The RSA Cryptosystem

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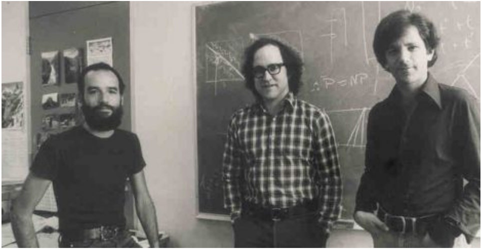
Plan

1. History and Uses
2. RSA Description
   - PKE
   - Decryption is Correct
   - Computational Issues
3. Security
   - Textbook RSA is not Semantically Secure
   - Fixes

RSA – History and Uses

History



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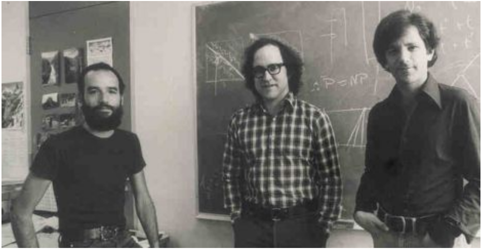

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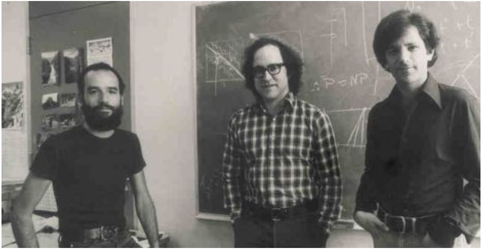

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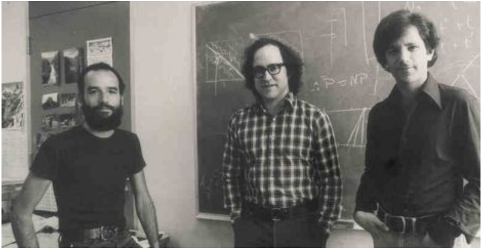

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Diffie and Hellman in 1976 propose a new method for key exchange

Used a so-called trap-door one-way function

Easy to compute in one direction but ‘hard’ to invert

Diffie and Hellman suggested that a similar idea may be used to provide public key encryption and authentication scheme

That set R-S-A in action

A public key encryption scheme

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Where is RSA Used?

Web servers and browsers to secure web traffic

Privacy and authenticity of email

Secure remote login sessions

Electronic payment system

Can you find out more and exactly how?

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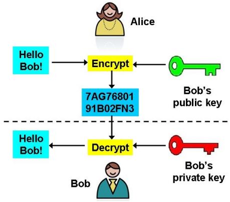
Public Key Encryption



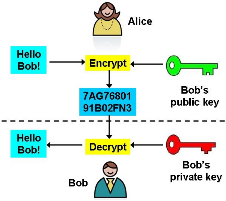
(WIKI)

Public Key Encryption Scheme

A triple of PPT algorithms (G, E, D)

1. $G(1^\lambda) \leftarrow (Pk, Sk)$
2. $E(Pk, m) \leftarrow CT$
3. $D(Sk, CT) \leftarrow m$. (E and D are consistent)

Public Key Encryption



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- Setup(1^λ): Choose two large primes p, q and set $n = pq$. Let $\phi(n) := (p-1)(q-1)$.

  Choose an odd number e s.t $(e, \phi(n)) = 1$. Compute d s.t $ed \equiv 1 \mod (\phi(n))$.

  Publish $Pk = (e, n)$ and keep secret $Sk = d$.

- Encrypt(Pk, m): The sender encrypts any $m < n$ as

$$CT = m^e \mod (n).$$

- Decrypt(Sk, CT): The receiver decrypts CT as

$$m = CT^d \mod (n).$$

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Euler ϕ function: Is a arithmetic function equalling the number of positive integers less than n and relatively prime with n.

For p, prime $\phi(p) = p - 1$, for $n = pq$, where $p,\ q$ are prime $\phi(n) = (p-1)(q-1)$.

Euler's Theorem: If n and a are coprime then

$$a^{\phi(n)} \equiv 1 \mod (n).$$

The set of all $y < n$ with $(y, n) = 1$, say $\{a_1, \ldots, a_{\phi(n)}\}$ equals $\{a \cdot a_1, \ldots, a \cdot a_{\phi(n)}\}$. (Why?)

So $\prod_i a_i = \prod_i a \cdot a_i = a^{\phi(n)} \prod_i a_i \mod (n)$.

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Decryption is Correct

RSA Decryption Theorem

For almost all m, we have $m = CT^d \mod (n)$.

Let $ed = k\phi(n) + 1$.

So $CT^d = m^{ed} = m^{k\phi(n)+1} \mod (n)$.

If $(m, n) = 1$, we can apply Euler theorem and obtain the required result.

Otherwise, decryption can not be done. But this happens with very less probability.

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Computational Issues

- Choosing Primes: Choose a random integer and use AKS primality test algorithm.
- Testing $(e, \phi(n)) = 1$ and inverting e to obtain d: Inverse of a modulo m exists if and only if $(a, m) = 1$.

  Use (extended) Euclidean algorithm. Find $x,\ y$ such that $ex + \phi(n)y = 1$.
- Modular Exponentiation: Square-and-multiply.

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Choose a random positive integer of requisite size

Use AKS (deterministic) polytime algorithm to test whether it is prime

There also exist some randomized algorithms which can be used

Theorems about distribution of primes guarantee that you are bound to succeed

$$\pi(N) \approx N/\log(N)$$

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Extended Euclidean Algorithm

Given $a > b$ GCD such that $(a, b) = ax + by$

$r_0 = a,\ r_1 = b,\ r_{i+1} = r_{i-1} - q_i r_i$

$s_1 = a,\ s_1 = 0,\ s_{i+1} = s_{i-1} - q_i s_i$ and
$t_0 = 0,\ t_1 = 1,\ t_{i+1} = t_{i-1} - q_i t_i$

End if $r_{k+1} = 0$. Then $(a, b) = r_k = s_k a + t_k b$

EEA Correctness

Given a, b EEA outputs x, y, d such that

$$d = (a, b) = ax + by.$$

$\{r_i\}$ is decreasing, so the algorithm terminates

$(r_{i-1}, r_i) = (r_i, r_{i+1})$ and $as_i + bt_i = r_i$

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Extended Euclidean Algorithm – Illustation

Let us compute $(240, 46)$

$240 = 5 \times 46 + 10, \; s = 1, \; t = -5$

$46 = 4 \times 10 + 6, \; s = -4, \; t = 21$

$10 = 1 \times 6 + 4, \; s = 5, \; t = -26$

$6 = 1 \times 4 + 2, \; s = -9, \; t = 47$

$4 = 2 \times 2 + 0, \; s = 23, \; t = -120$

$2 = -9 \times 240 + 47 \times 46$

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Modular Exponentiation

Say you want to compute $5^{37} \mod (19)$

$37 = 2^5 + 2^2 + 2^0$

So $5^{37} = 5^{2^0} \times 5^{2^2} \times 5^{2^5} \mod (19)$

Keep squaring $5 \mod (19)$ and multiply requisite terms: 5^1, 5^2, $(5^2)^2 = 5^4$, $(5^4)^2 = 5^8$, $(5^8)^2 = 5^{16}$, $(5^{16})^2 = 5^{32}$ all modulo 19

Write down the algorithm and estimate the complexity of this method

Other methods like Motgomery laddering are also used

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RSA Security

Semantic Security

Textbook RSA is not even semantically secure

In fact, semantic security is a weak notion

It is *deterministic*

The adversary can distinguish ciphertext for 1 and 0, as he can himself encrypt

Ciphertext is malleable

Particular attacks for $e = 3$

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Even if the message space is small non-deterministic encryption will ensure different ciphertexts for same message

Padding: Encrypt padded message and remove padding after decryption

PKCS (Public Key Cryptography Standard), OAEP (Optimal Asymmetric Encryption Padding) provide padding schemes

Making RSA secure is a huge research area

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RSA vs. Factoring

RSA Problem

Given (n, e) and a $CT = m^e \mod (n)$ from the RSA encryption scheme, determine m.

Factoring

Given $n = pq$, find p.

Integer factorization is hard

If an algorithm for factoring is known, then $\phi(n)$ can be computed, the d can be found and RSA problem solved

Is factoring as hard as RSA problem? – Not known

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No poly-time algorithm known for factoring

Many recent advances in so-called index calculus methods for factoring. Number Field Sieve (NFS) and its generalizations yield subexponential methods

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https://factorable.net/index.html

Look at RSA public keys (N, e) available on the internet

If not sufficient randomness is used while generating n, results in weakness

They could succeed in finding private keys of 0.5% of TLS hosts and 0.03% SSH hosts because the modulii shared a non-trivial factor

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Ron was wrong, Whit is right. Arjen K. Lenstra and James P. Hughes and Maxime Augier and Joppe W. Bos and Thorsten Kleinjung and Christophe Wachter, http://eprint.iacr.org/2012/064

Thank You!