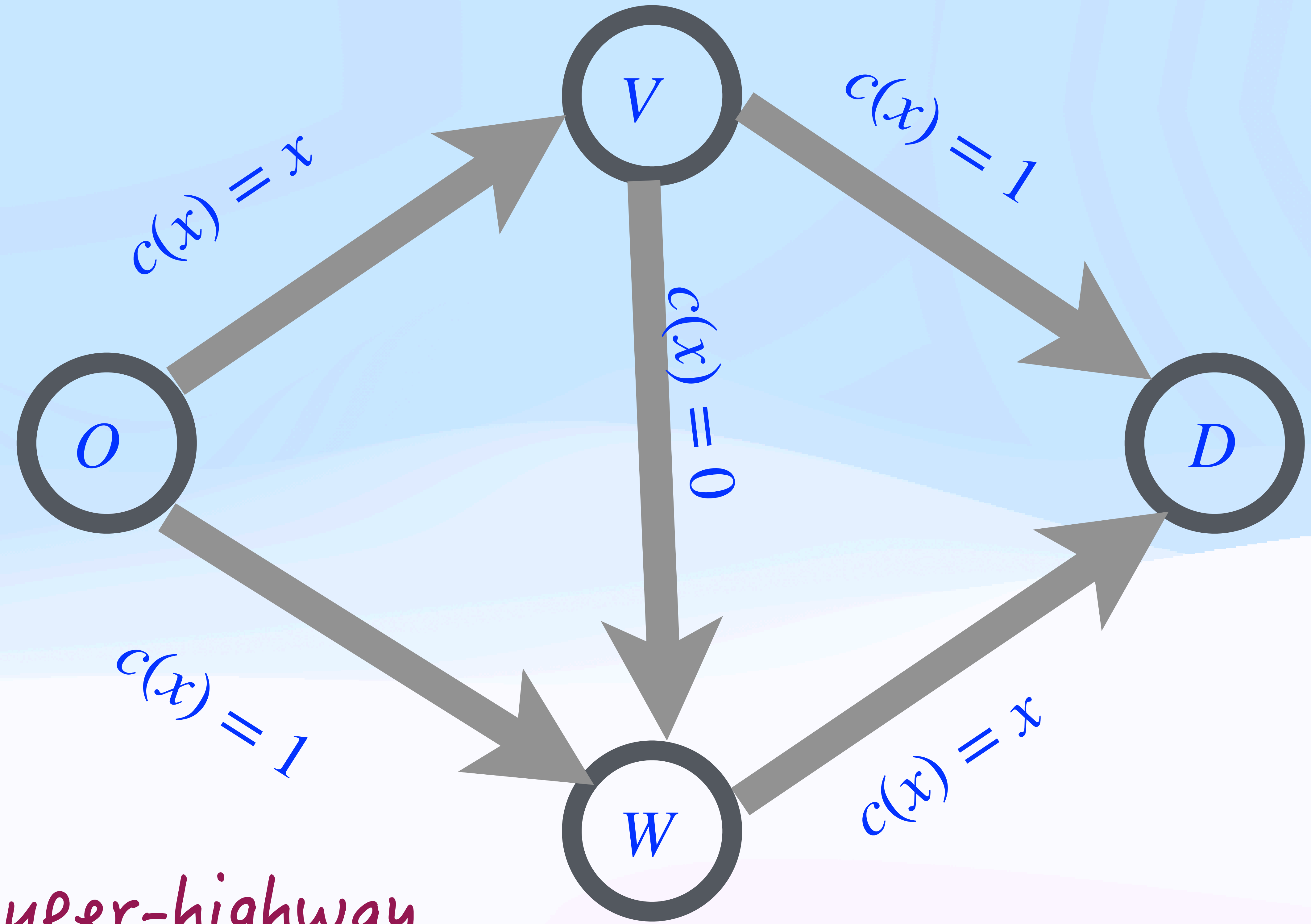
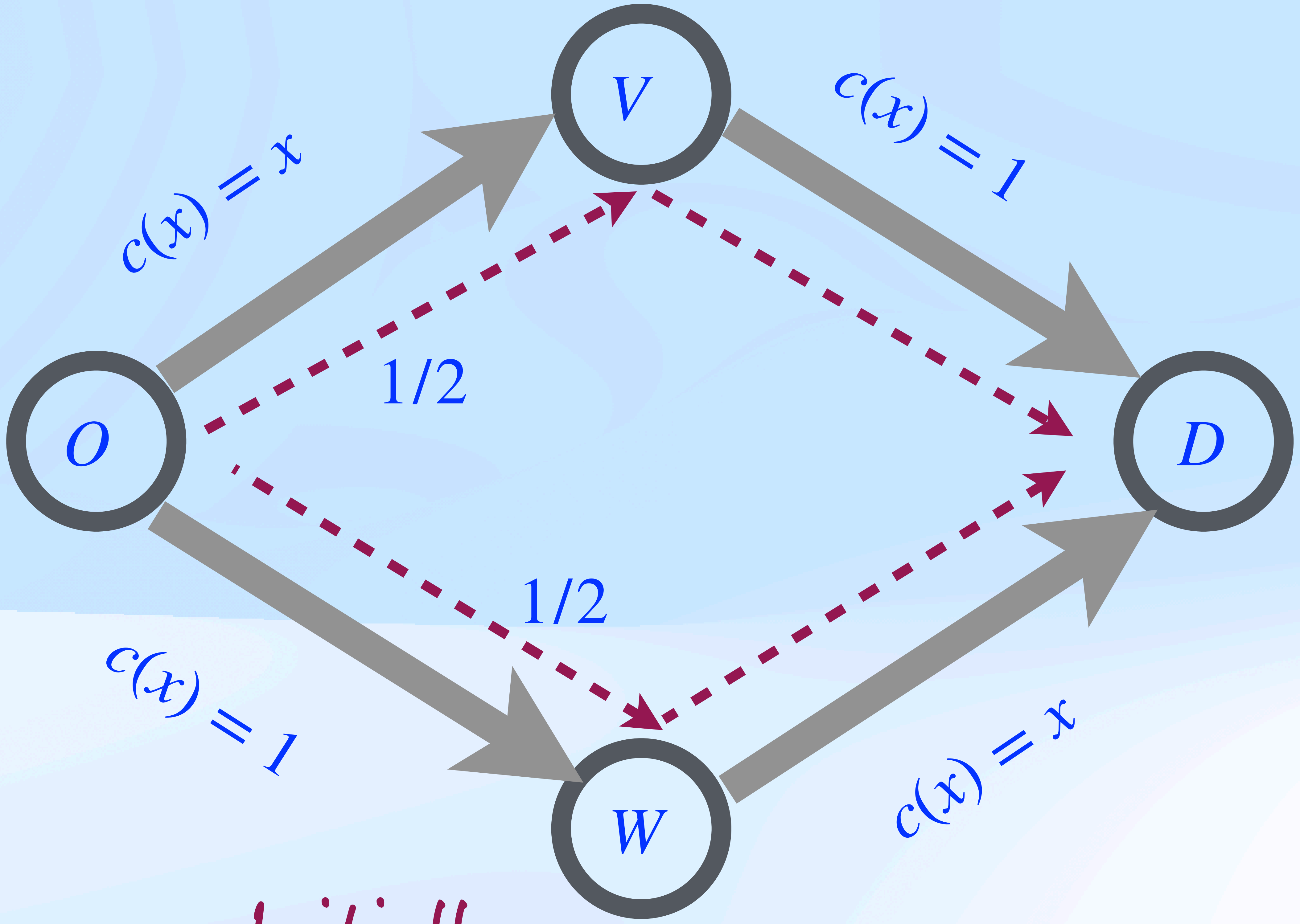


After super-highway

Braess Network, 1965

1 unit of traffic



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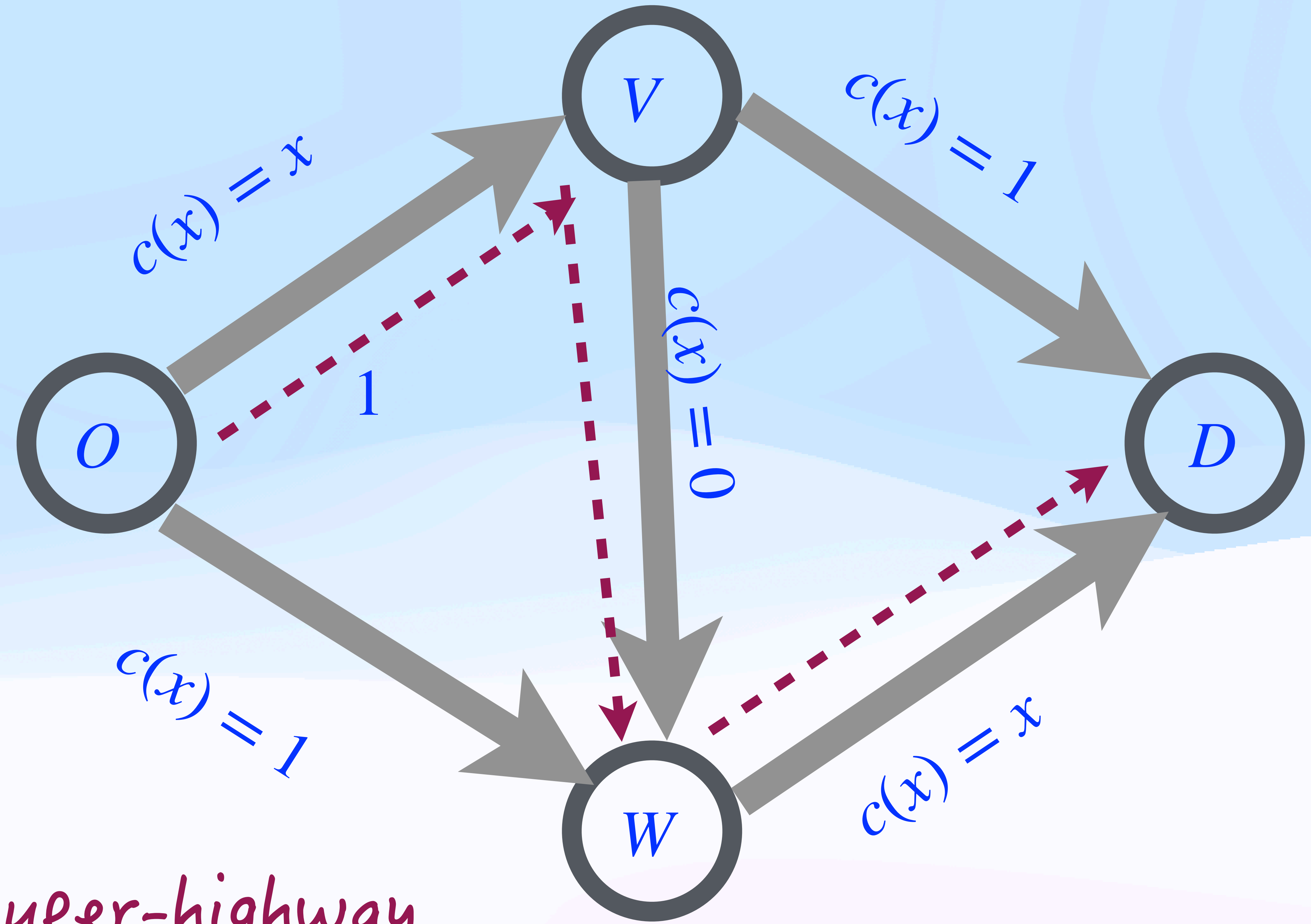

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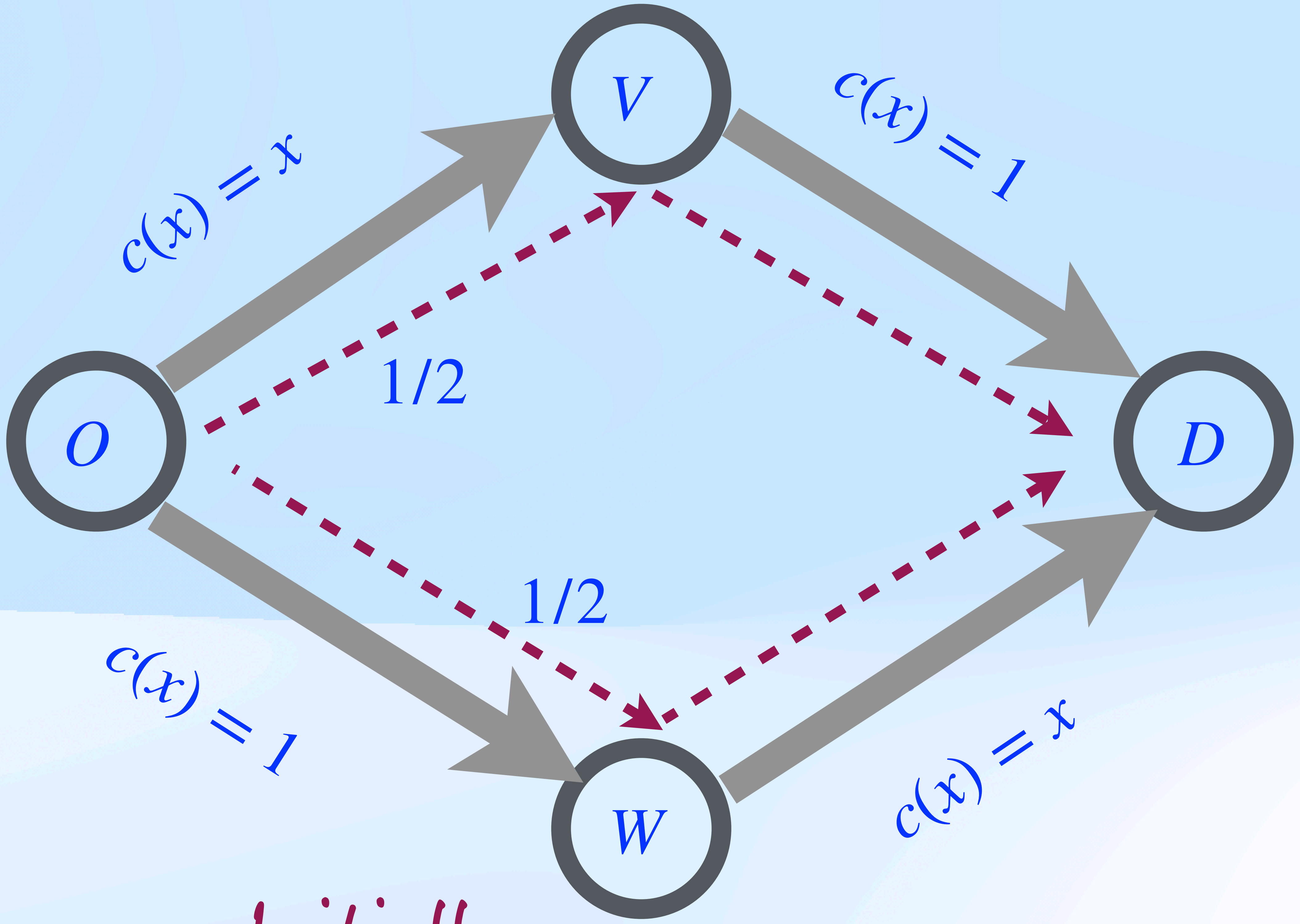

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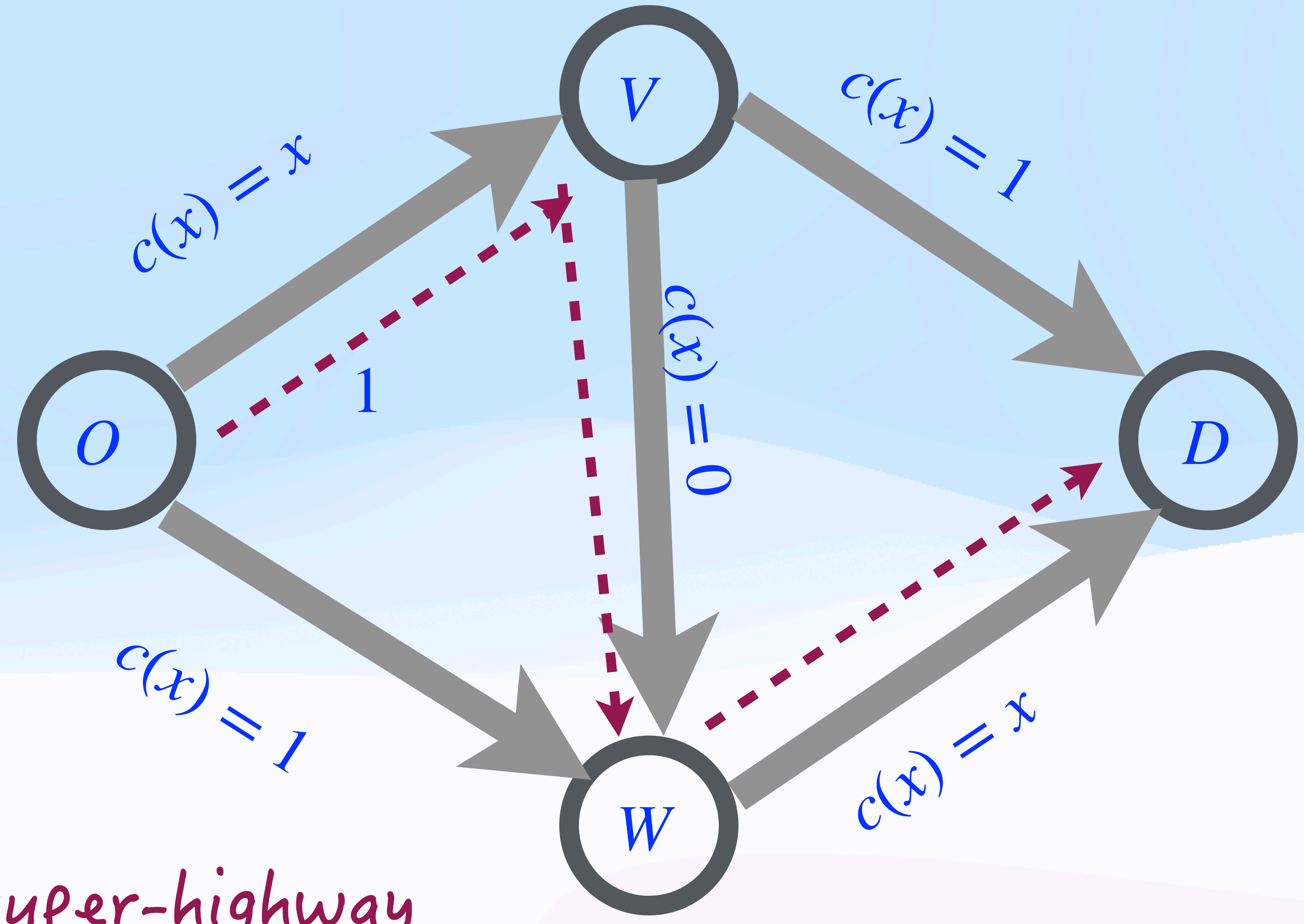
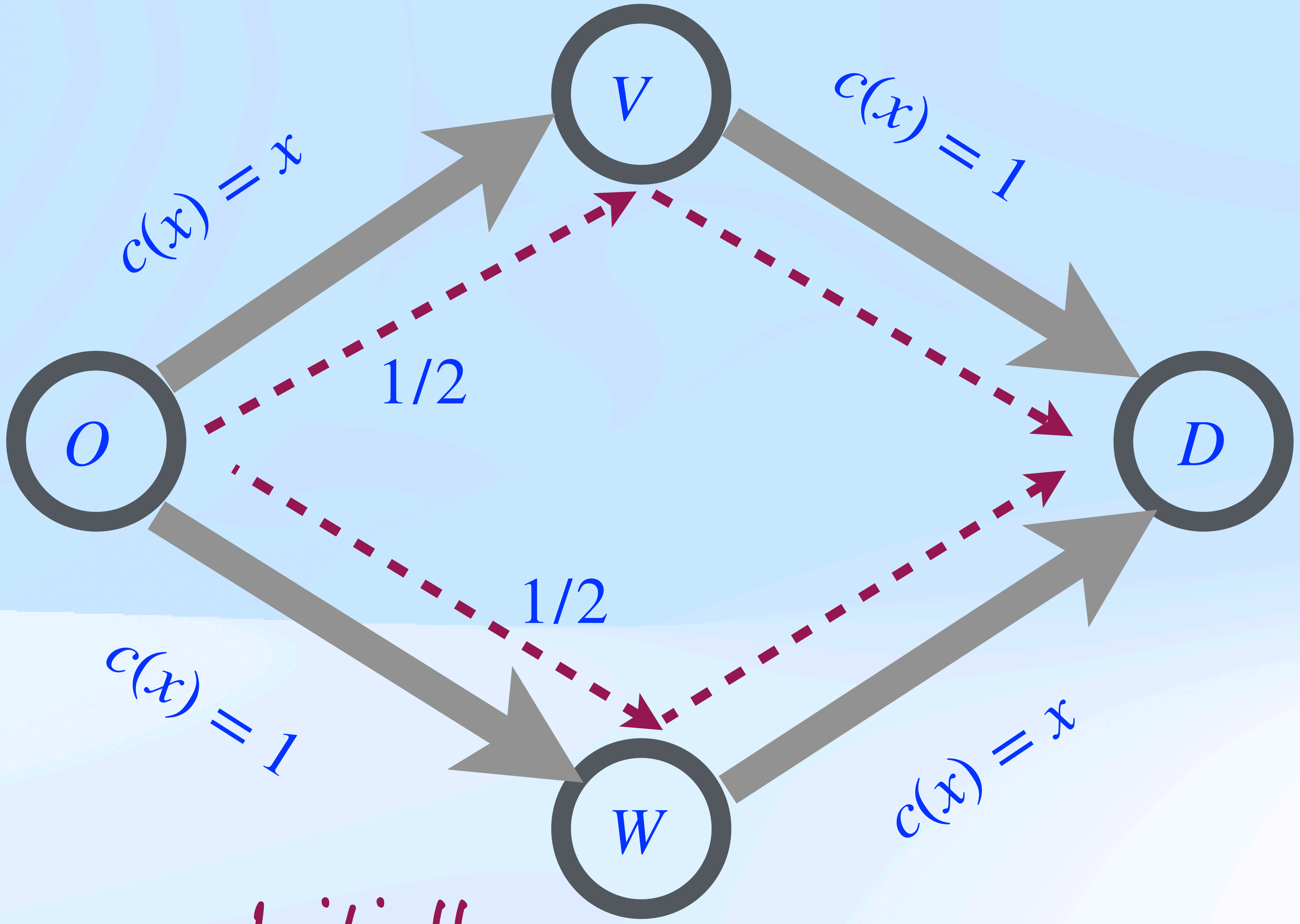


After super-highway

Travel time in DS

Braess Network, 1965

1 unit of traffic



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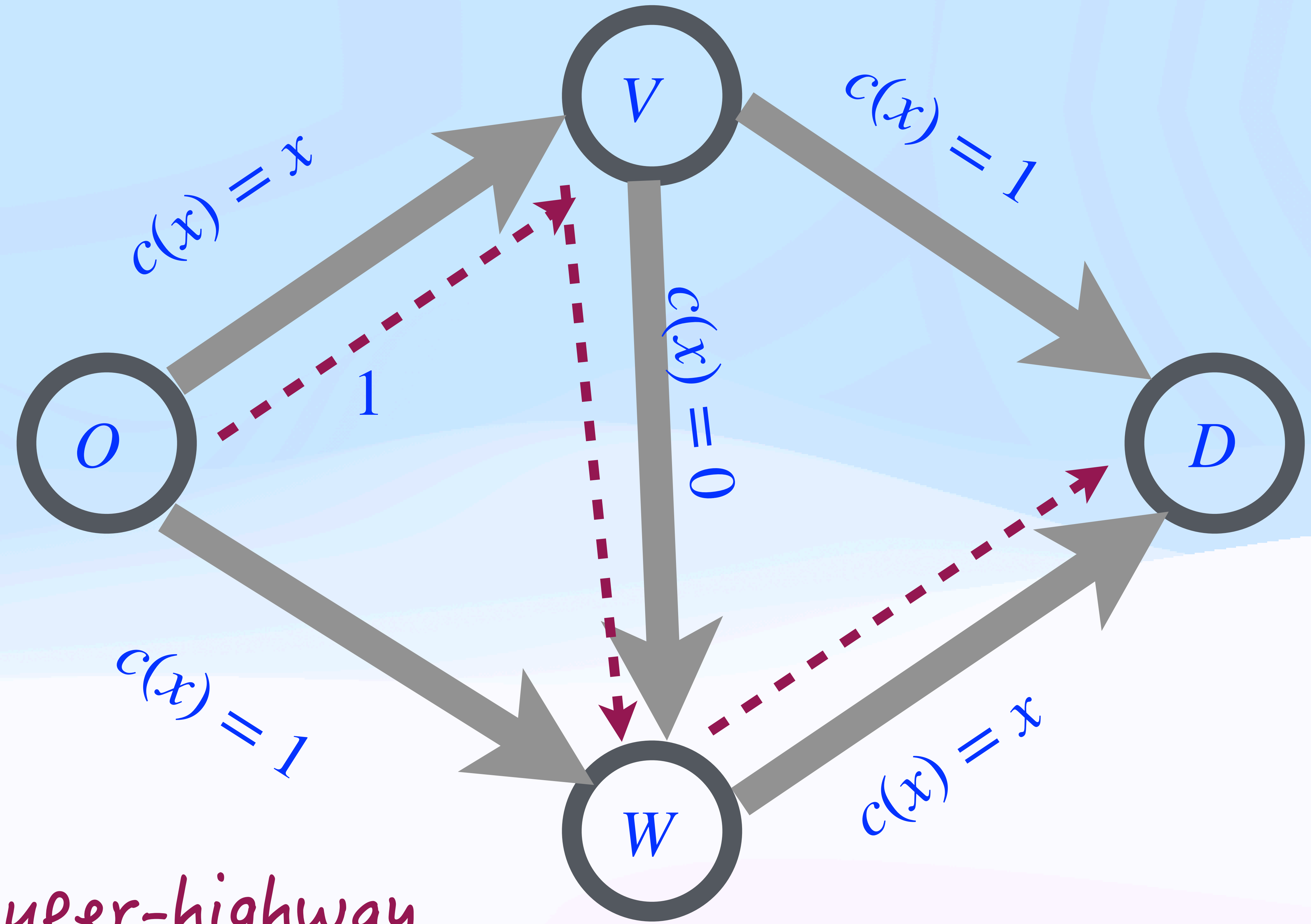

After super-highway

Travel time in DS $= 1\cdot 1 + 0 + 1\cdot 1 = 2$

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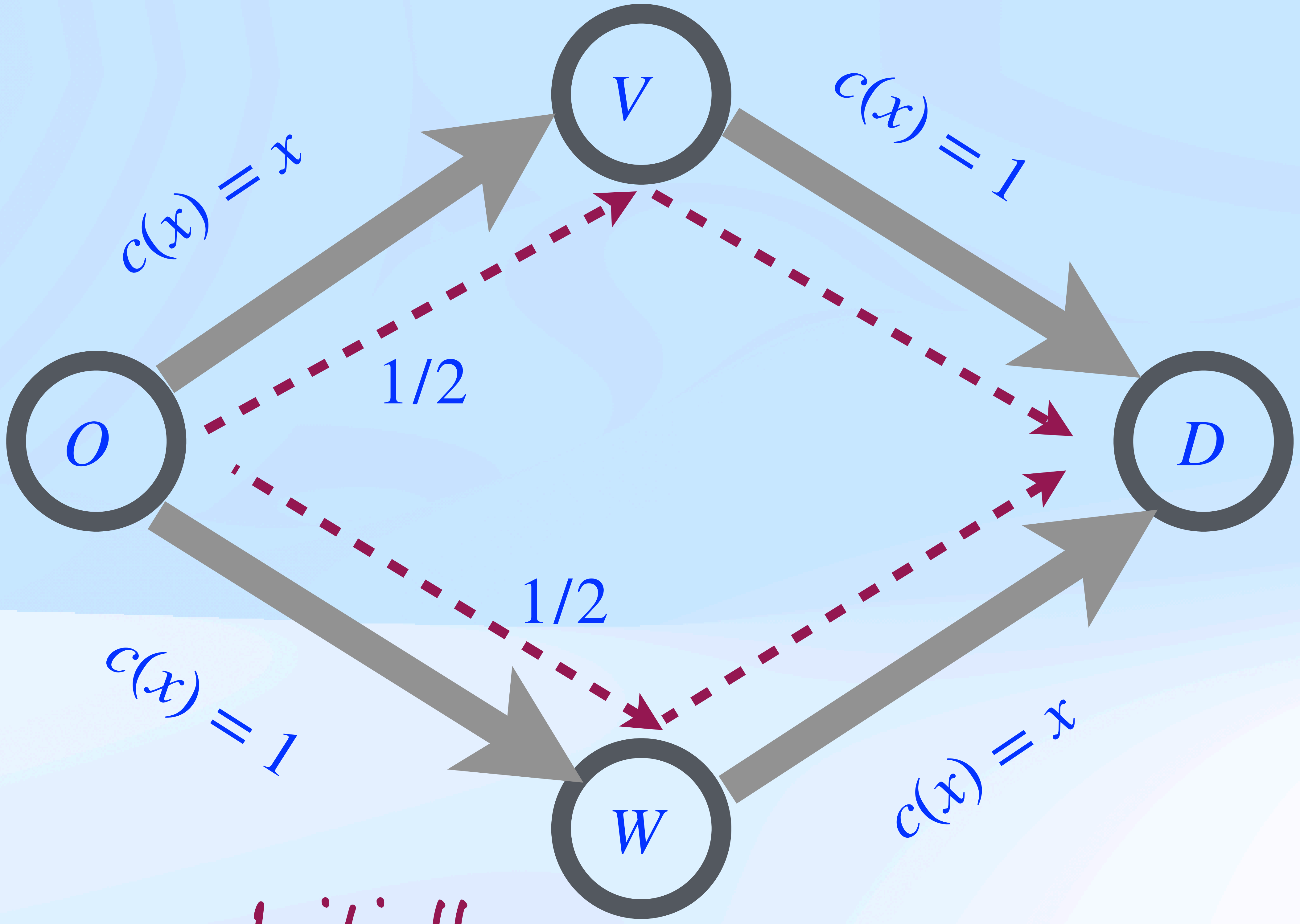

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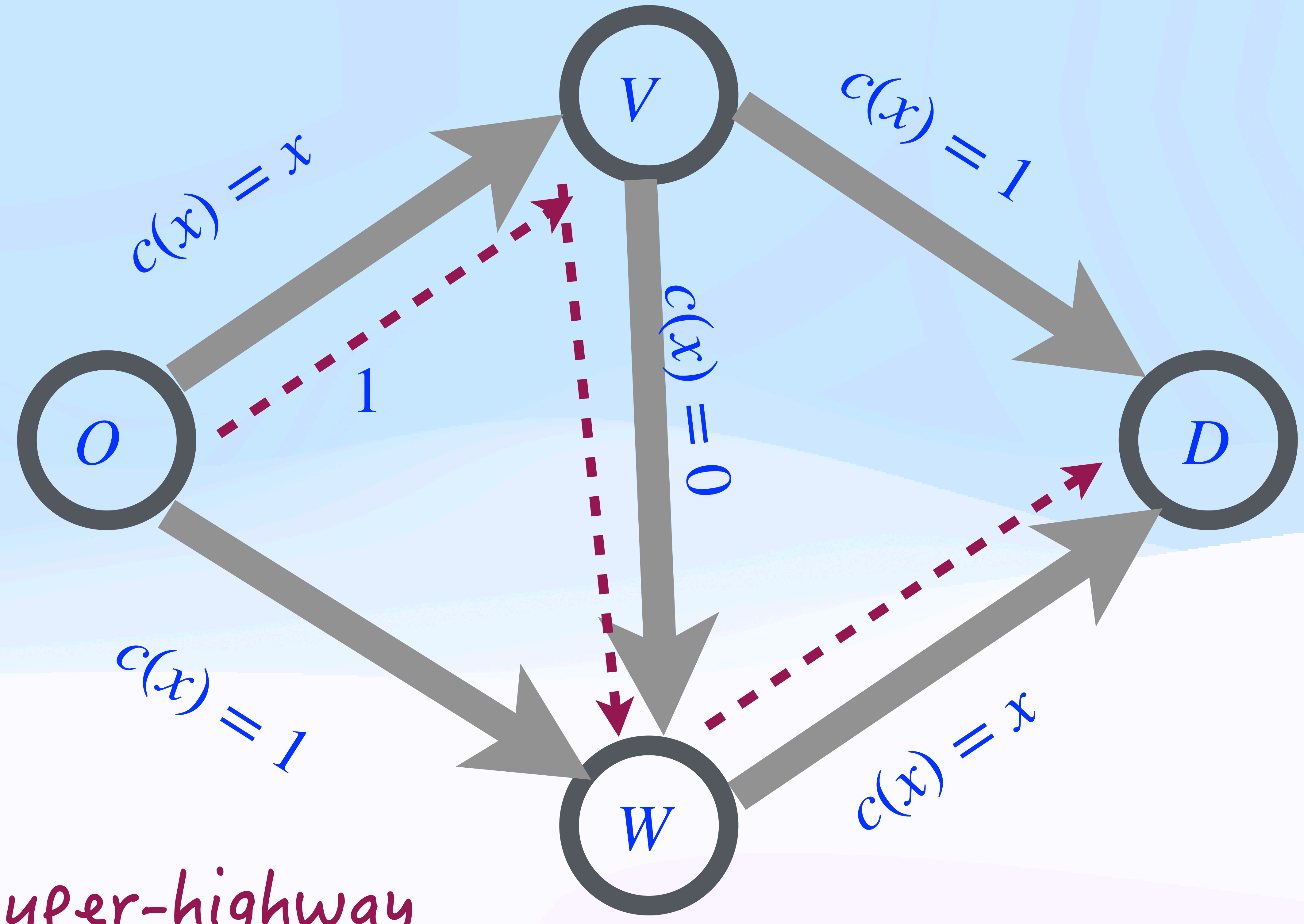

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It is taking more time!!

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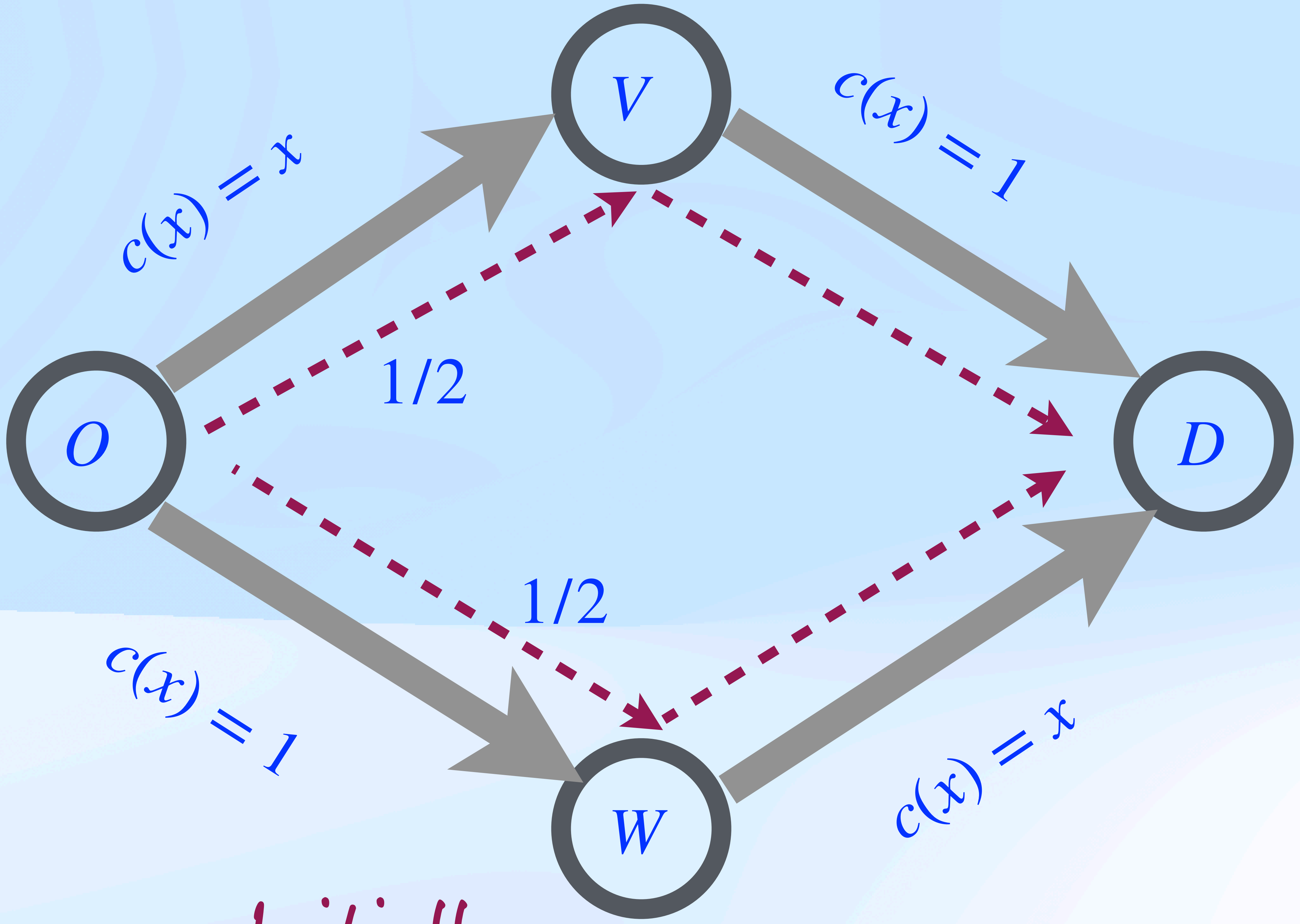

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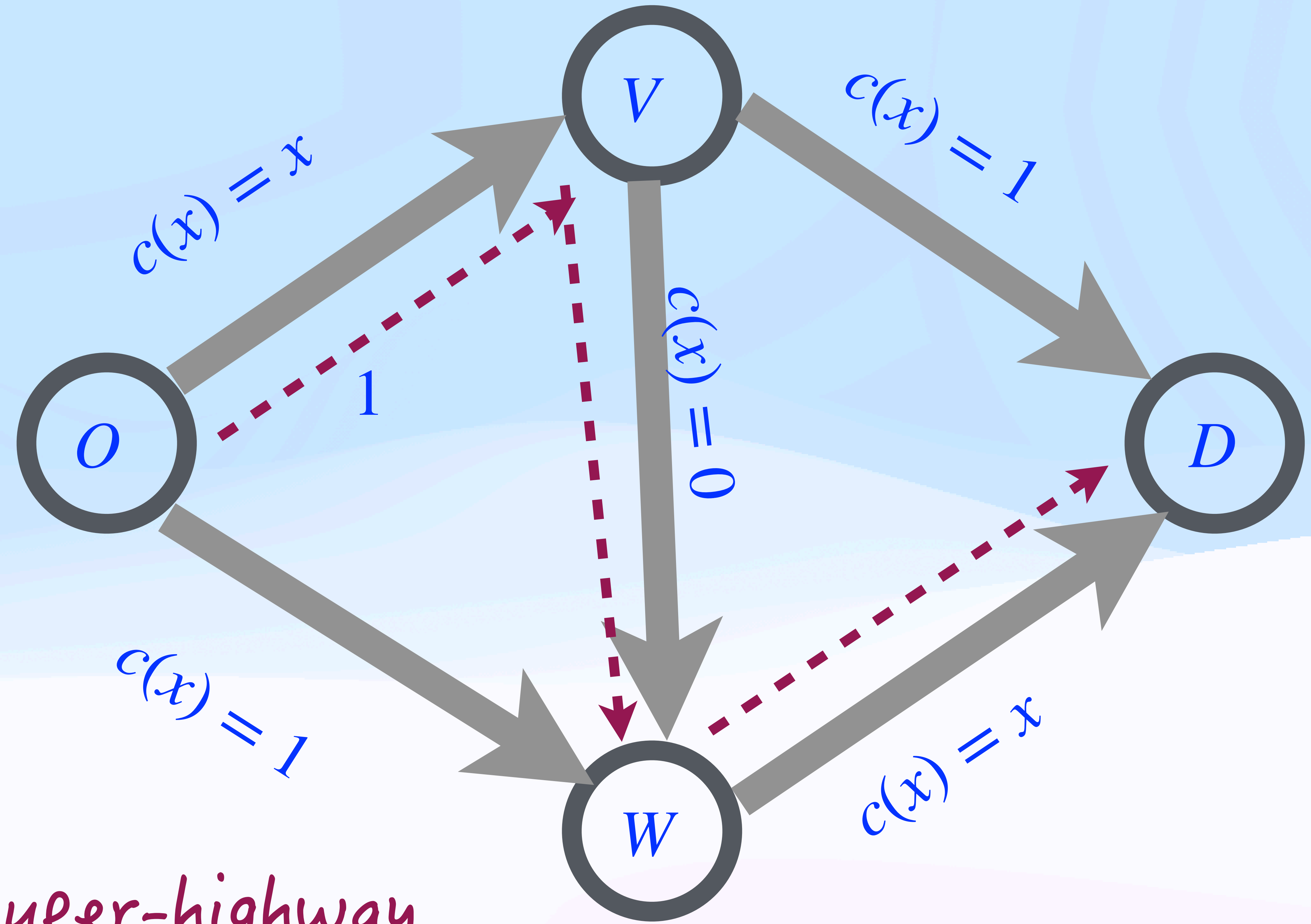

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Travel time in DS $= 1\cdot 1 + 0 + 1\cdot 1 = 2$

Price Of Selfish behavior $= \frac{2}{3/2} = \frac{4}{3}$

It is taking more time!!

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Can this get worse ?

What if the network is more complicated ??

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Roughgarden and Tardos 2002

In *every* network routing game with *linear* (or affine) cost function the price of selfishness is $\frac{4}{3}$

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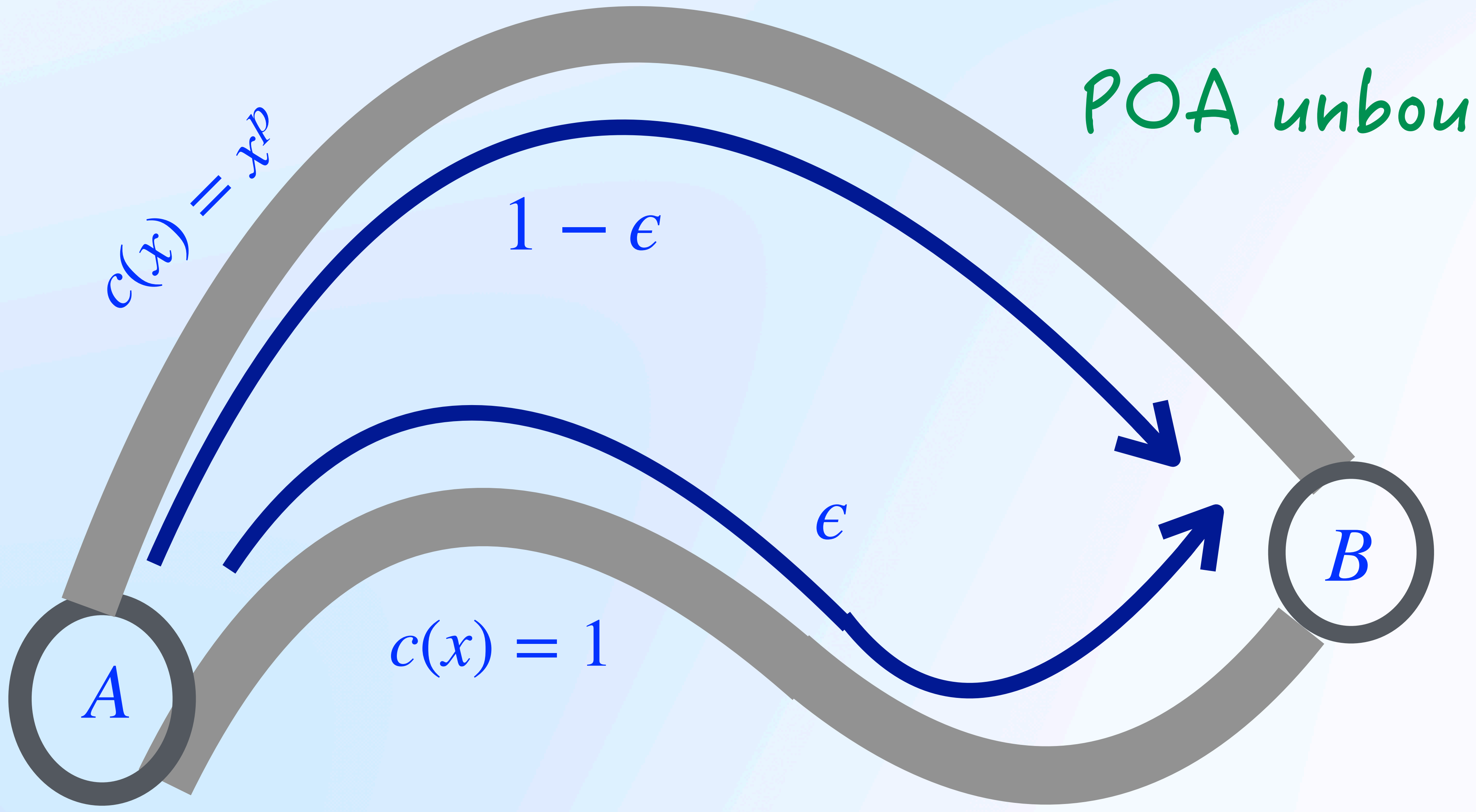

POA unbounded!

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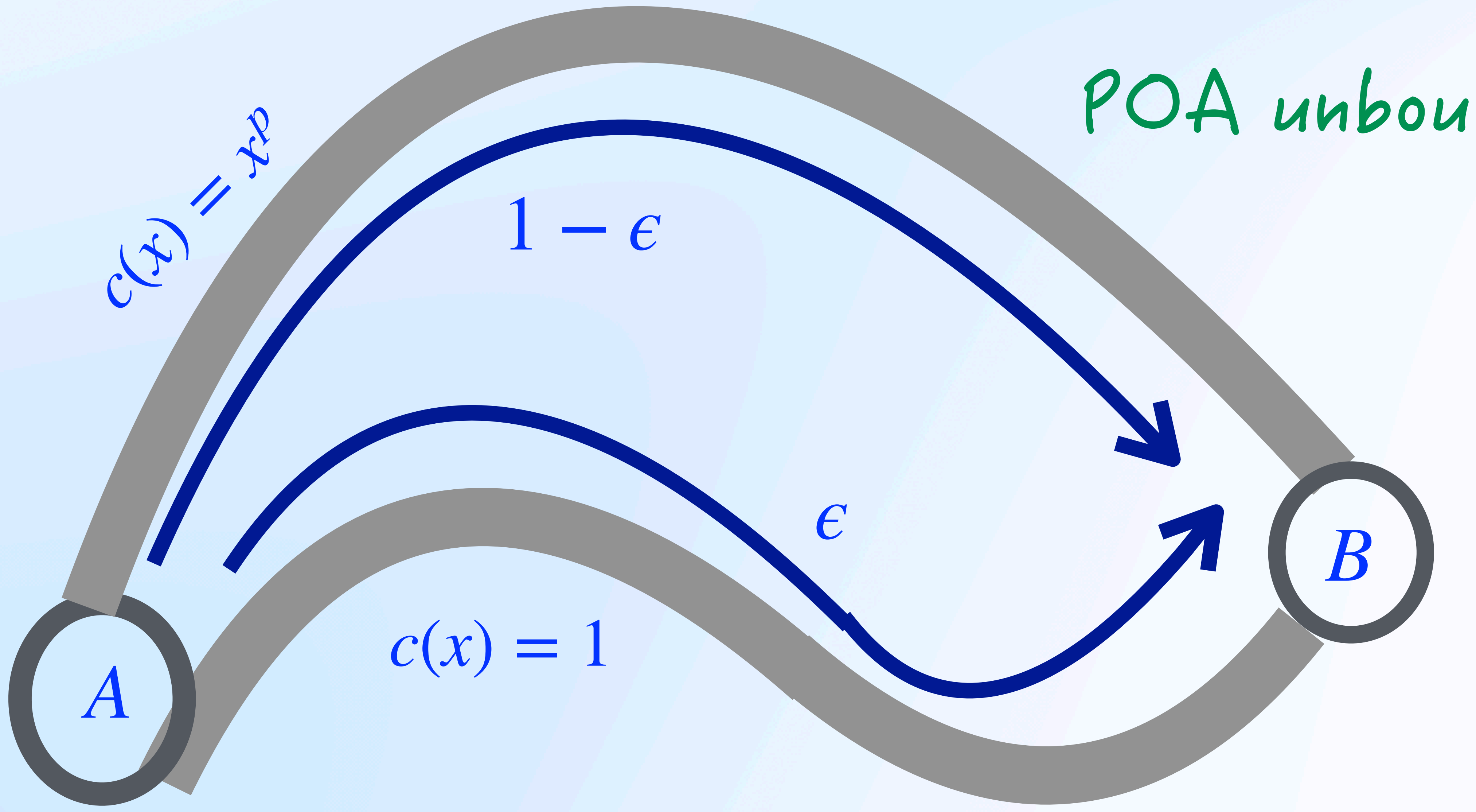

POA unbounded!

Journey of 80 years!

Moral of the story: Culprits are non-linear cost functions!

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Each agent controls a significant fraction of the overall traffic.

Eg: an agent could represent an ISP responsible for routing the data of a large number of end users

Some cool facts about congestion games

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What is the rate of convergence ?

What is the computational complexity of finding an equilibrium?

Thank You!

Proof Details

We will prove

Among all networks with cost function $\mathscr{C}$, the largest POA is achieved in a Pigou-like network.

More formally

For every set $\mathscr{C}$ of cost function and every selfish routing network with cost function in $\mathscr{C}$, the $POA = \dfrac{\text{Flow in DSE}}{\text{Optimum flow}}$ is at most $\alpha(\mathscr{C})$, where

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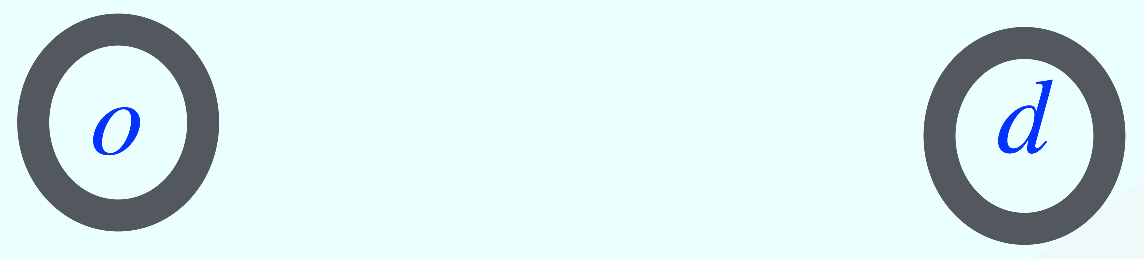

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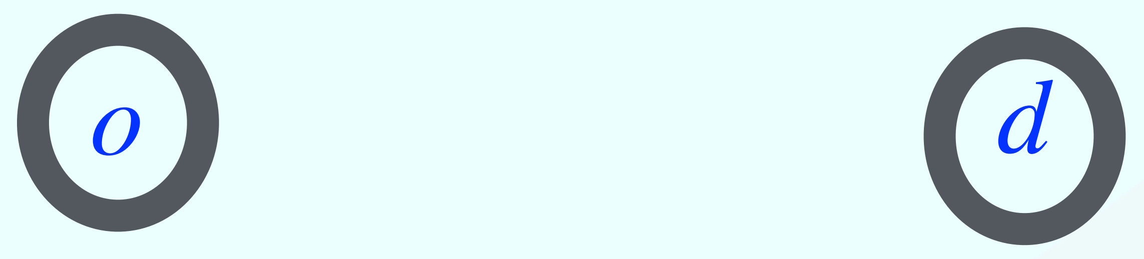

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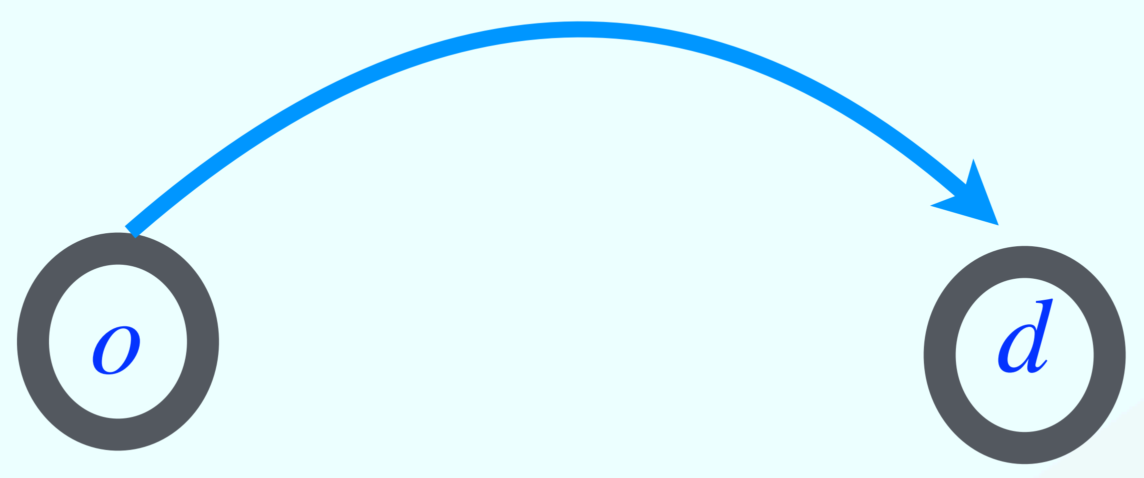

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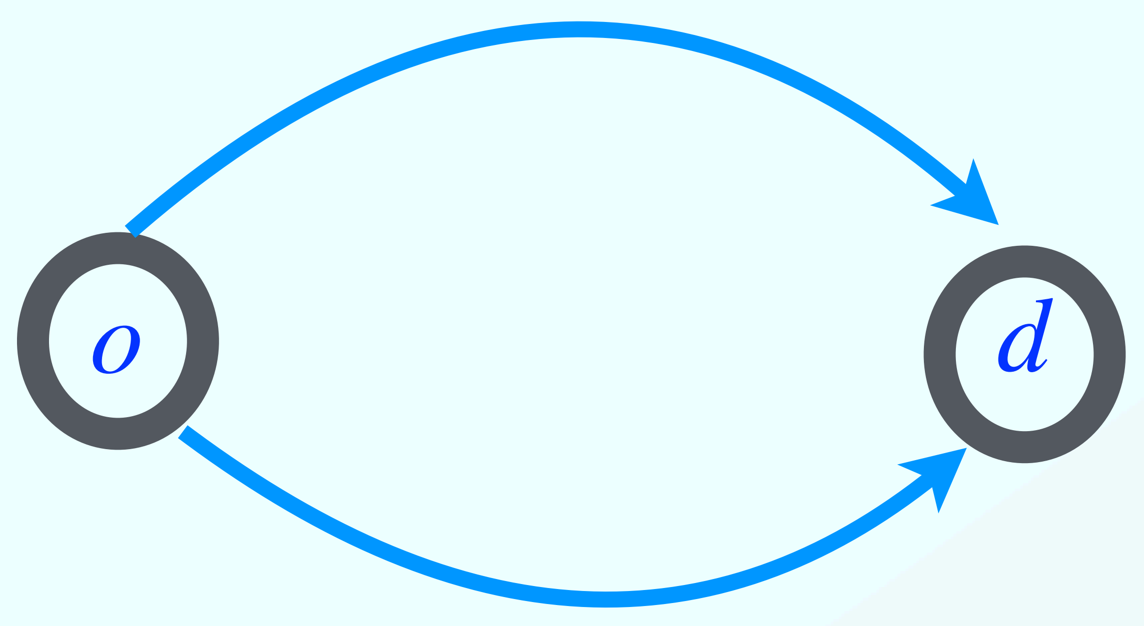

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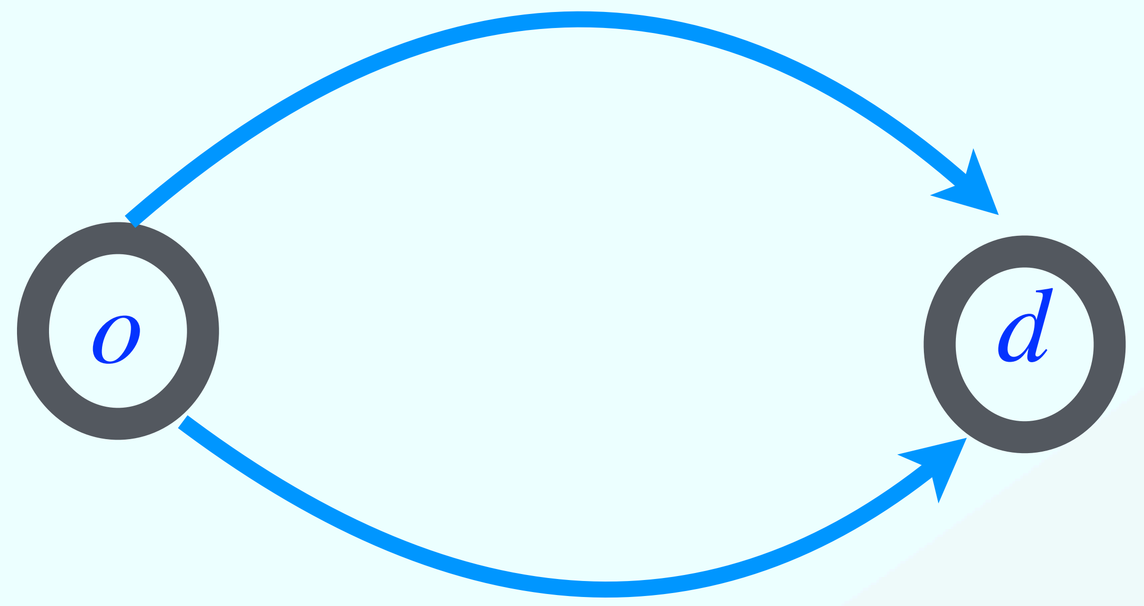

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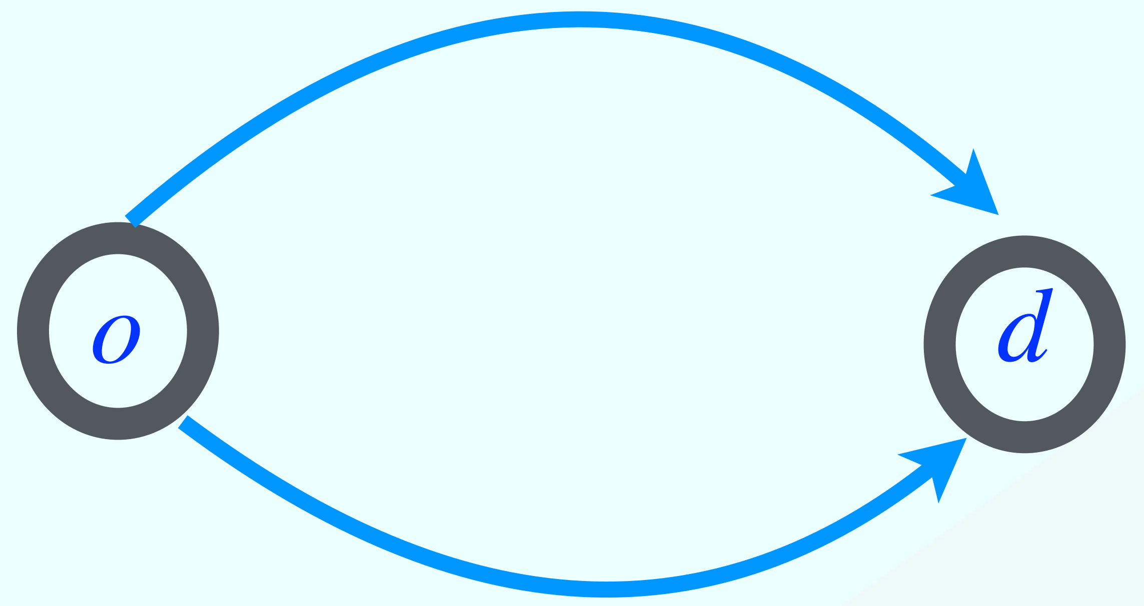

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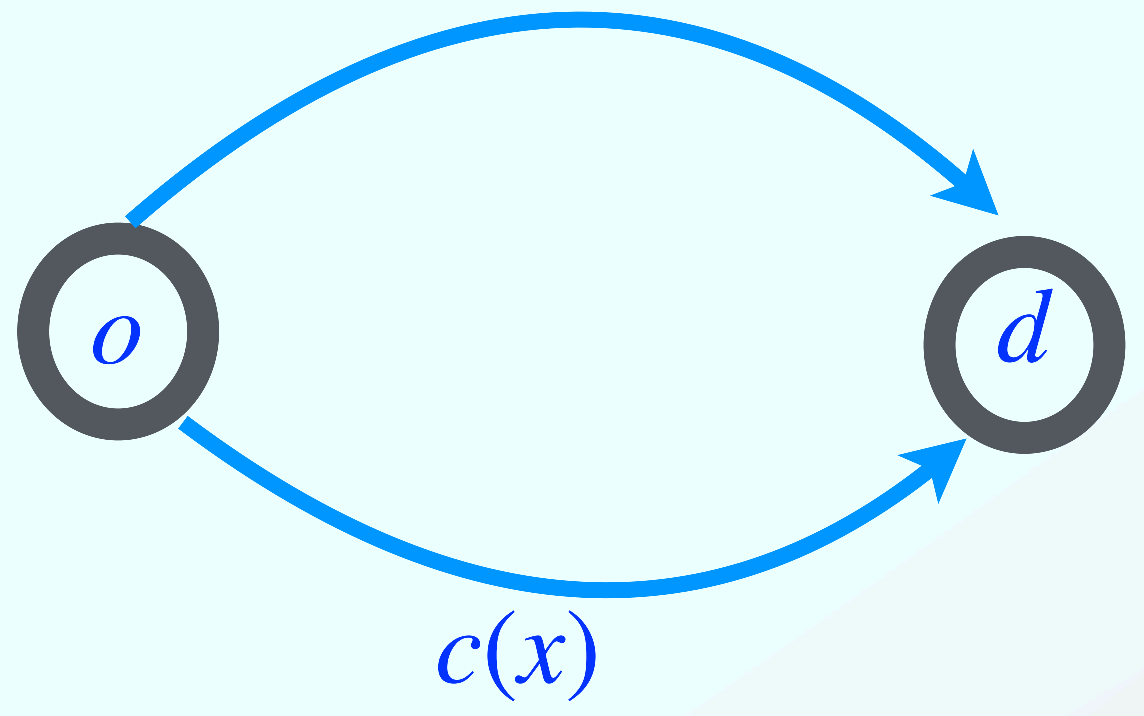

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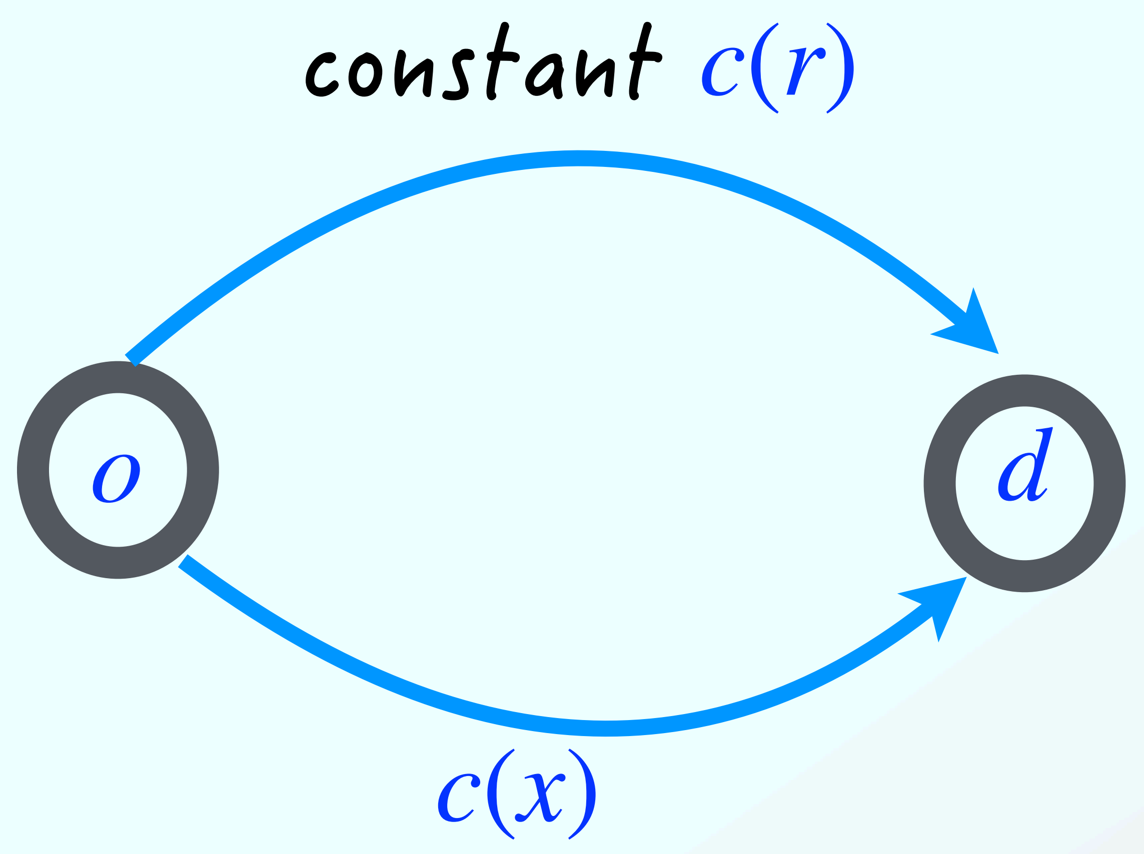

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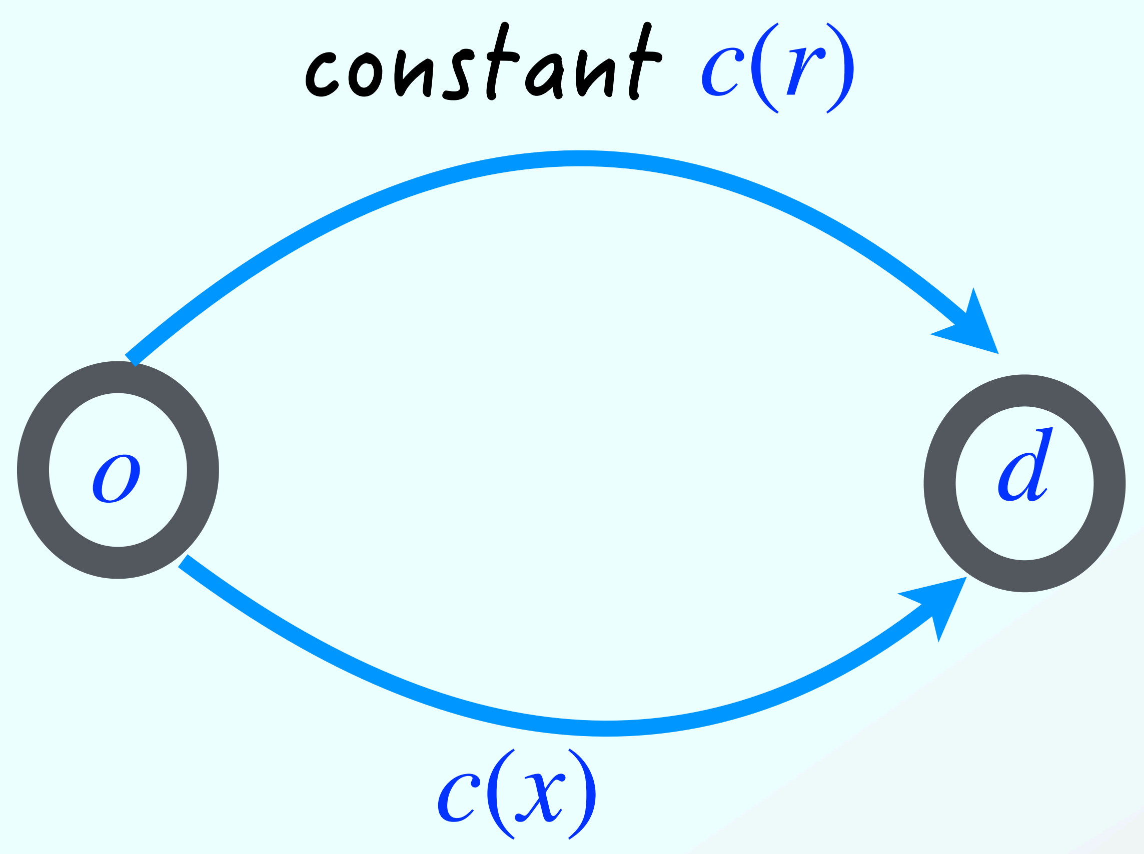

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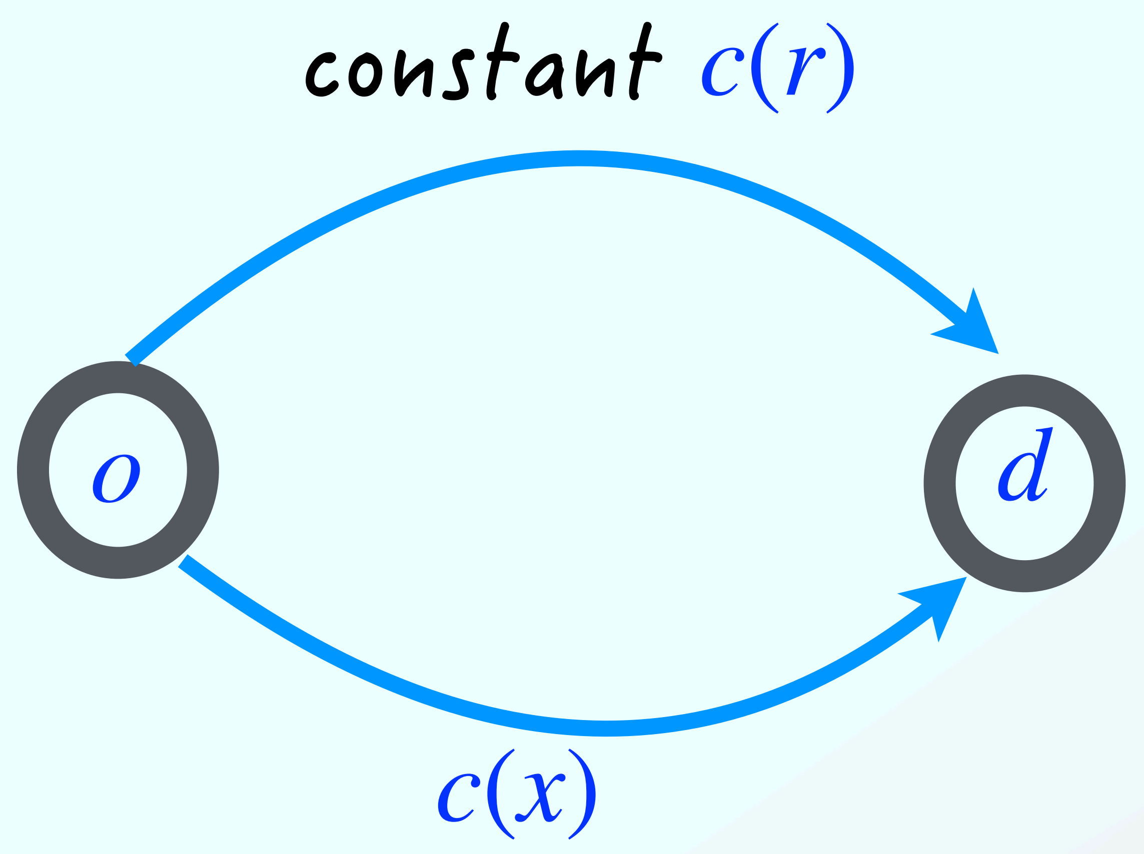

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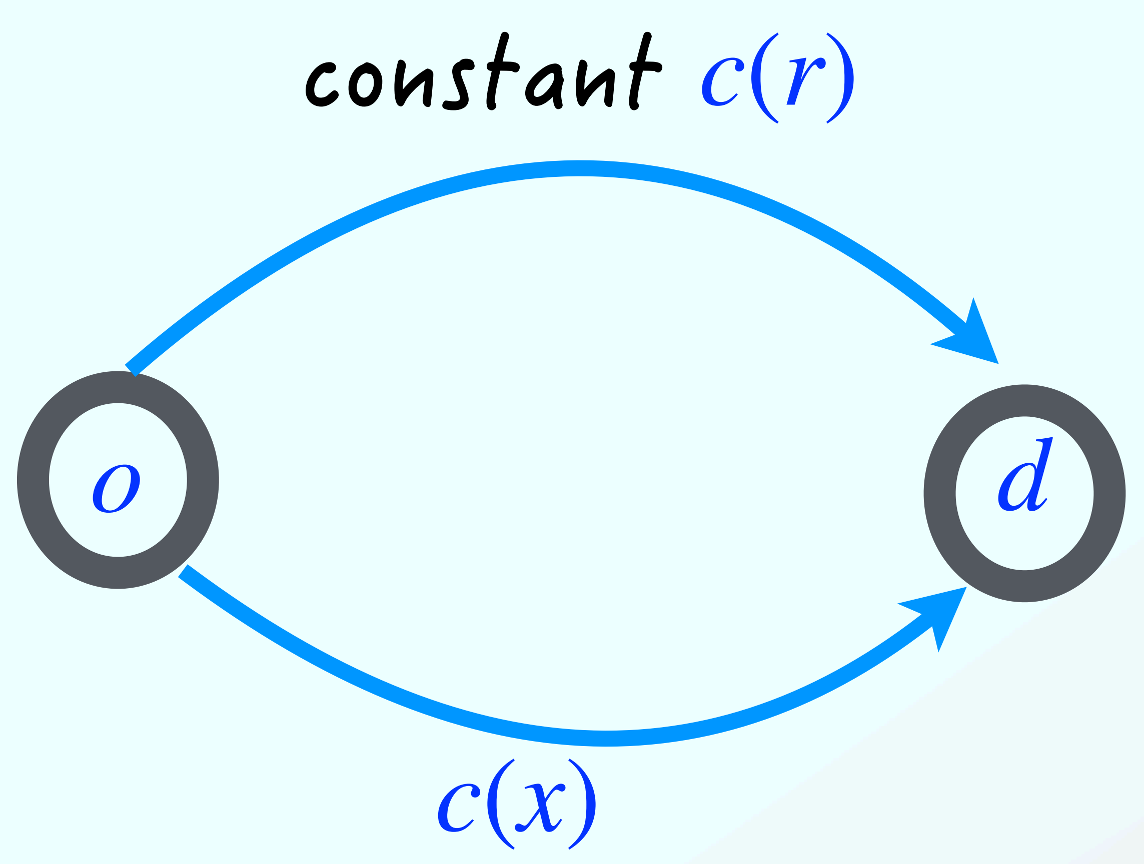

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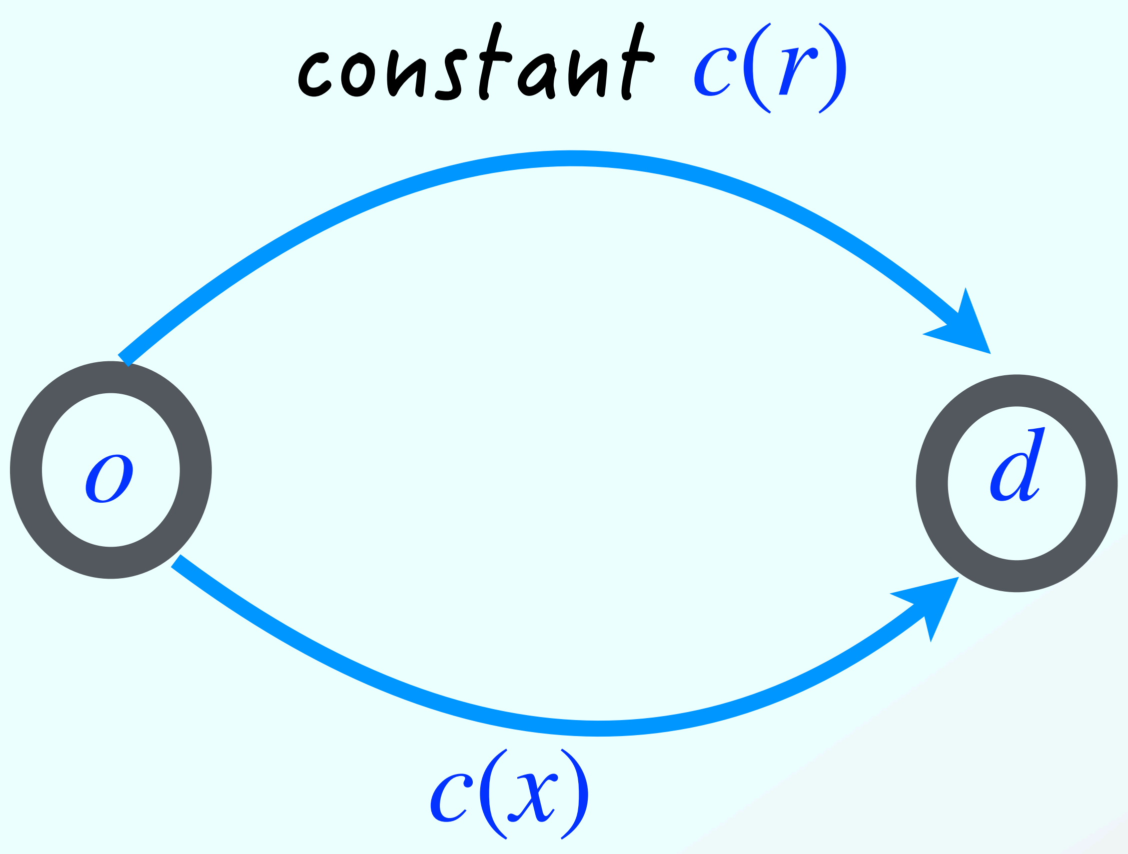

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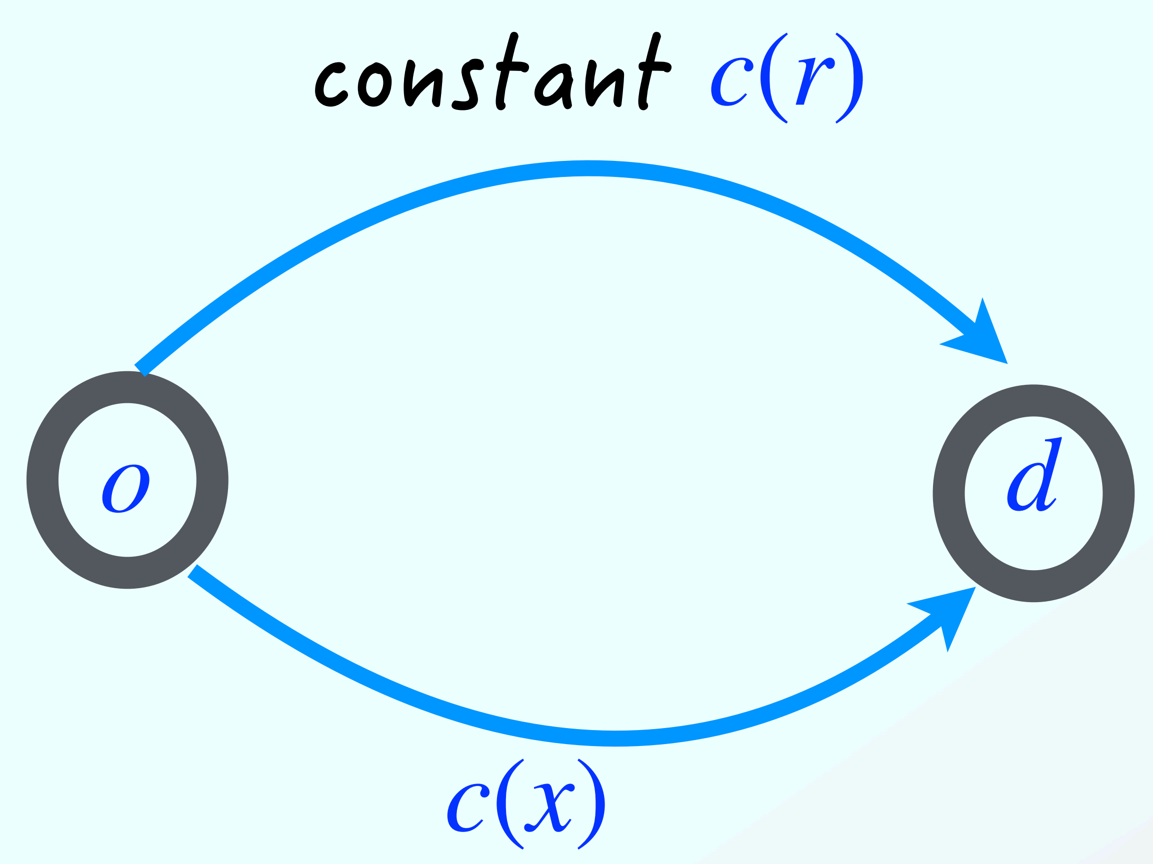

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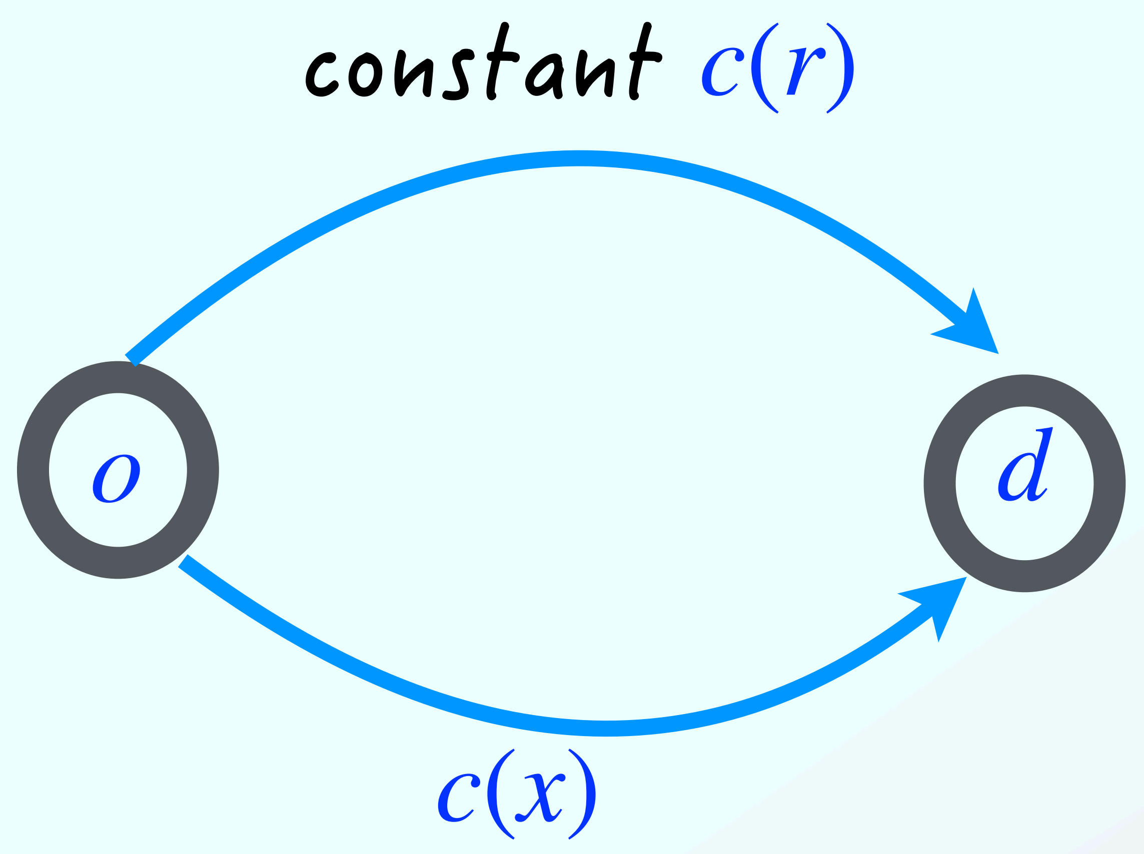

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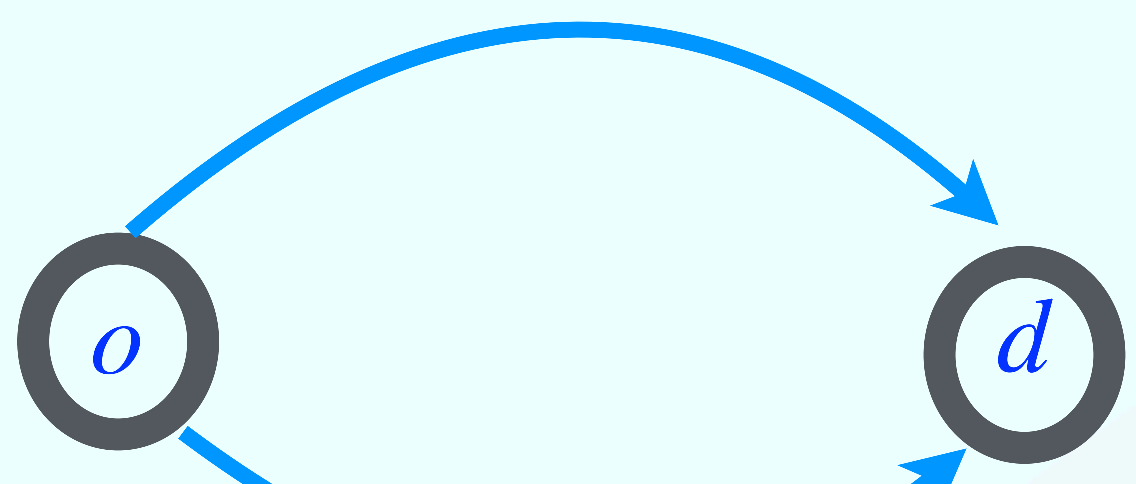

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Worst POA with polynomial cost functions with positive coefficient

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$4/3$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4 + bx^3 + cx^2 + dx + e$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} \approx \frac{p}{\ln p}$

For every set $\mathscr{C}$ of cost function and every selfish routing network with cost function in $\mathscr{C}$, the *POA* is at most $\alpha(\mathscr{C})$, where $\alpha(\mathscr{C}) = \sup_{c \in \mathscr{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

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Proof sketch: Let $G = (V, E)$ be a SRN with r unit of traffic between o to d.

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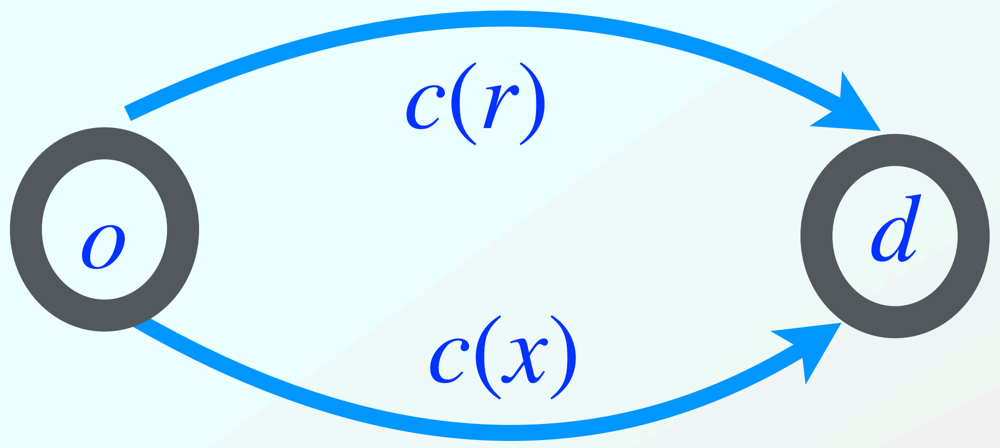

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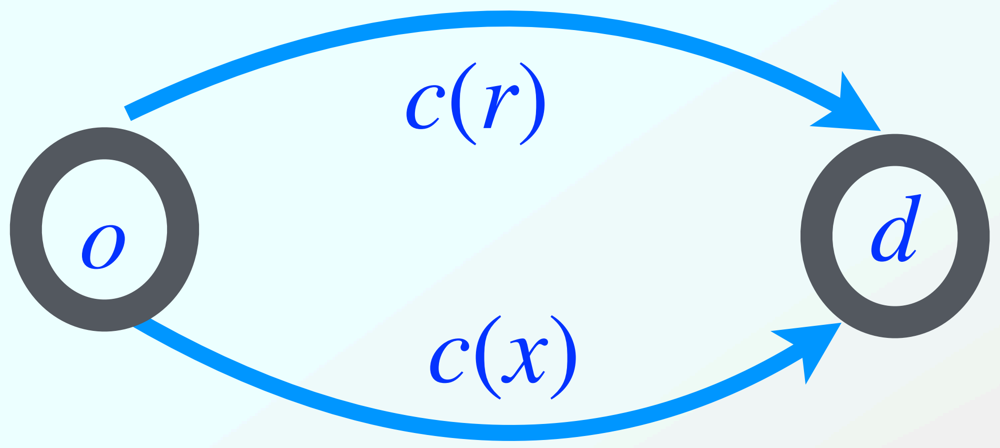

Let $\mathscr{P}$ denote the set of non-empty o-d paths of G.

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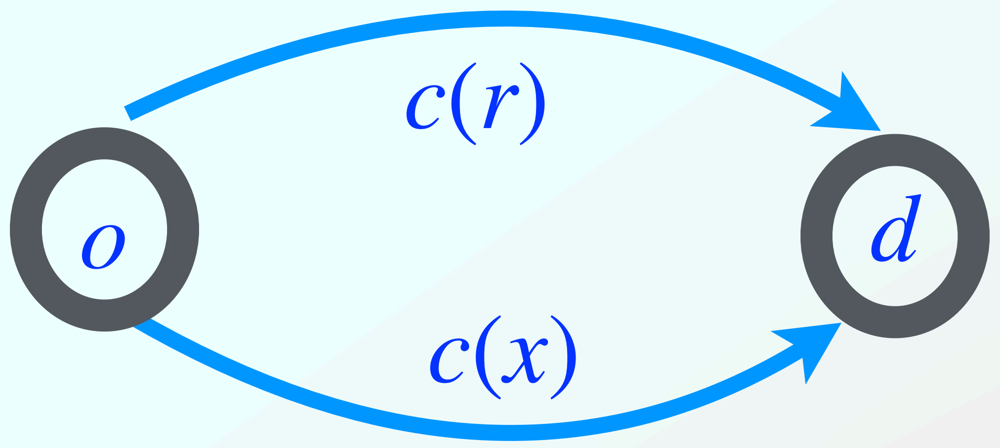

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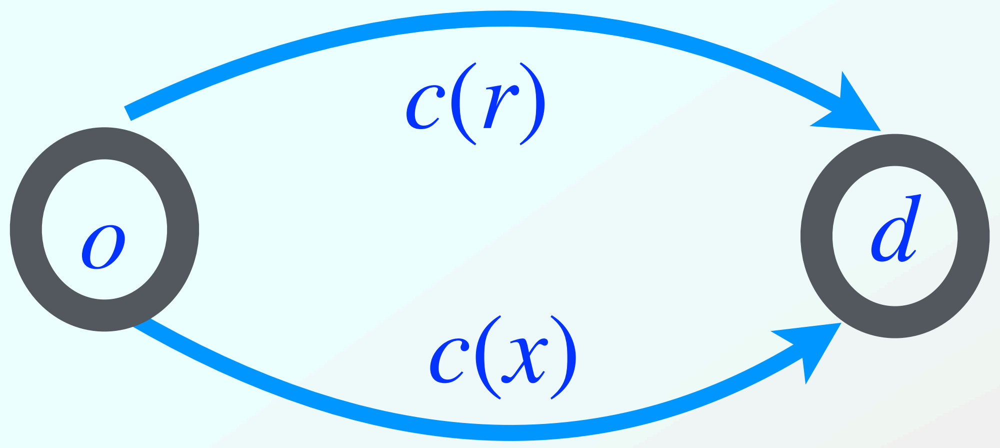

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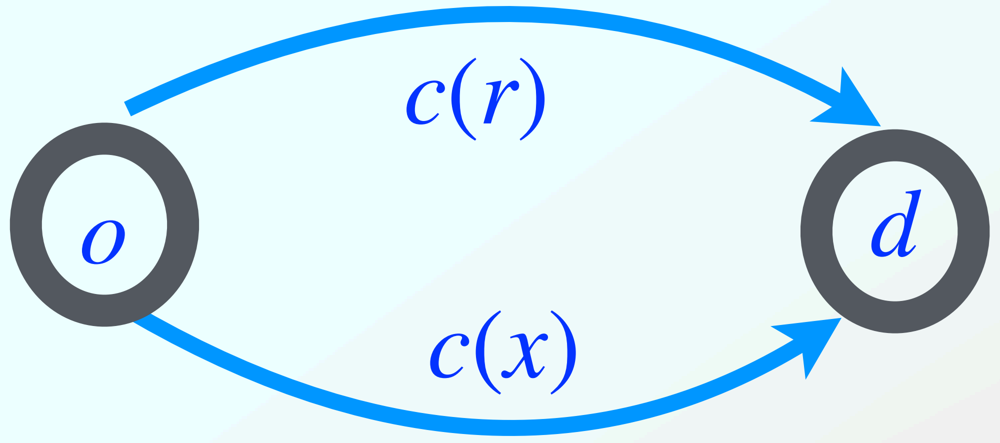

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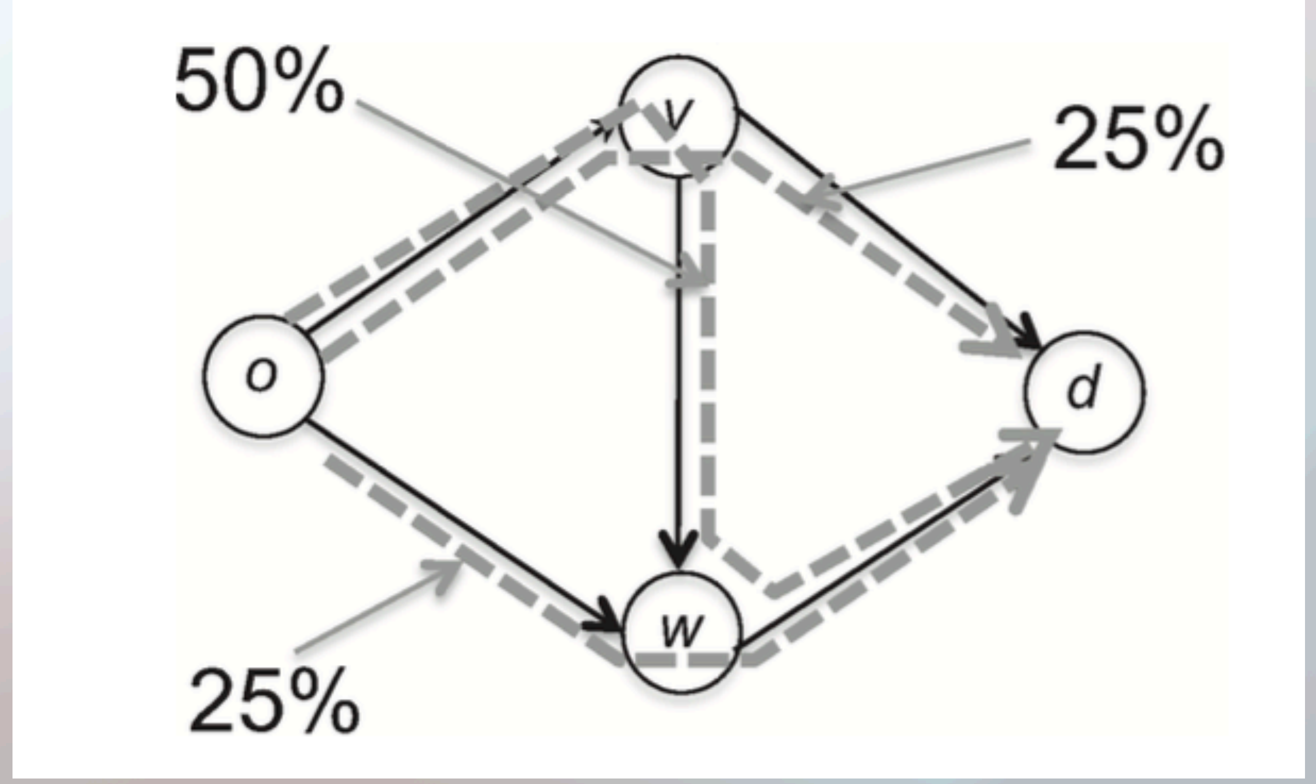

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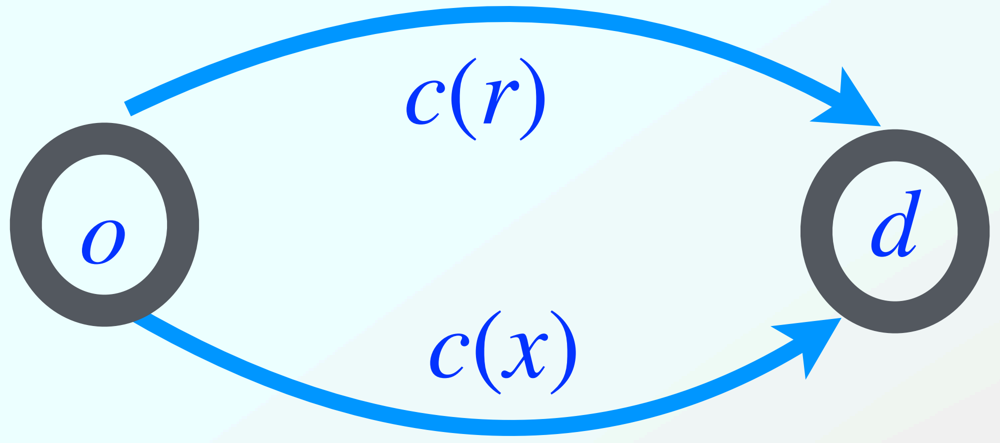
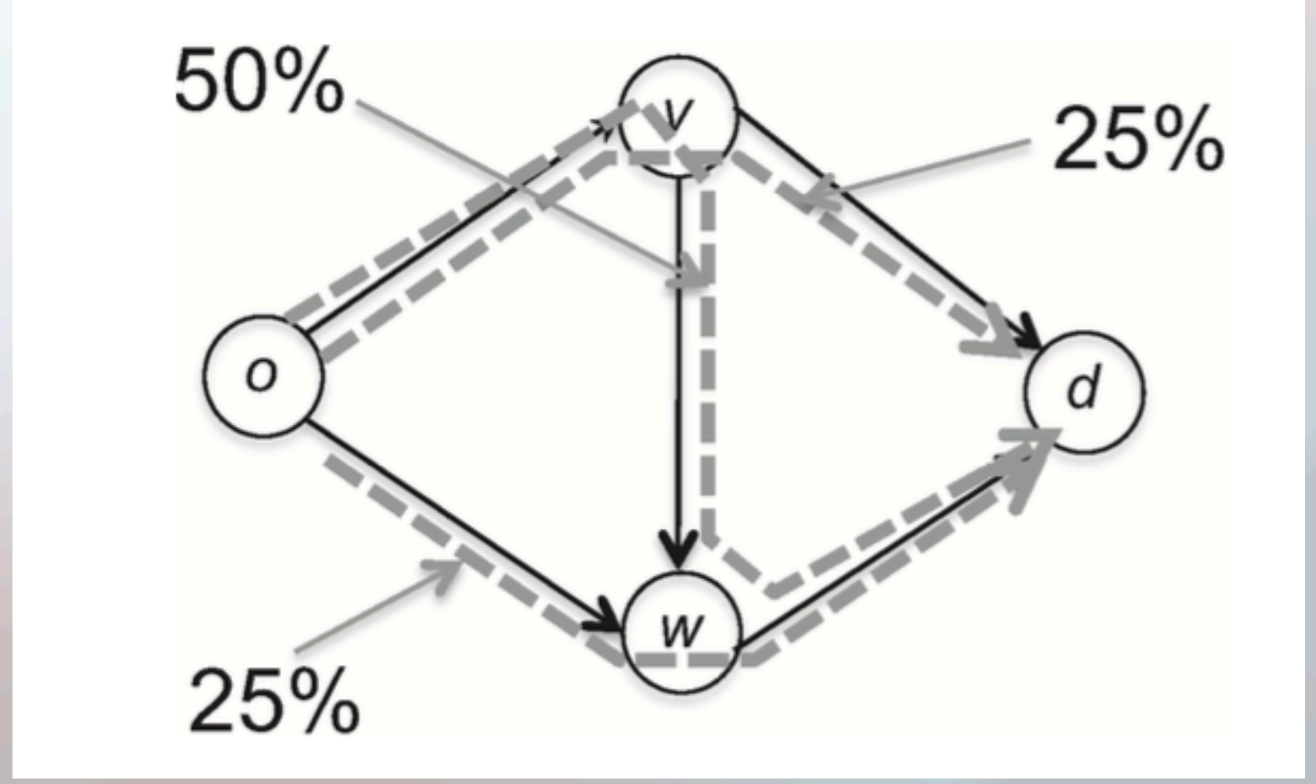


Let $\mathscr{P}$ denote the set of non-empty o-d paths of G.

A flow describes how traffic is split over the $o-d$ paths Non-negative vector $\{f_P\}_{P\in\mathscr{P}}$ with $\sum_{p\in\mathscr{P}} f_P = r$



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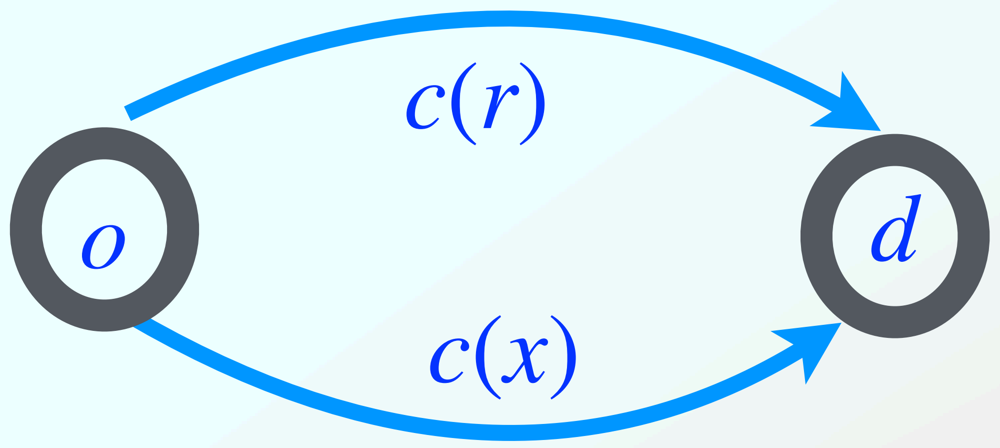

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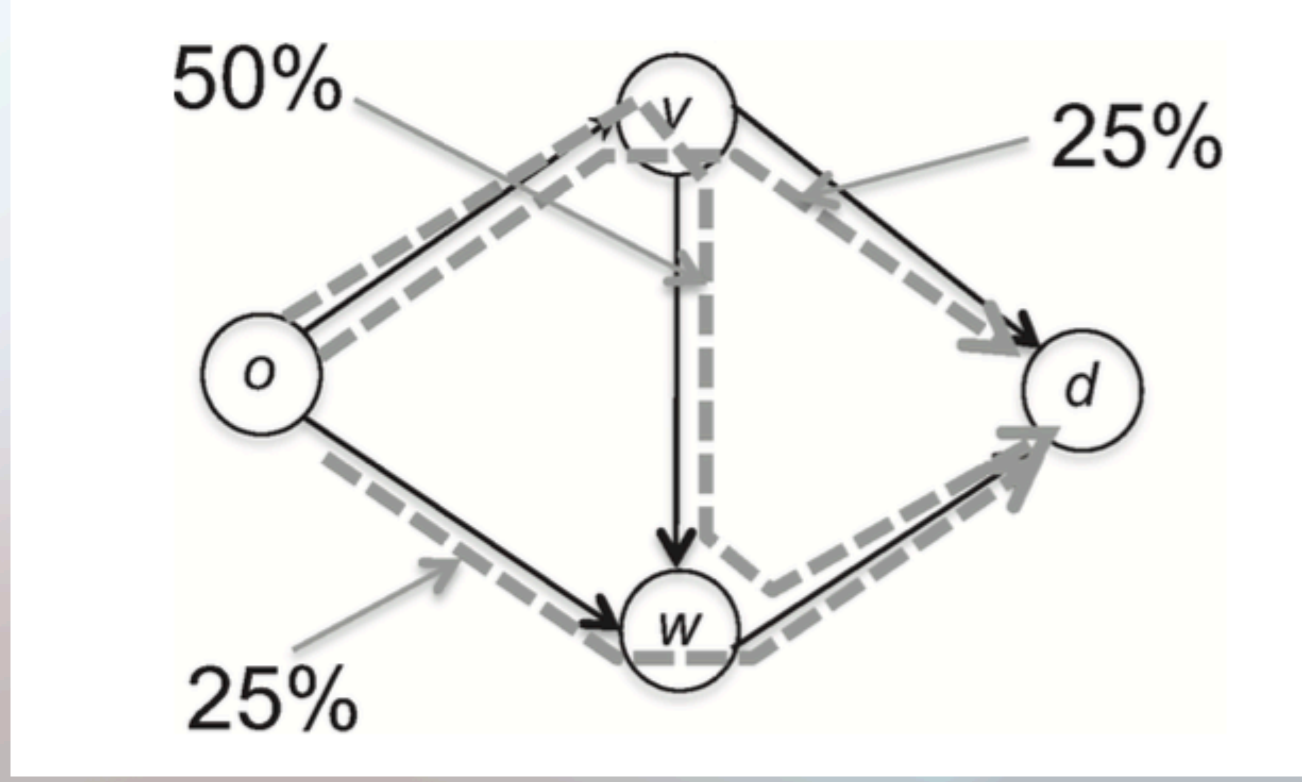

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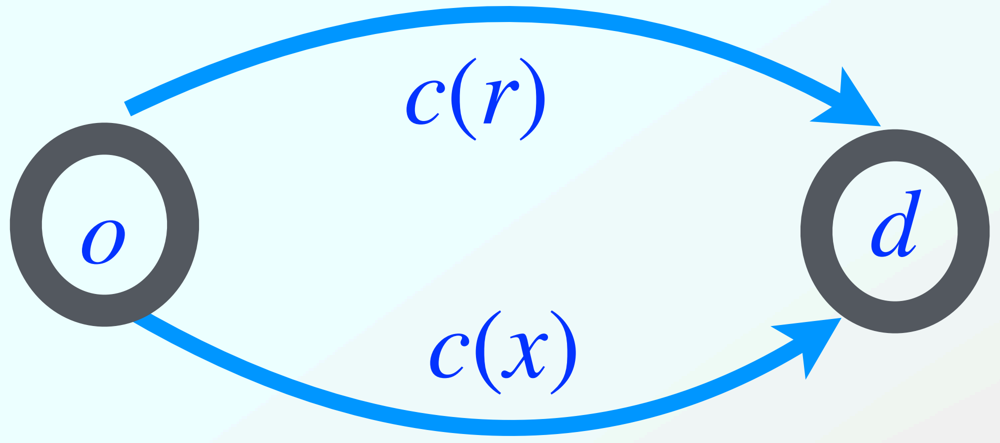

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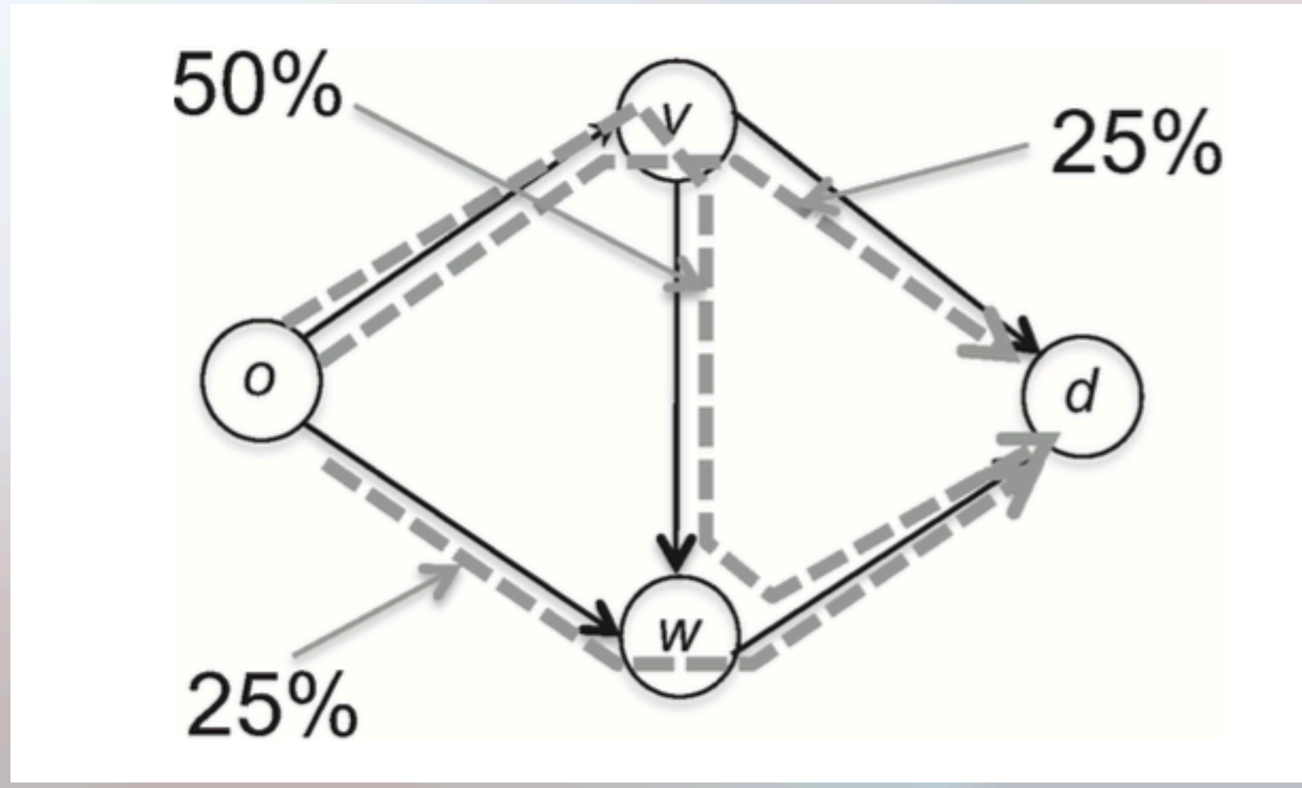

In equilibrium flow, traffic travels only along fastest $o-d$ paths induced by the flow

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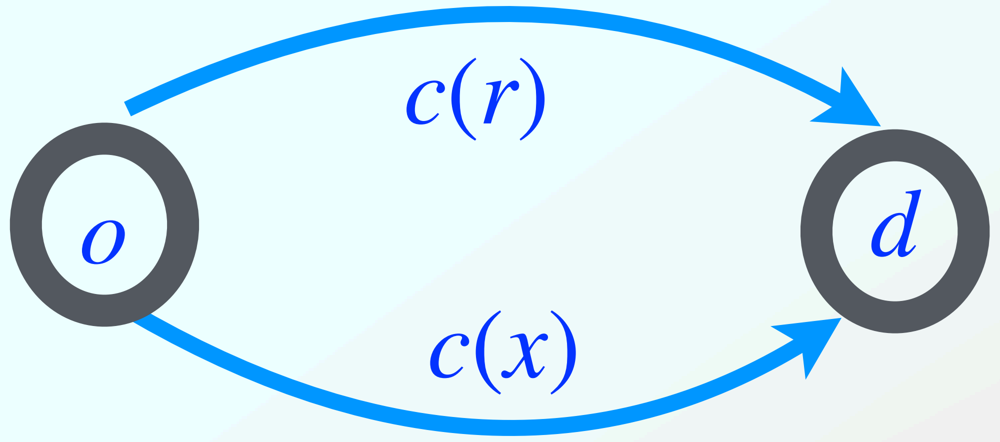

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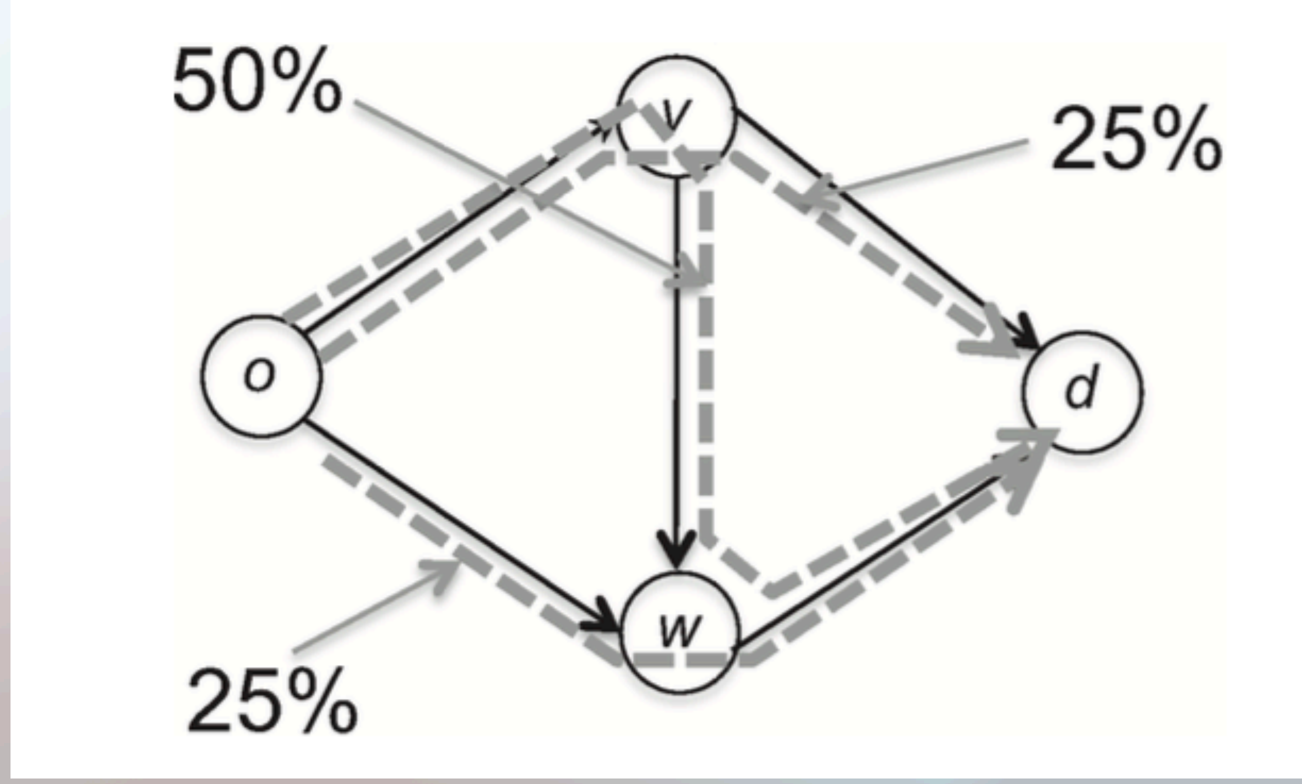

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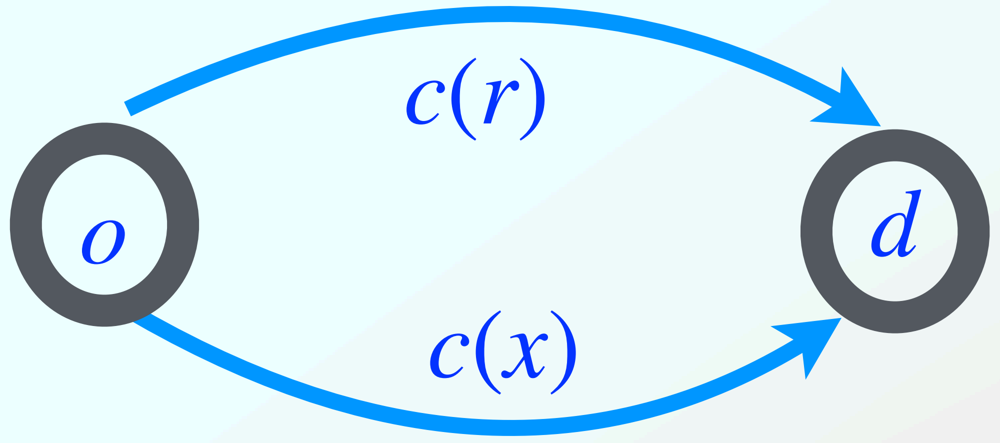

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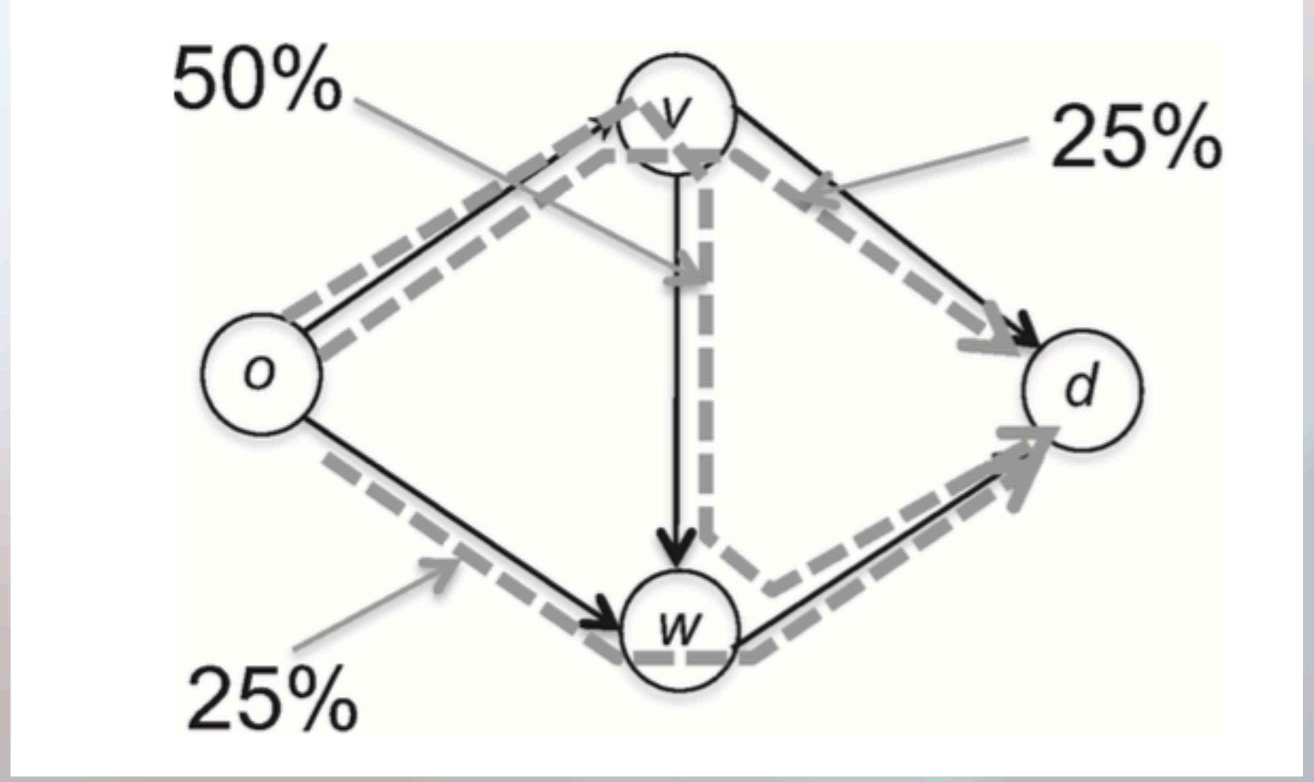

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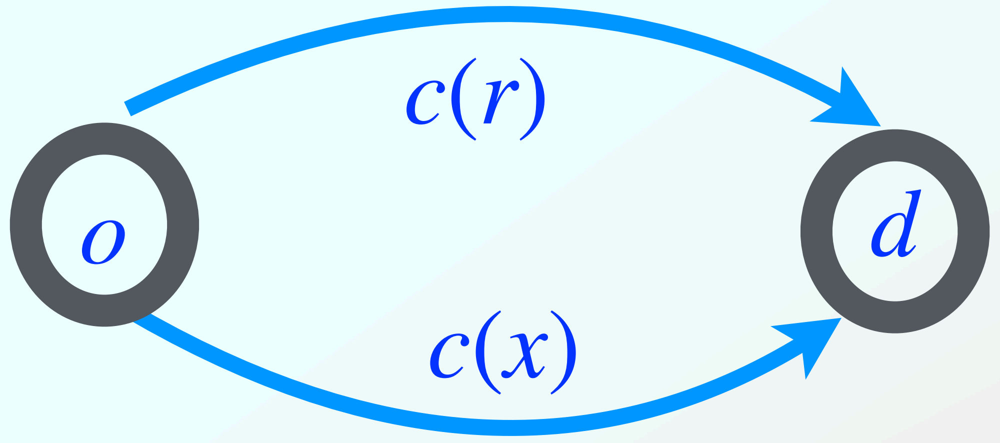

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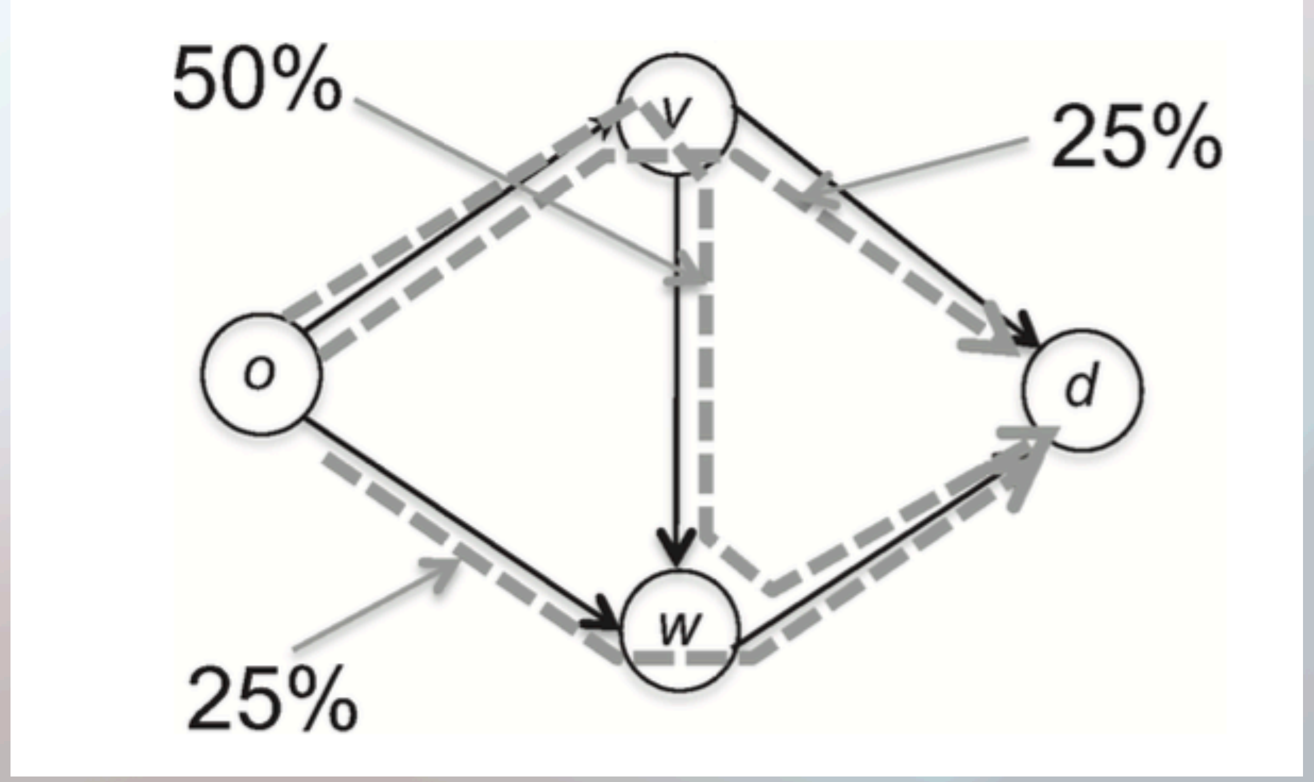

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NOT equilibrium flow! Shortest path is zig-zag & $f_{(o,w,d)} \neq 0$

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Let f and $f^\star$ denote equilibrium and min-cost flows in the network, respectively.

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Let f and $f^\star$ denote equilibrium and min-cost flows in the network, respectively.

Sufficient to prove: $\alpha(\mathscr{C}) \geq \frac{C(f)}{C(f^\star)} = POA$

Part 1: Freezing the cost of every edge e at equilibrium value $c_e(f_e)$ makes f optimal

Step 2: Quantify how much can $f^\star$ be better than f ?

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Claim: $\sum_{e \in E} f_e^\star \cdot c_e(f_e) \geq \sum_{e \in E} f_e \cdot c_e(f_e)$

Step 2: Quantify how much can $f^\star$ be better than f ?

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Think about : POA across an edge $e \in E$

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Replace c with c_e, flow f_e for r, and $f_e^\star$ for x

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$$C(f^\star) \geq \frac{C(f)}{\alpha(\mathscr{C})} + \sum_{e \in E} (f_e^\star - f_e) \cdot c_e(f_e) \geq \frac{C(f)}{\alpha(\mathscr{C})}$$

Proof sketch (contd):

For every set $\mathscr{C}$ of cost function and every selfish routing network with cost function in $\mathscr{C}$, the POA is at most $\alpha(\mathscr{C})$, where $\alpha(\mathscr{C}) = \sup_{c\in\mathscr{C}} \sup_{r\geq 0} \sup_{x\geq 0} \left\{ \frac{r\cdot c(r)}{x\cdot c(x) + (r-x)\cdot c(r)} \right\}$

Think about : POA across an edge $e \in E$

Replace c with c_e, flow f_e for r, and $f_e^\star$ for x

Since $\alpha(\mathscr{C})$ is supremum over all choices of c, r, and x, so over $(c_e, f_e, f_e^\star)$ as well

By rearranging

$$f_e^\star \cdot c_e(f_e^\star) \geq \frac{1}{\alpha(\mathscr{C})} f_e \cdot c_e(f_e) + (f_e^\star - f_e)\cdot c_e(f_e)$$

Summing over all the edges yields

$$C(f^\star) = \sum_{e\in E} f_e^\star \cdot c_e(f_e^\star) \geq \frac{1}{\alpha(\mathscr{C})} \sum_{e\in E} f_e \cdot c_e(f_e) + \sum_{e\in E} (f_e^\star - f_e)\cdot c_e(f_e)$$

$$C(f^\star) \geq \frac{C(f)}{\alpha(\mathscr{C})} + \sum_{e\in E} (f_e^\star - f_e)\cdot c_e(f_e) \geq \frac{C(f)}{\alpha(\mathscr{C})}$$

Using claim

Proof sketch (contd):

For every set $\mathscr{C}$ of cost function and every selfish routing network with cost function in $\mathscr{C}$, the POA is at most $\alpha(\mathscr{C})$, where $\alpha(\mathscr{C}) = \sup_{c \in \mathscr{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$

Think about : POA across an edge $e \in E$

Replace c with c_e, flow f_e for r, and $f_e^\star$ for x

Since $\alpha(\mathscr{C})$ is supremum over all choices of c, r, and x, so over $(c_e, f_e, f_e^\star)$ as well

By rearranging

$$f_e^\star \cdot c_e(f_e^\star) \geq \frac{1}{\alpha(\mathscr{C})} f_e \cdot c_e(f_e) + (f_e^\star - f_e) \cdot c_e(f_e)$$

Summing over all the edges yields

$$C(f^\star) = \sum_{e \in E} f_e^\star \cdot c_e(f_e^\star) \geq \frac{1}{\alpha(\mathscr{C})} \sum_{e \in E} f_e \cdot c_e(f_e) + \sum_{e \in E} (f_e^\star - f_e) \cdot c_e(f_e)$$

$$C(f^\star) \geq \frac{C(f)}{\alpha(\mathscr{C})} + \sum_{e \in E} (f_e^\star - f_e) \cdot c_e(f_e) \geq \frac{C(f)}{\alpha(\mathscr{C})}$$

Using claim QED