Non-linear

Part I

Sitabhra Sinha
Center for Complex Systems and Data Science
The Institute of Mathematical Sciences, Chennai

William Shakespeare's Romeo + Juliet (1996)

Romeo & Juliet

as Linear Dynamical System

$R(t)$: Romeo's love / hate for Juliet at time *t*

$J(t)$: Juliet's love / hate for Romeo at time *t*

In general,

$$\frac{dR}{dt} = a\,R + b\,J$$

$$\frac{dJ}{dt} = c\,R + d\,J$$

Parameters *a*, *b*, *c*, *d* can be +ve or –ve

a, *d* : measures of cautiousness

each try to avoid throwing themselves at the other

b, *c* : measures of responsiveness

each get excited by the other's advances

Different choices will yield different dynamics

Romeo & Juliet

The Eternal Cycle

$a = 0, b = 1, c = -1, d = 0$

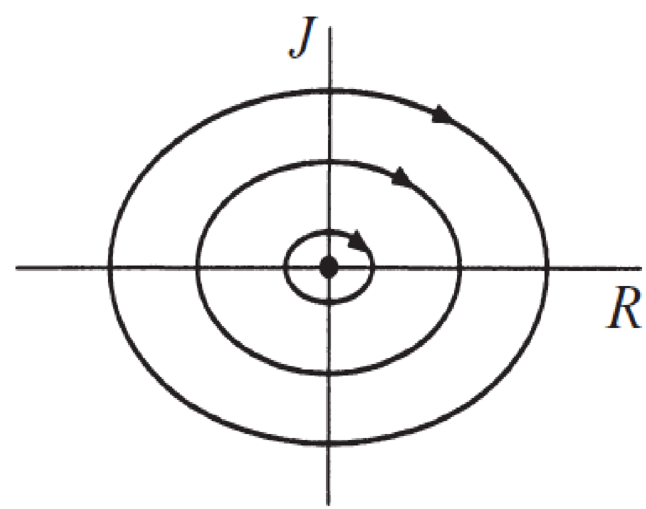
$$\frac{dR}{dt} = J$$

$$\frac{dJ}{dt} = -R$$

Juliet adopts hard-to-get strategy.
The more Romeo loves her, the more Juliet avoids him.
When Romeo gets discouraged and backs off, Juliet begins to attract him.

Romeo tries to coordinate with Juliet's actions
Romeo warms up when she loves him, and grows distant when she avoids him.

Neutrally stable equilibrium at (0,0) : a center



In general, the nature of behaviour around the equilibrium is given by the eigenvalues and eigenvectors of the Jacobian

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (\text{trace})\lambda + \text{determinant}$$

The eigenvalues are $\lambda = \dfrac{\text{trace} \pm \left[(\text{trace})^2 - 4\det\right]^{1/2}}{2}$.

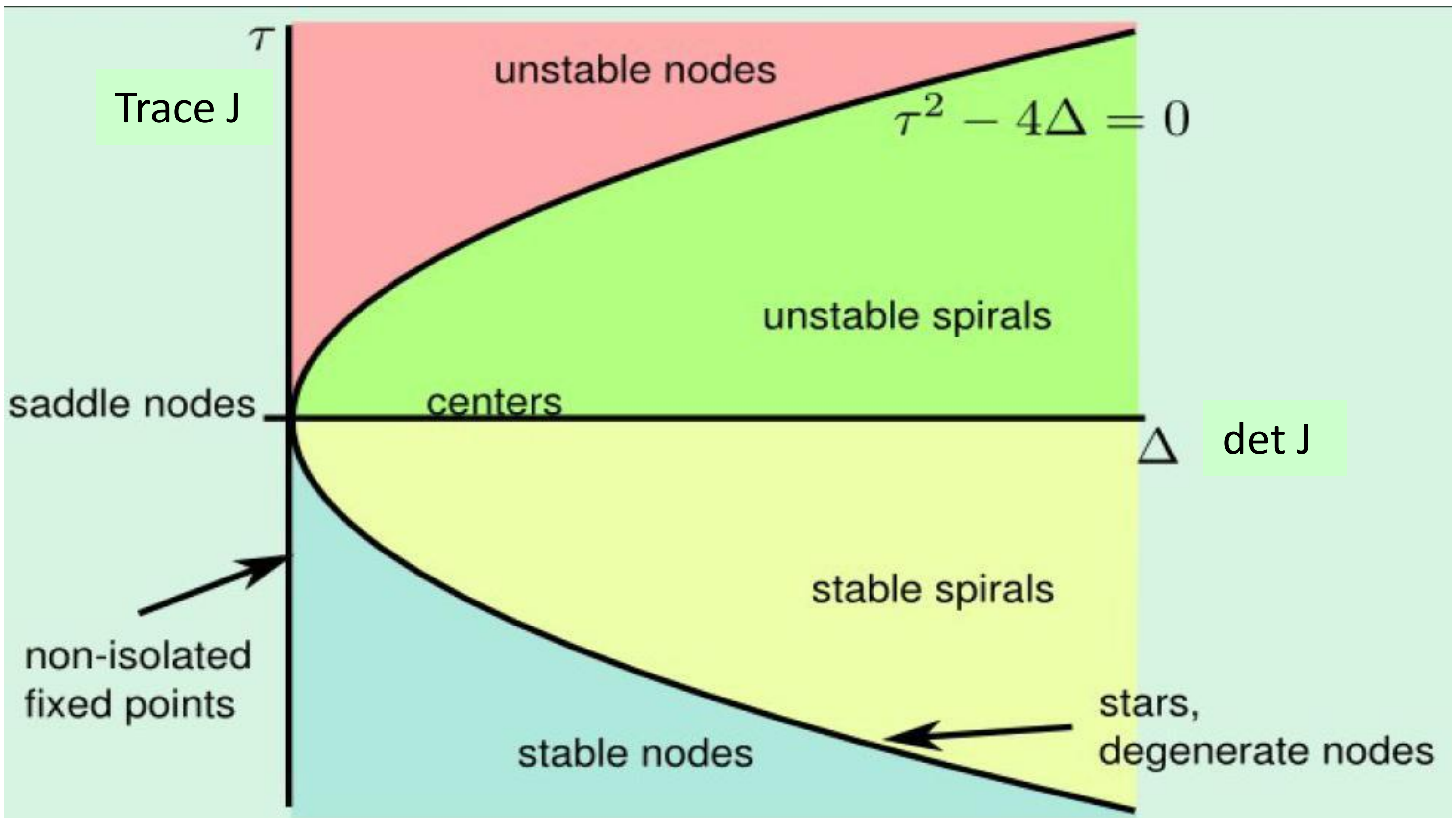


Image: Wikimedia Commons

Romeo & Juliet

Identically Cautious Lovers

$$a = d < 0, \qquad b = c > 0$$

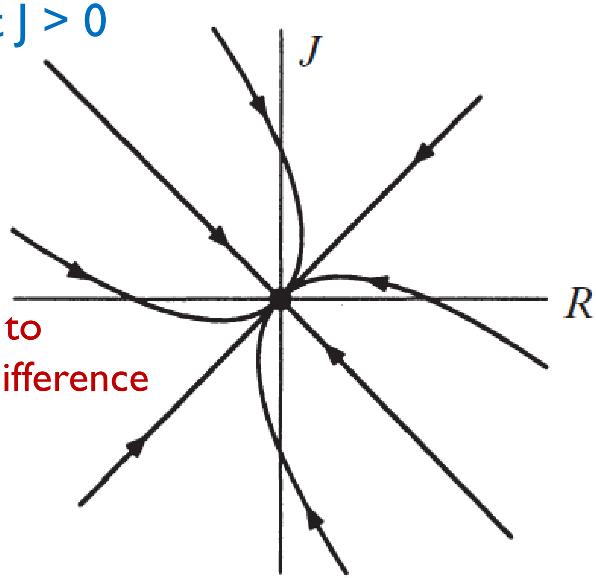
Romeo & Juliet

Identically Cautious Lovers

$$a = d < 0, \qquad b = c > 0$$

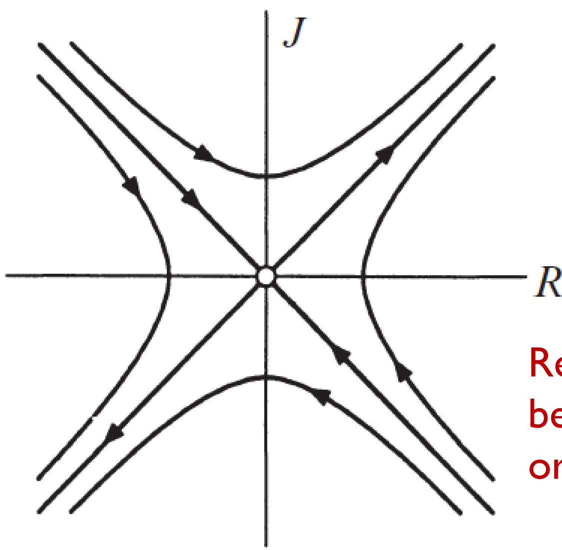
$\lambda_1 = a + b$, $\mathbf{v}_1 = (1, 1)$, $\lambda_2 = a - b$, $\mathbf{v}_2 = (1, -1)$ Tr J = 2a, det J = $(a + b)(a - b) = a^2 - b^2$

det J > 0



fizzles out to mutual indifference

det J < 0



Relationship either becomes a love fest or a war

a + b > a – b ⇒ eigenvector (1, 1) spans the slow eigendirection when equilibrium (0,0) is a stable node and spans the unstable manifold when equilibrium (0,0) is a saddle point

Romeo & Juliet

Identical “Eager Beavers” in Love

$a = d > 0,$ $b = c > 0$ encouraged by their own feelings (a > 0) and excited by their partner's love (b > 0).

$$\frac{dR}{dt} = aR + bJ$$

$$\frac{dR}{dt} = bR + aJ$$

Do it and find out

Tr J = ?, det J = ?

Romeo & Juliet

"Eager Beaver" vs "Cautious Lover"

$$a = -d > 0, b = c > 0$$

Romeo is encouraged by his own feelings ($a > 0$)
But Juliet doesn't want to throw herself at him ($d < 0$)
Both reciprocate their partner's love ($b, c > 0$).

$$\frac{dR}{dt} = aR + bJ$$

$$\frac{dR}{dt} = bR - aJ$$

Do it and find out

Tr J = ?, det J = ?

Laura and Petrarch : Love and Nonlinear Dynamics

Marie Spartali Stillman, The First Meeting of Petrarch and Laura, 1889

Introducing Nonlinearity

SIAM J. APPL. MATH.
Vol. 58, No. 4, pp. 1205–1221, August 1998

LAURA AND PETRARCH: AN INTRIGUING CASE OF CYCLICAL LOVE DYNAMICS*

SERGIO RINALDI†

$L\ (t)$: Laura's love for Petrarch at time t

$P\ (t)$: Petrarch's love for Laura at time t

$Z\ (t)$: Petrarch's poetic inspiration for writing the songs, sonnets & ballads collected in the *Canzoniere*

Introducing Nonlinearity

SIAM J. APPL. MATH.
Vol. 58, No. 4, pp. 1205–1221, August 1998

LAURA AND PETRARCH: AN INTRIGUING CASE OF CYCLICAL LOVE DYNAMICS*

SERGIO RINALDI†

$L(t)$: Laura's love for Petrarch at time t

$P(t)$: Petrarch's love for Laura at time t

$Z(t)$: Petrarch's poetic inspiration for writing the songs, sonnets & ballads collected in the *Canzoniere*

$$\frac{dL(t)}{dt} =$$
$$\frac{dP(t)}{dt} =$$
$$\frac{dZ(t)}{dt} =$$

?

Introducing Nonlinearity

SIAM J. APPL. MATH.
Vol. 58, No. 4, pp. 1205–1221, August 1998

LAURA AND PETRARCH: AN INTRIGUING CASE OF CYCLICAL LOVE DYNAMICS*

SERGIO RINALDI†

$L(t)$: Laura's love for Petrarch at time *t*

$P(t)$: Petrarch's love for Laura at time *t*

$Z(t)$: Petrarch's poetic inspiration for writing the songs, sonnets & ballads collected in the *Canzoniere*

$$\frac{dL(t)}{dt} = -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P,$$

$$\frac{dP(t)}{dt} = -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_L}{1+\delta Z(t)},$$

$$\frac{dZ(t)}{dt} = -\alpha_3 Z(t) + \beta_3 P(t),$$

Decay or forgetting ($-\alpha_1 L(t)$, $-\alpha_2 P(t)$, $-\alpha_3 Z(t)$)

Petrarch's appeal (A_P)

Laura's appeal (A_L)

Reaction function to Petrarch

$$R_L(P) = \beta_1 P \left(1 - \left(\frac{P}{\gamma}\right)^2\right)$$

Reaction function to Laura

$$R_P(L) = \beta_2 L$$

Inspiration is a function of Petrarch's love ($\beta_3 P(t)$)

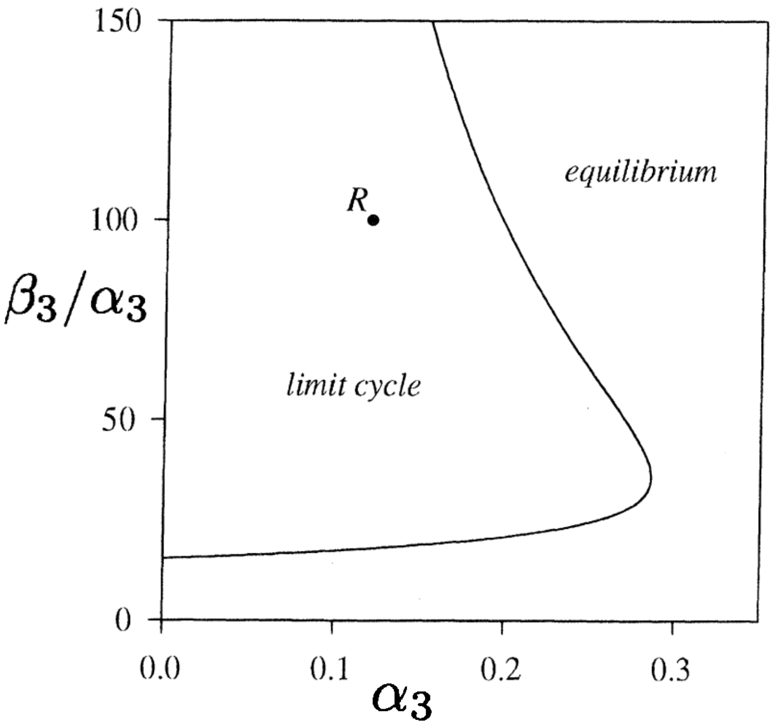
Cyclical Love Dynamics

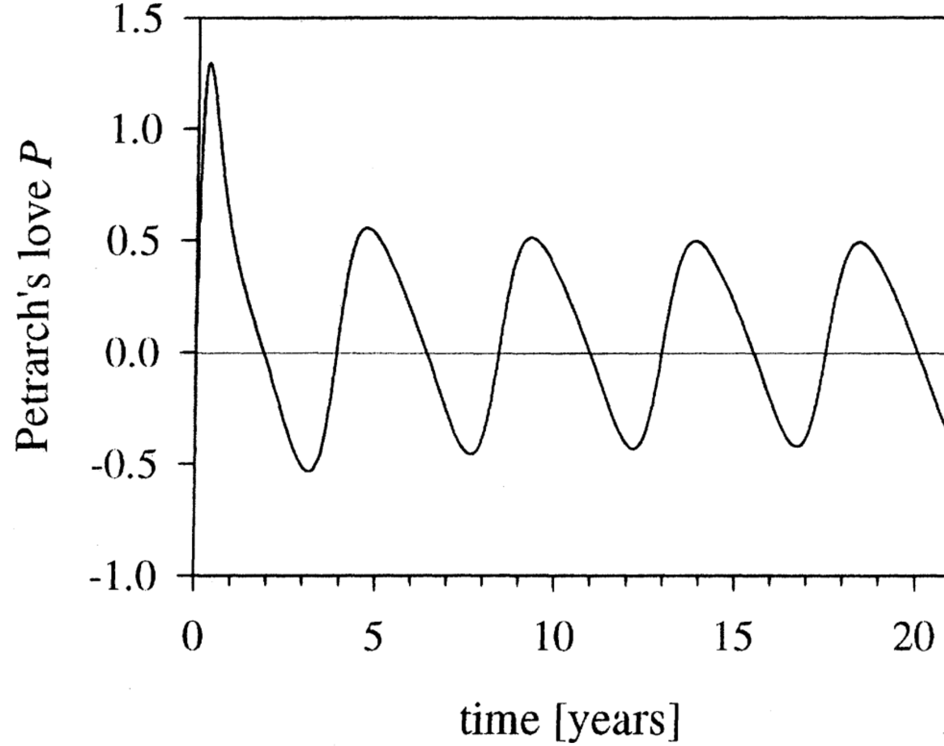
Parameters

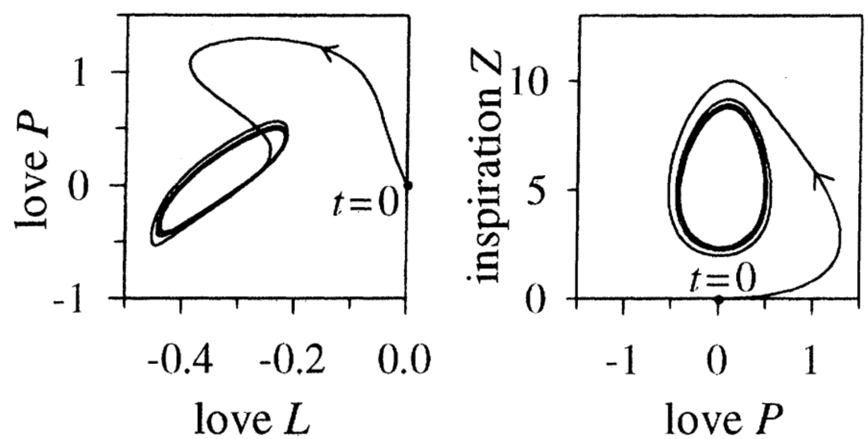
$$\alpha_2 = 3, \quad \alpha_2 = 1, \quad \alpha_3 = 0.1,$$
$$\beta_1 = 1, \quad \beta_2 = 5, \quad \beta_3 = 10,$$
$$\gamma = \delta = 1, \quad A_L = 2, \quad A_P = -1$$

Initial Conditions

$$L(0) = 0, \quad P(0) = 0, \quad Z(0) = 0.$$







Laura & Petrarch's love was unconsummated…

… but the reason evolution came up with this complex emotion is so that…

Organisms can breed & population increases