CROWDS IN MOTION:
HERDING, FLOCKING,
SWARMING

THE SHAPE OF FLOCKS TO COME:
MODELLING COLLECTIVE MOTION

Behaviour, Evolution & Emergence (BEE) 2026

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Video: The Fish Guy https://www.youtube.com/watch?v=DeBd6pmkWuU

“As if a cast of grain leapt back to the hand,
A landscapeful of small black birds, intent
On the far south, convene at some command
At once in the middle of the air, at once are gone
With headlong and unanimous consent
From the pale trees and fields they settled on.”

— Richard Wilbur, *An Event* (1956)

“If you want to be a different fish, you've got to
jump out of the school”

— Don Van Vliet (1978)

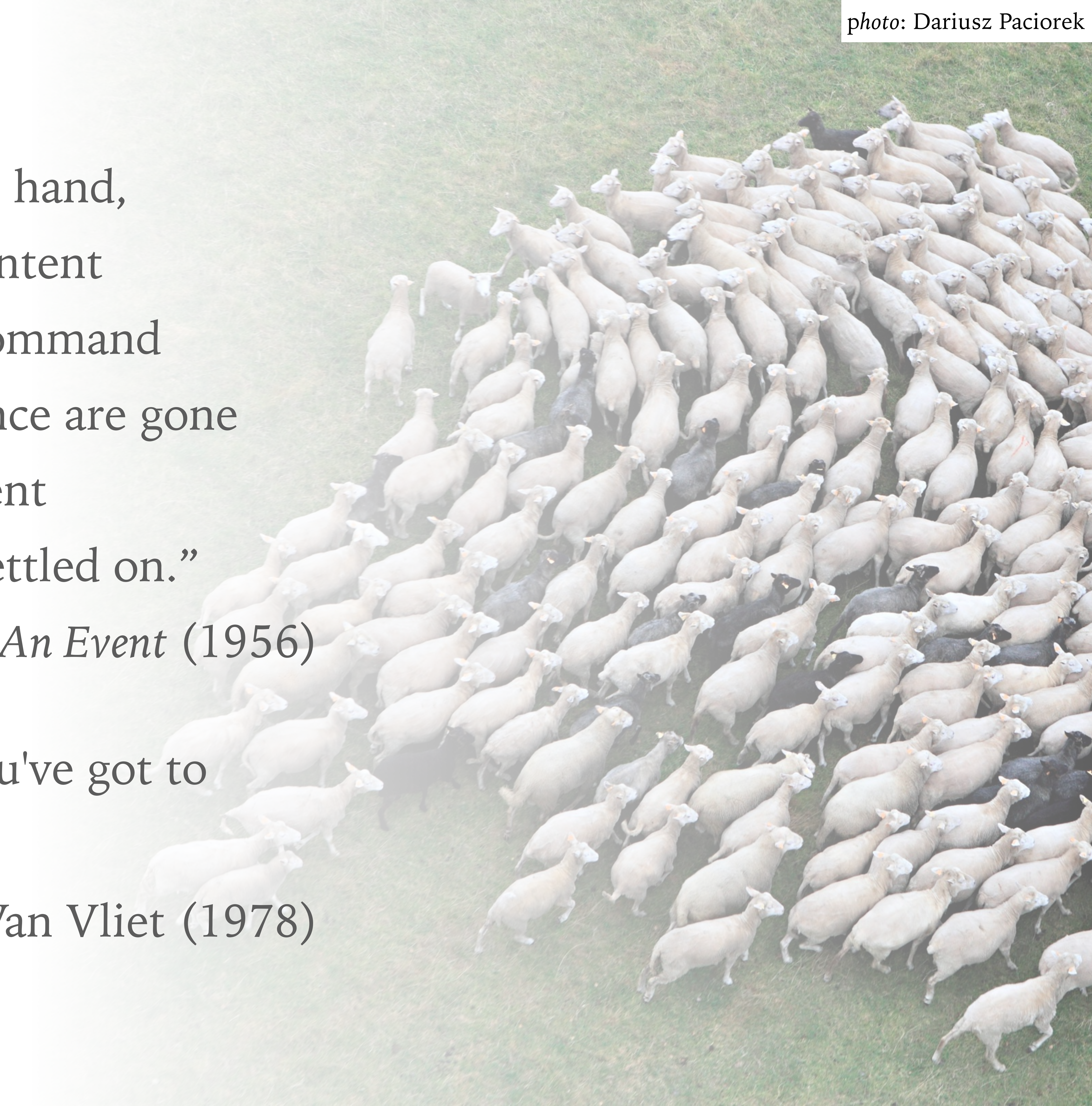
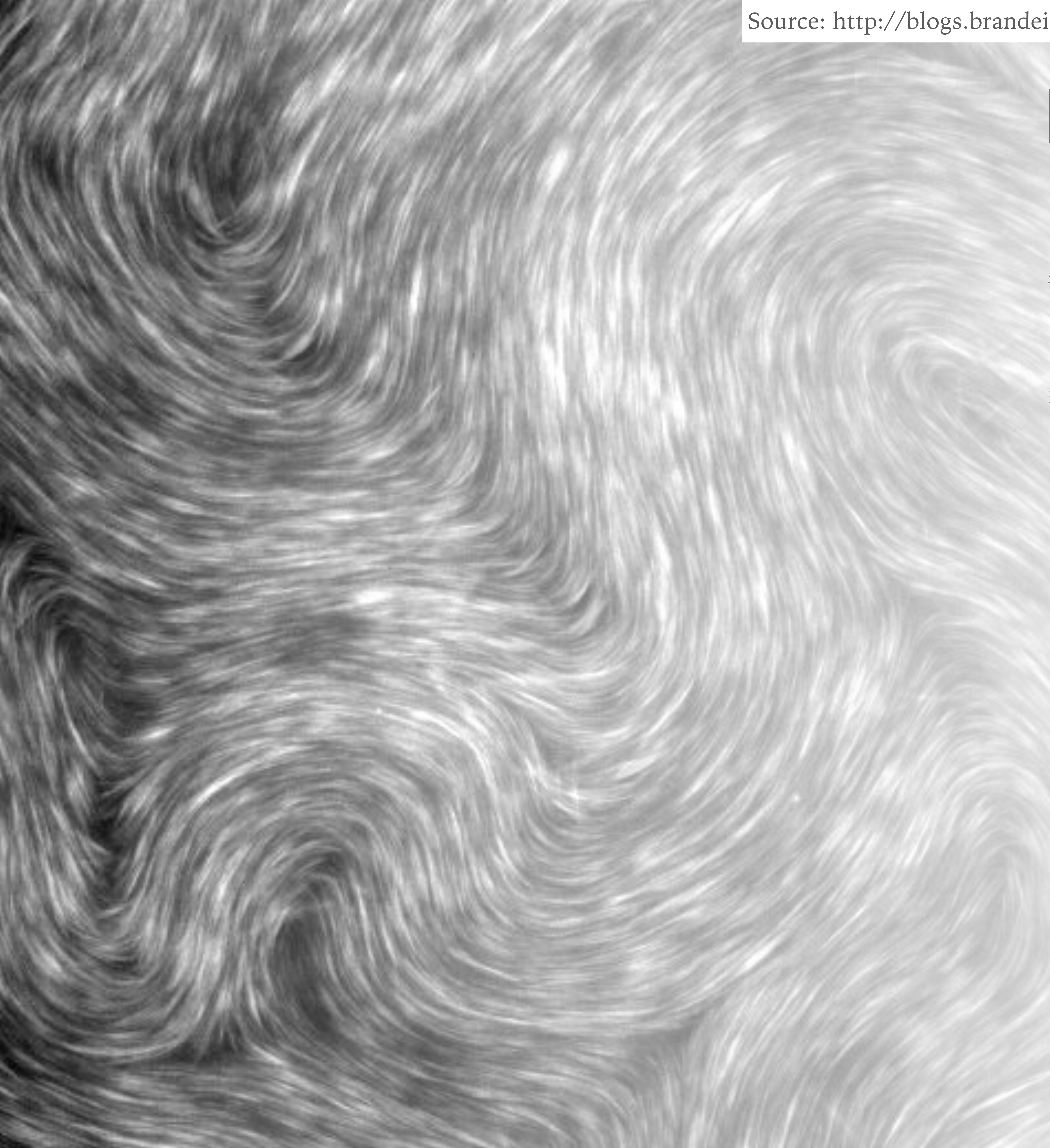


photo: Dariusz Paciorek



Source: http://blogs.brandeis.edu/science/2016/01/06/nature-news-feature-highlights-dogic-lab-active-matter-research/

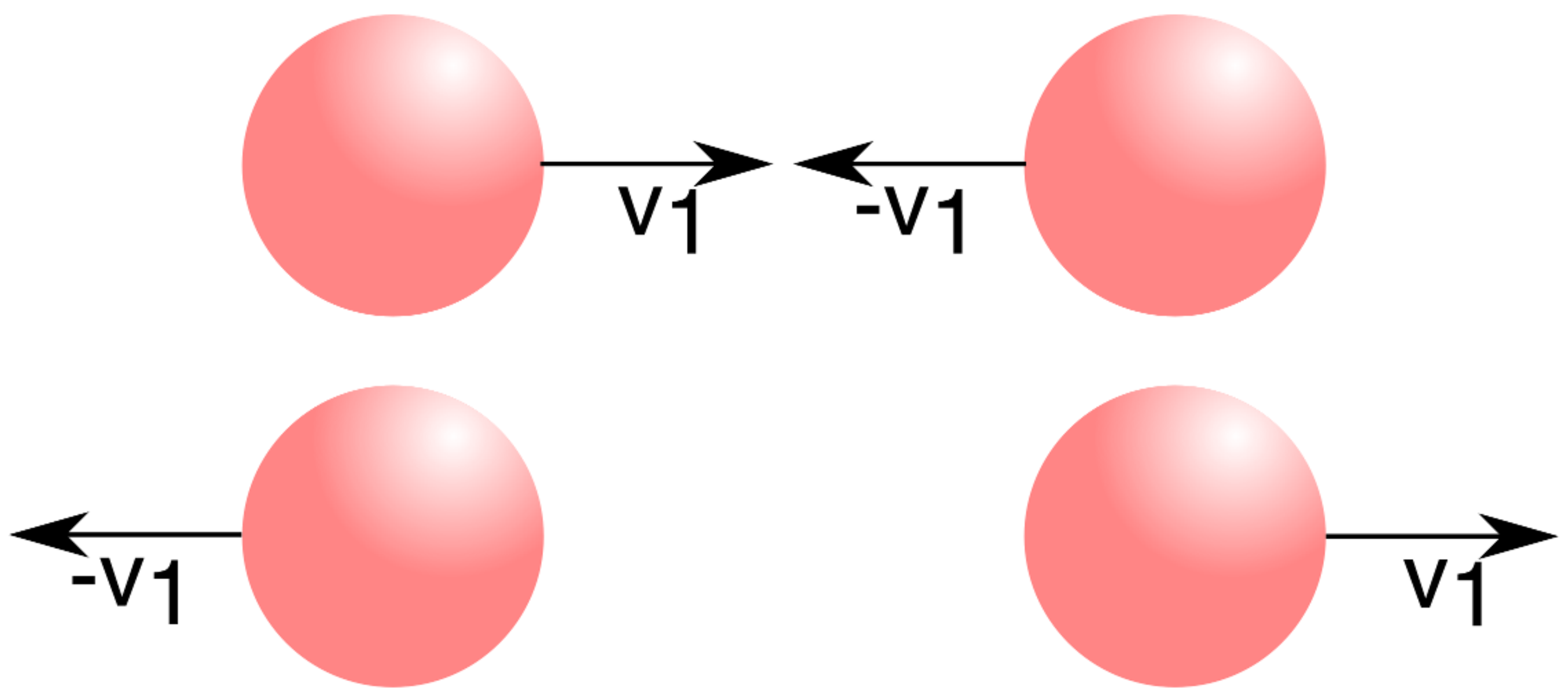
FLOCKS AS “DRY” ACTIVE MATTER

- Flocks can be viewed as a subset of a class of nonequilibrium condensed systems known as *active matter*.
- Active matter constitutes units/particles that are assumed to be “self-propelled”, i.e. they utilise stored or ambient free energy for movement, and classified as:
  - **Dry**: Particles move through or on an inert medium that acts as a momentum sink and only provides local friction (e.g. flocks).
  - **Wet**: Particles are embedded in a fluid medium where momentum is conserved globally. Viscosity mediates long-range hydrodynamic interactions (e.g. bacterial/colloid suspensions).
  - **Damp (Varuni et al, 2022)**: Particles alter their substrate as they move (e.g. by secreting a slime trail). This introduced an additional indirect short-range time-delayed interaction (e.g. cyanobacteria colonies).

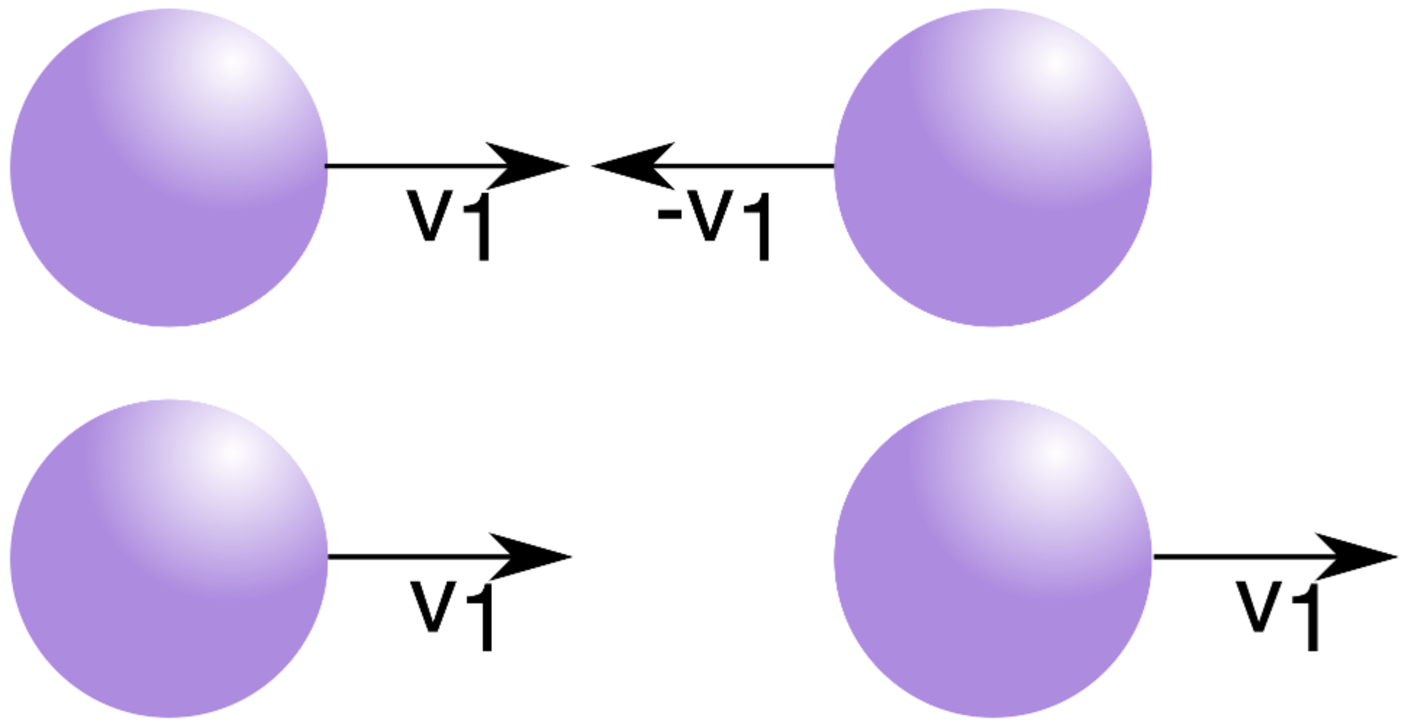
SELF-PROPELLED PARTICLES (SPP)

- Unlike in equilibrium systems, momentum is **not conserved** in **dry** active matter systems of self-propelled particles due to continuous energy exchange with the environment (Fily, 2017).
- SPPs break *Galilean invariance* because their dynamics are tethered to the environment in which they're embedded.
- The motion of SPPs may have a stochastic component if:
    - they are in a complex environment, or
    - there may be complexities associated with internal processes and imperfect execution of self-propulsion.
- Alignment interactions are often ferromagnetic and tend to promote *polar order*, i.e. particles are aligned head to head and tail to tail.
- Motility-induced phase separation can occur in SPPs with/without alignment interactions.

CONSERVATION OF MOMENTUM



SELF-PROPELLED PARTICLES



C Bechinger et al, Rev. Mod. Phys. **88**, 045006 (2016)

ACTIVE BROWNIAN PARTICLES

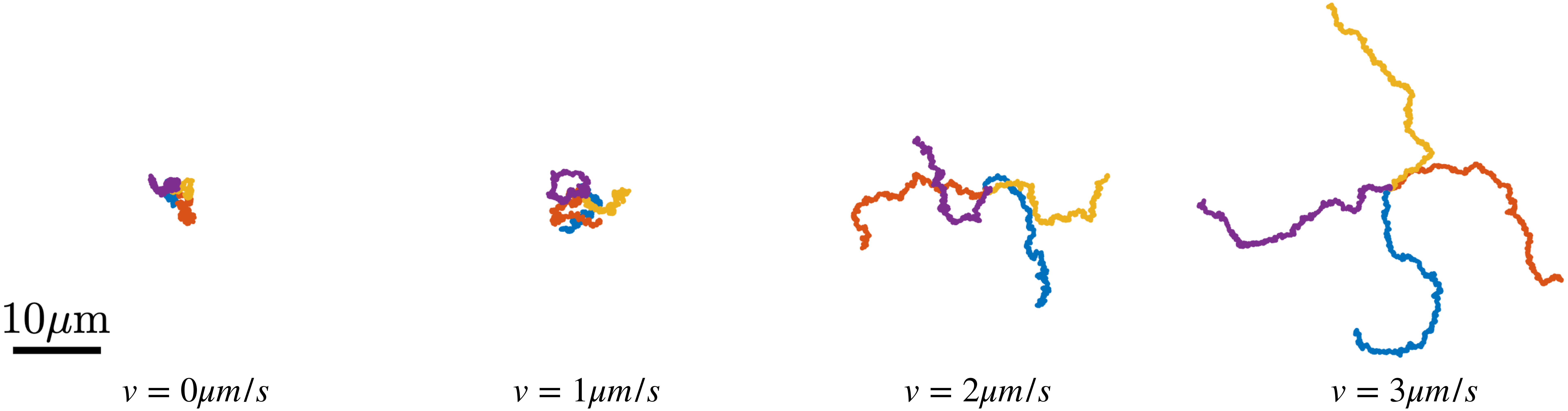
Passive Brownian particles are purely diffusive and have independent translational and rotational motions:

$$\dot{x} = \sqrt{2D_T}\,\xi_x\,,\ \dot{y} = \sqrt{2D_T}\,\xi_y\,,\ \dot{\varphi} = \sqrt{2D_R}\,\xi_\varphi$$

Where $\xi_x(t)$, $\xi_x(t)$ & $\xi_x(t)$ are independent white noise stochastic processes.

Active Brownian particles are propelled with velocity v, and so the direction of motion is subject to rotational diffusion:

$$\dot{x} = v\cos\varphi + \sqrt{2D_T}\,\xi_x\,,$$
$$\dot{y} = v\sin\varphi + \sqrt{2D_T}\,\xi_y\,,\ \dot{\varphi} = \sqrt{2D_R}\,\xi_\varphi$$



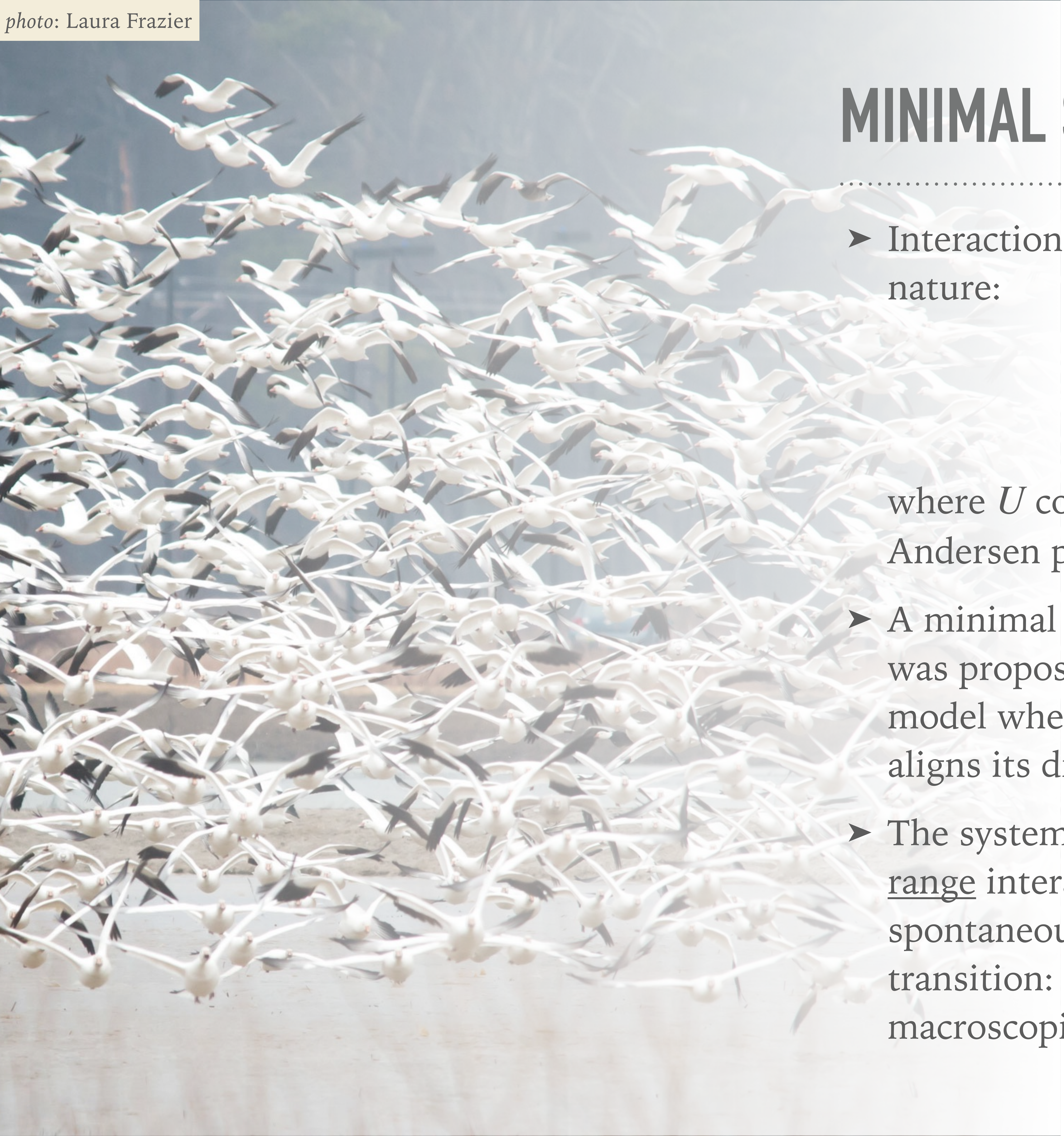


photo: Laura Frazier

MINIMAL SPP MODEL WITH ALIGNMENT

- Interactions between ABPs are typically assumed to be steric in nature:

$$\dot{\mathbf{r}}_i = v\mathbf{n}_i - \sum_{j\neq i} \frac{dU(r_{ij})}{dr_{ij}}\hat{\mathbf{r}}_{ij} + \sqrt{2D_T}\,\xi_{\mathbf{r}_i}(t),$$

  where U could, for example, be a Weeks–Chandler–Andersen potential.

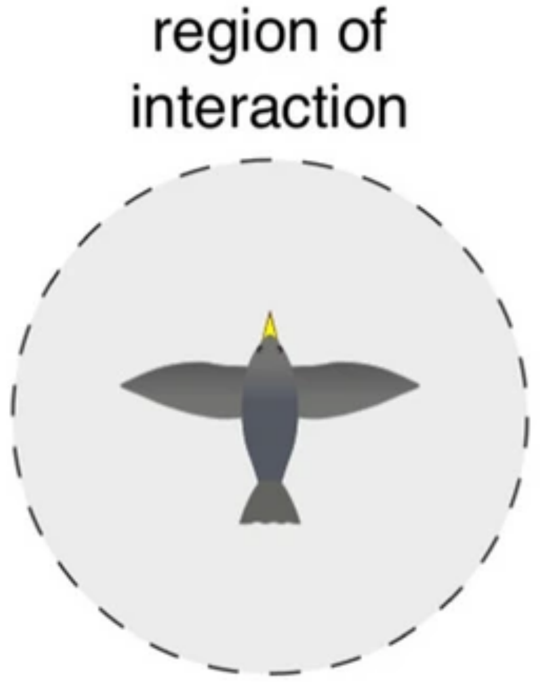
- A minimal model for SPPs that exhibit alignment interactions was proposed by Vicsek et al (1995). This is a discrete time model where, at each step, an SPP surveys its neighbourhood and aligns its direction with that of the average direction of motion.
- The system develops long-range order, even with only short-range interaction. Thus, the rotational symmetry is spontaneously broken, giving rise to a non-equilibrium phase transition: from an isotropic gas to a polar flock with macroscopic drift.

T. Viscek et al, *Phys. Rev. Lett.* **75**, 1226 (1995).

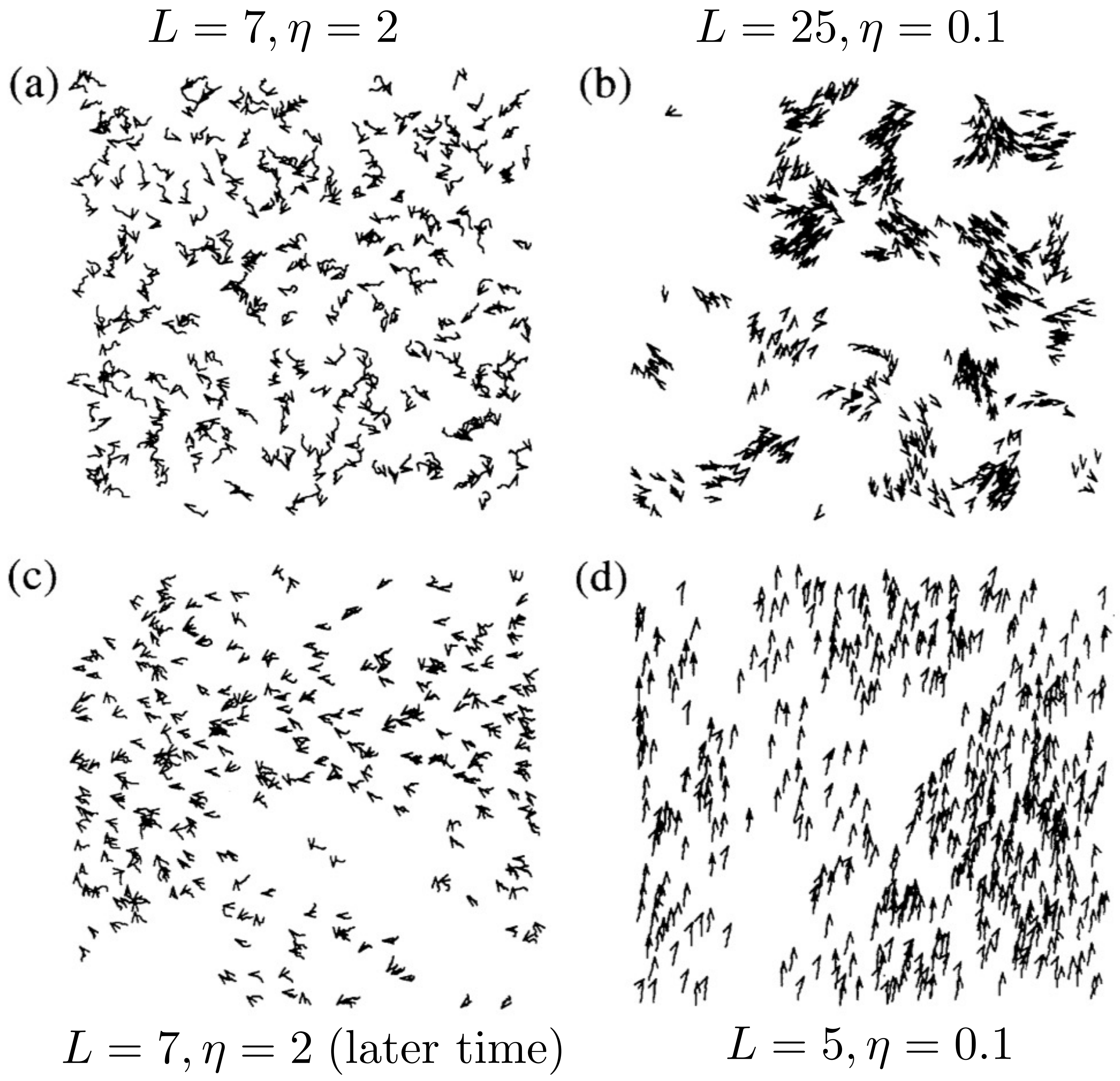
THE MODEL BY VICSEK ET AL (1995)

- The system comprises N particles that are propelled with a time-invariant absolute velocity v.
- Particle i moves in a direction $\theta_i(t)$, at time t.
- At each time step the position $\mathbf{x}_i$ of each particle i is updated via: $\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t$
- Their directions are also updated via:

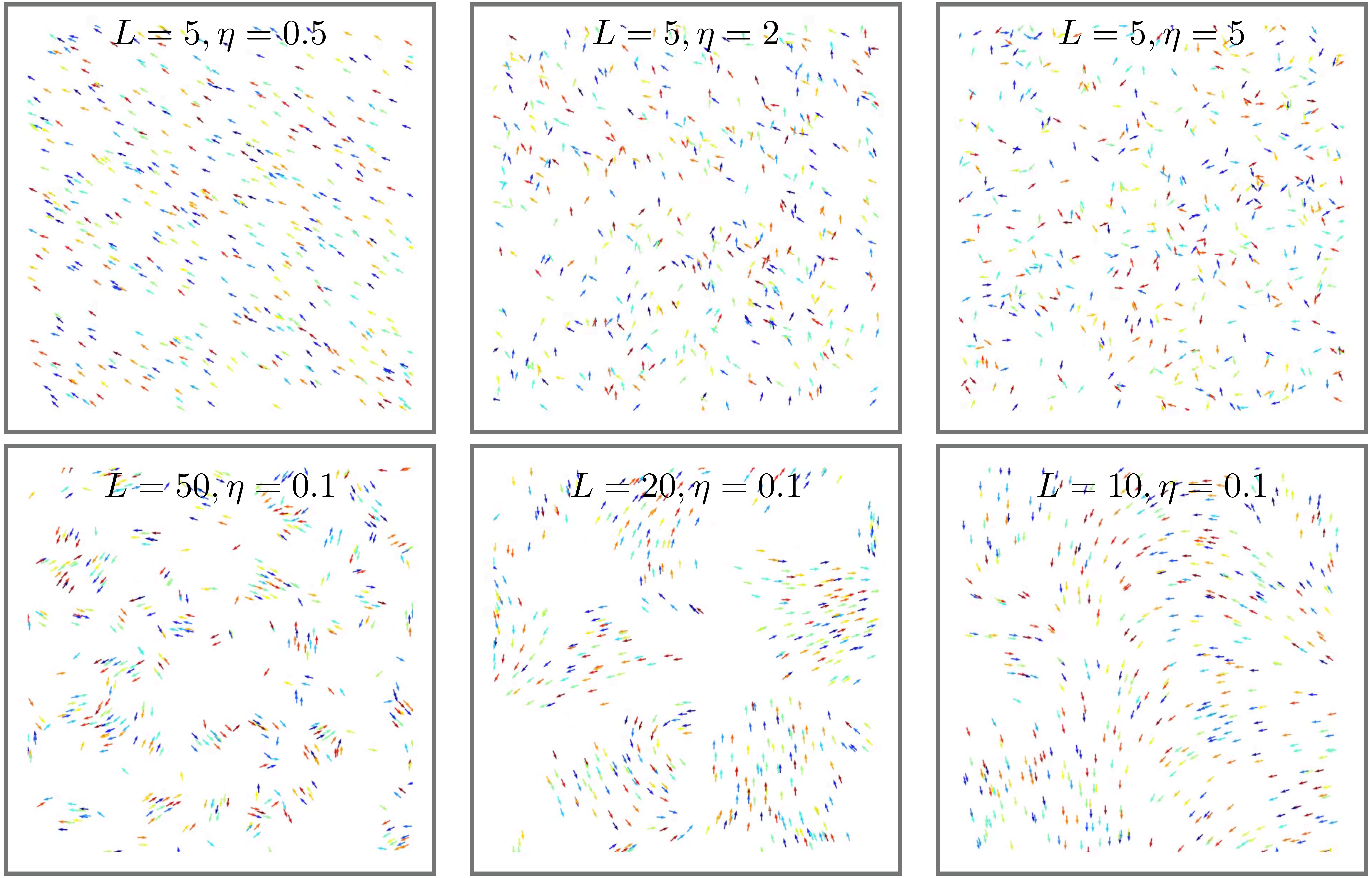
$$\theta_j(t+1) = \arctan\left(\frac{\langle \sin\theta(t)\rangle_r}{\langle \cos\theta(t)\rangle_r}\right) + \Delta\theta$$



- Where $\Delta\theta \in \left[-\eta/2, \eta/2\right]$, r is the radius of the neighbourhood and the noise level is η.
- Here we have “angular noise”, where the error is made when trying to move in a perfectly calculated direction.



Simulations of the model for N particles in a box of size L with periodic boundary conditions and noise strength η.



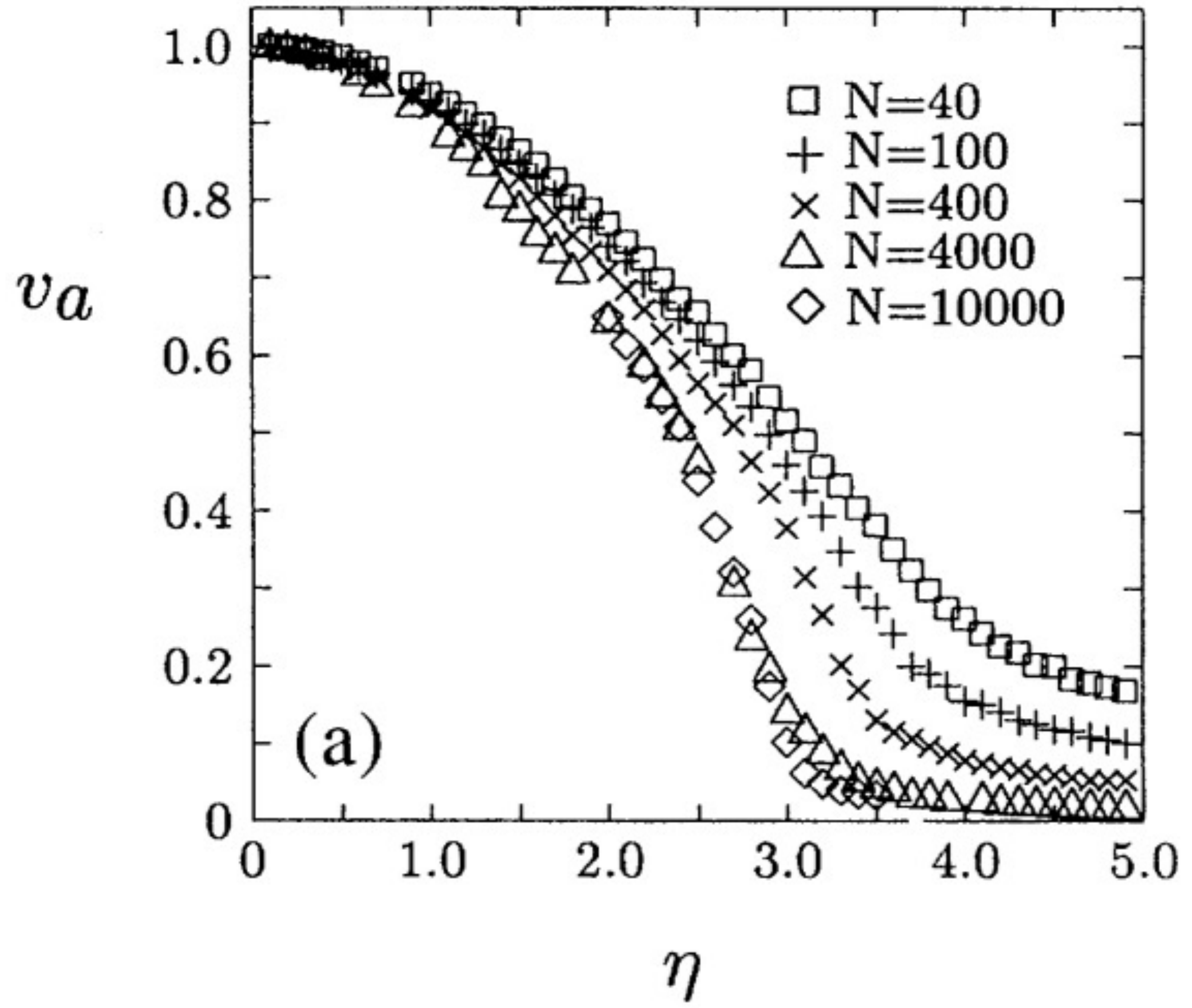
T. Viscek et al, *Phys. Rev. Lett.* **75**, 1226 (1995).

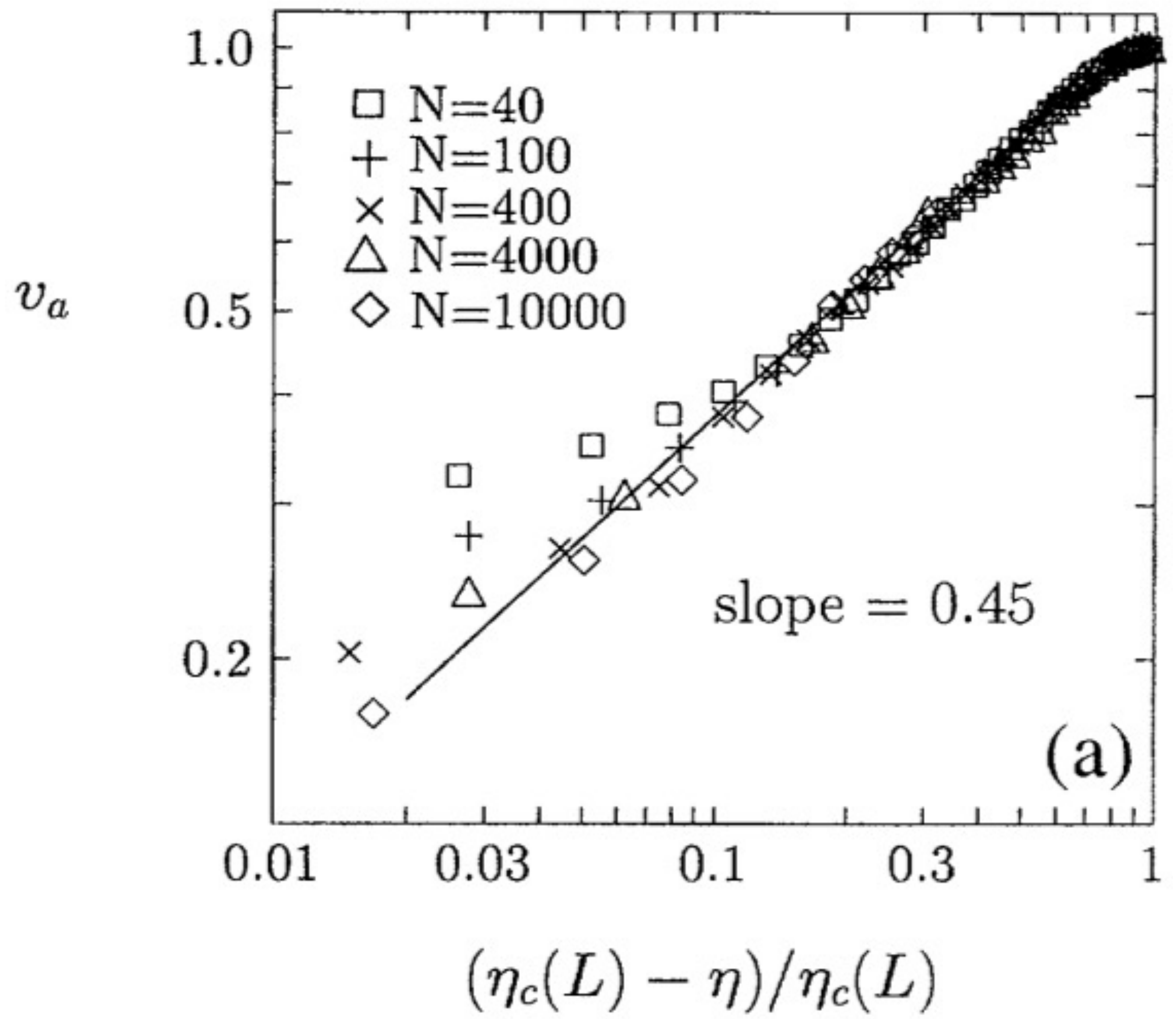
PHASE TRANSITION IN THE VICSEK MODEL

- For large density and small noise, the motion is ordered at the macroscopic scale.
- On increasing noise, the system undergoes a phase transition from an ordered to a disordered state.
- To characterize this, the average normalised velocity is considered:

$$v_a = \frac{1}{Nv}\left|\sum_{i=1}^{N}\mathbf{v}_i\right|$$

- In the fully disordered case $v_a = 0$ and in the fully ordered case $v_a = 1$.
- They find that the average velocity scales as $v_\alpha \sim \left[\eta_c(\rho) - \eta\right]^\beta$ where $\beta \approx 0.5$.
- While they suggested that phase transition is second order, it was later shown to be first order (Solon, 2015).
- The Vicsek model is a non-equilibrium dynamical analogue of the ferromagnetic type of models. Here, in place of spin alignment, there is direction alignment, and perturbations play the role of temperature. However, there is a crucial difference…





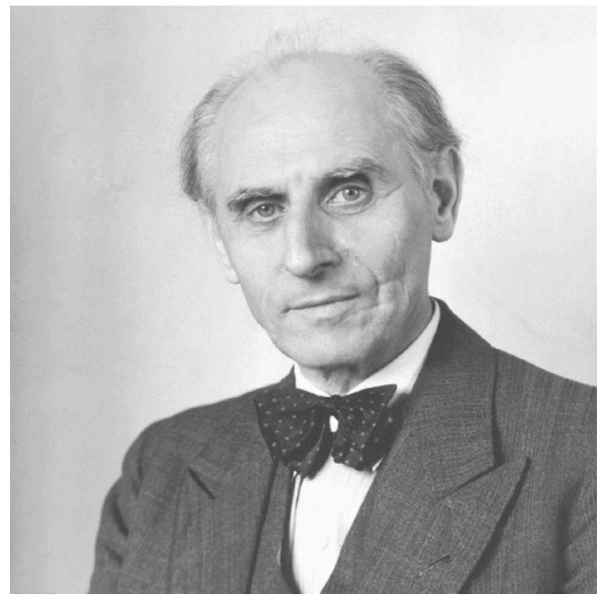
THE XY MODEL

- This model was first studied by Yôichirô Nambu in 1950, but was given its name in a later work by Lieb et al (1961).
- The model describes the energy configuration of a system of spins that can point in any direction over the range $(0,2\pi]$. The spins are located at every site of a given graph (e.g. a d-dimensional lattice).
- The spins at each site i are represented by the unit vector $\mathbf{s}_i = \left(\cos\theta_i, \sin\theta_i\right)$. If a pair of neighbouring spins i and j interact with strength J_{ij} , then the energy of the configuration of spins on the graph is given by:

$$\mathscr{H} = -\sum_{\langle i,j\rangle} J_{ij}\,\mathbf{s}_i \cdot \mathbf{s}_j = -\sum_{\langle i,j\rangle} J_{ij} \cos(\theta_i - \theta_j)$$

- This system does not show long range order at finite temperatures due to the Mermin-Wagner theorem.
- In the Ising model, where $\mathscr{H} = -\sum_{\langle i,j\rangle} J_{ij}\, s_i\, s_j - \mu \sum_j h_j\, s_j$, the cost of alignment of a pair of spins (i, j) is $\Delta E = 2J_{ij}$ whereas in the XY model you can have infinitesimally small differences between neighbouring spins. So it's very easy to create long-range disorder at non-zero temperatures.


Ernst Ising


Wilhelm Lenz


Yôichirô Nambu

WHAT'S MISSING?

- Here particles have access to information regarding the states of all neighbours in a surrounding radius. In reality there may be “blind zones”.
- Particles are assumed to be able to accurately compute the instantaneous average direction of all neighbours, and their decisions are subject to some uniform “implementation error”. In reality individuals are typically unable to make such computations.
- A Langevin dynamics approach, where the deterministic and fluctuating components of motion are assumed to be separable, is not necessarily valid in flocking.
- While some models have incorporated cohesion explicitly (Grégoire & Chaté, 2004) might cohesion be achieved with velocity alignments alone?

photo: Arni Einarsson

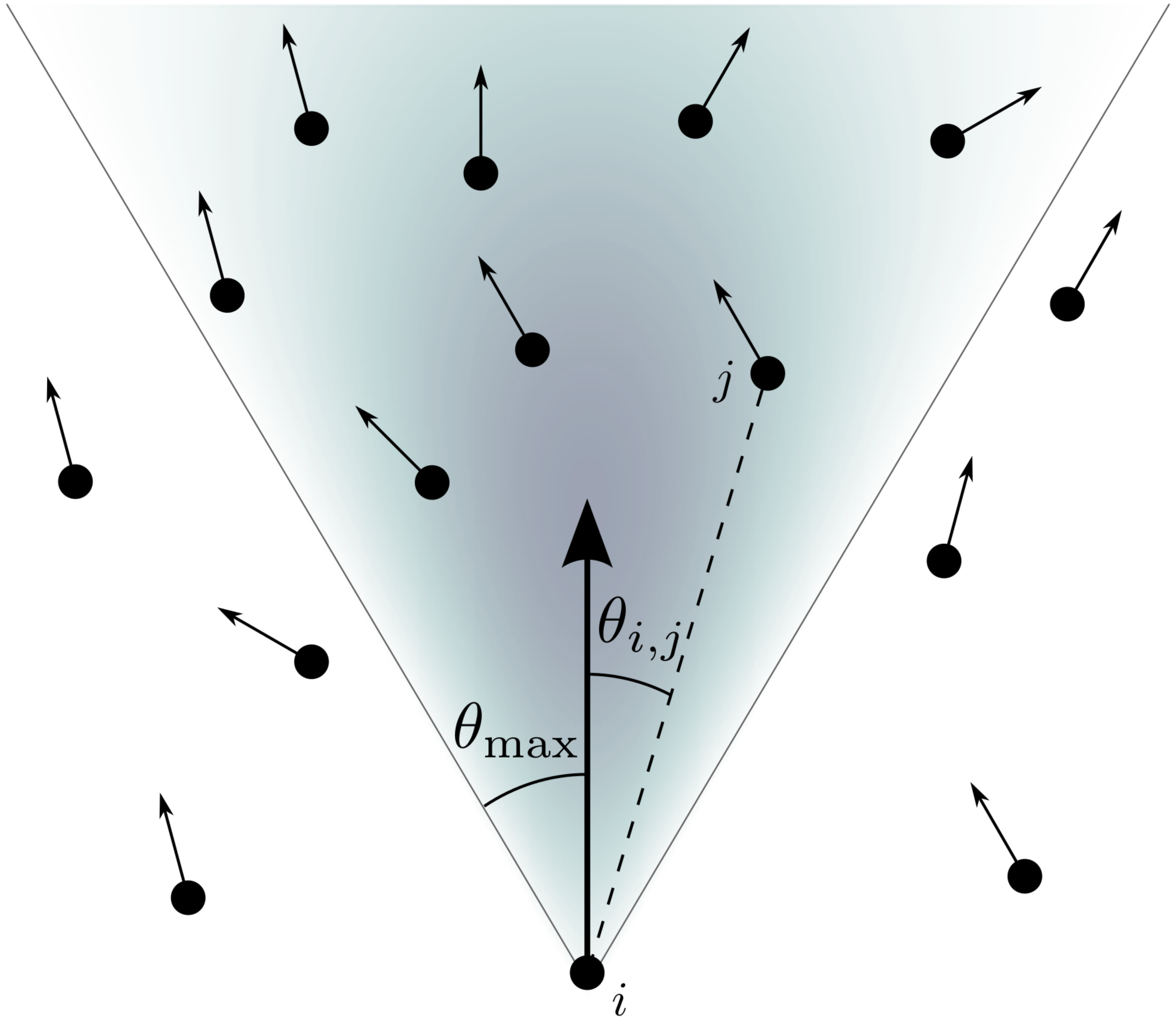
T. Bagarti & S. N. Menon, Phys. Rev. E 100, 012609 (2019).

FLOCKING OF STOCHASTICALLY INTERACTING AGENTS WITH A FIELD OF VIEW

- To address all of these issues, we developed a model (Bagarti & Menon, 2019) that described the flocking dynamics of SPPs that exhibit nonlinear velocity-alignment interactions with neighbors within their “field of view”. Furthermore, the stochasticity is heterogeneous and linked to dynamics of individual agents.
- Each agent *i* has a field of view that is delimited by a maximum bearing angle $\theta_{\max}$.
- The agent selects a randomly chosen neighbour *j* that lies within its field of view.
- The probability of selecting agent *j* is related to the weight function

$$\omega_{i,j} = |\mathbf{x}_i - \mathbf{x}_j|\, e^{-\frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{2\sigma^2}} \left(1 - \theta_{i,j}^2/\theta_{\max}^2\right) ,$$

- Here $\mathbf{x}_i$ and $\mathbf{x}_j$ are the positions of *i* and *j*, while σ is the mean interaction length and $\theta_{i,j}$ is the angle between the velocity $\mathbf{v}_i$ of agent *i* and the vector $\mathbf{x}_j - \mathbf{x}_i$.



T. Bagarti & S. N. Menon, Phys. Rev. E 100, 012609 (2019).

DETAILS OF THE MODEL

- We assume that the velocity and position of each particle are updated using the following rules:

$$\mathbf{v}_t(t+1) = \mathbf{v}_i(t) + \mathbf{a}_i(t)$$

$$\mathbf{x}_t(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1)$$

- Here, the agent's acceleration is given by

$$\mathbf{a}_t(t) = \begin{cases} -\mathbf{v}_i(t) + |\,\mathbf{v}_i(t)\,|\,\hat{\eta} & \text{if } \Omega_i = \emptyset, \\ \alpha[\mathbf{v}_j(t) - \mathbf{v}_i(t) + f(\mathbf{v}_j(t) + \mathbf{v}_i(t))] & \text{otherwise,} \end{cases}$$

where Ω_i is the set of all agents with which agent i may interact with, the coefficient $\alpha(< 1)$ represents the interaction strength and $\hat{\eta}$ is chosen from a uniform random distribution of vectors on the unit circle.

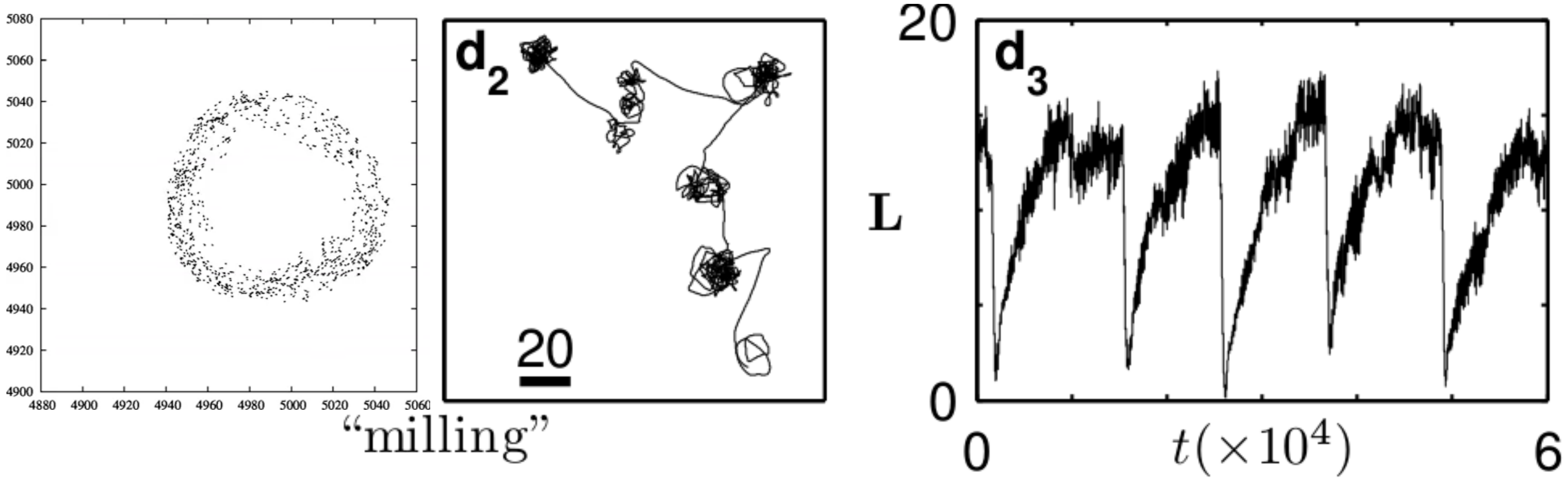
- The linear term $\alpha[\mathbf{v}_j(t) - \mathbf{v}_i(t)]$ describes an alignment interaction, while the nonlinear term $\alpha f(\,.\,)$ keeps the speed in the vicinity of a critical value.
- If the agent has no neighbours in its field of view, it performs a *random rotation*.

T. Bagarti & S. N. Menon, Phys. Rev. E 100, 012609 (2019).

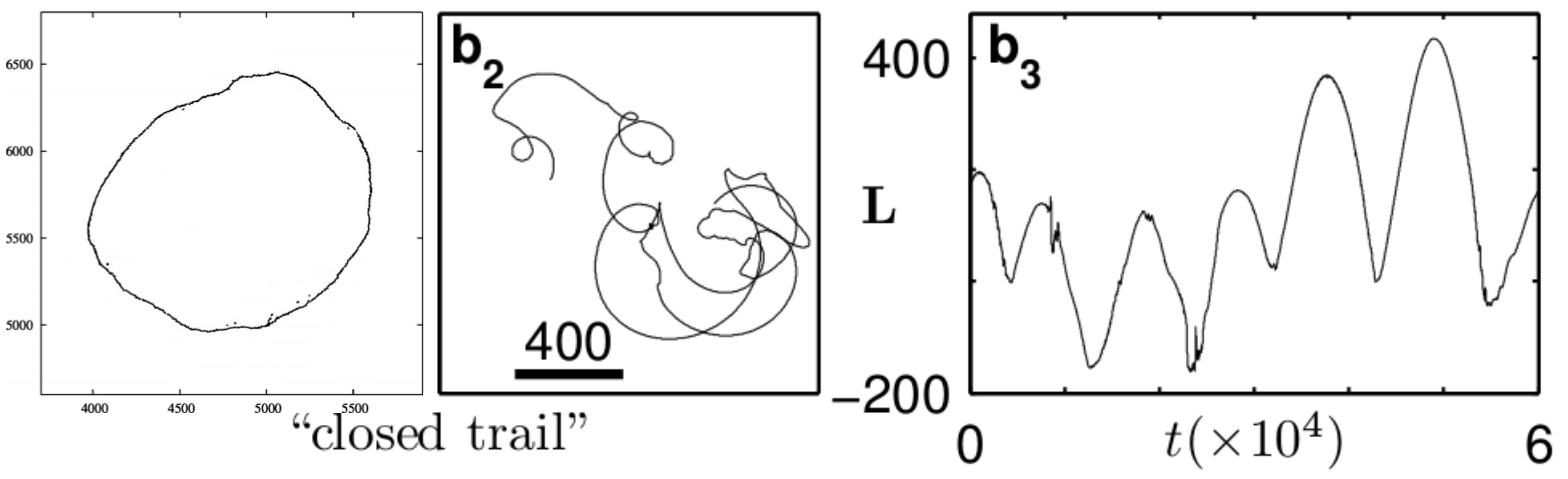
FLOCKING OF STOCHASTICALLY INTERACTING AGENTS WITH A FIELD OF VIEW

We vary the interaction strength α, mean interaction length σ, and the maximum bearing angle $\theta_{\max}$ over a range of values and observe a wide range of patterns, some of which exhibit spontaneous reversals in rotation.

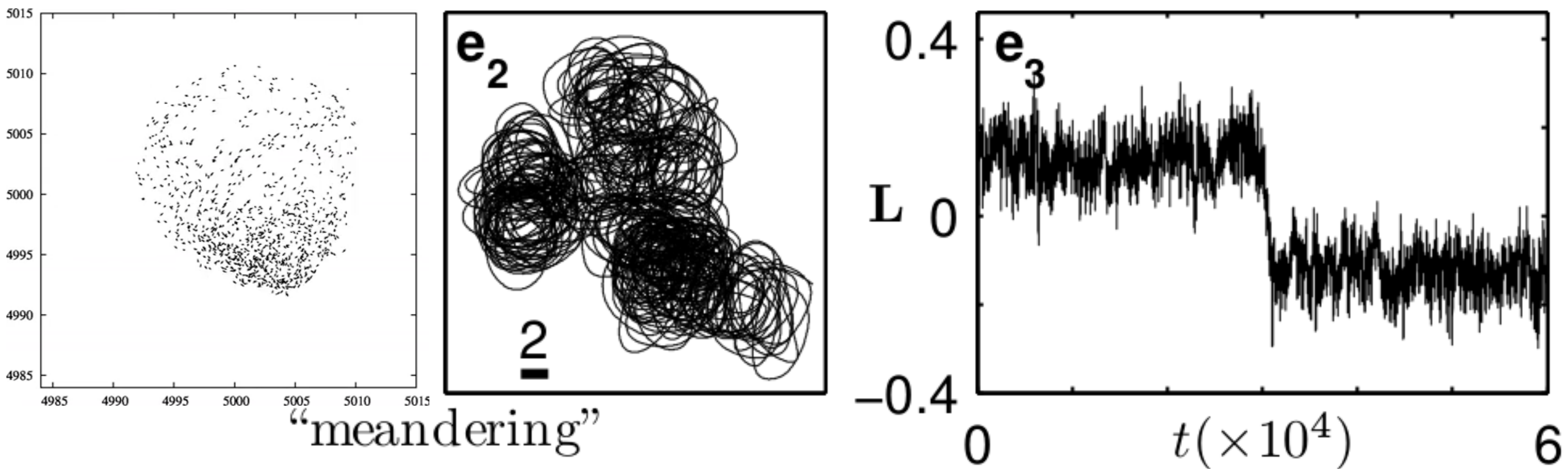
Most notably, flocks exhibit milling behaviour with overall *run-and-tumble* like characteristics.



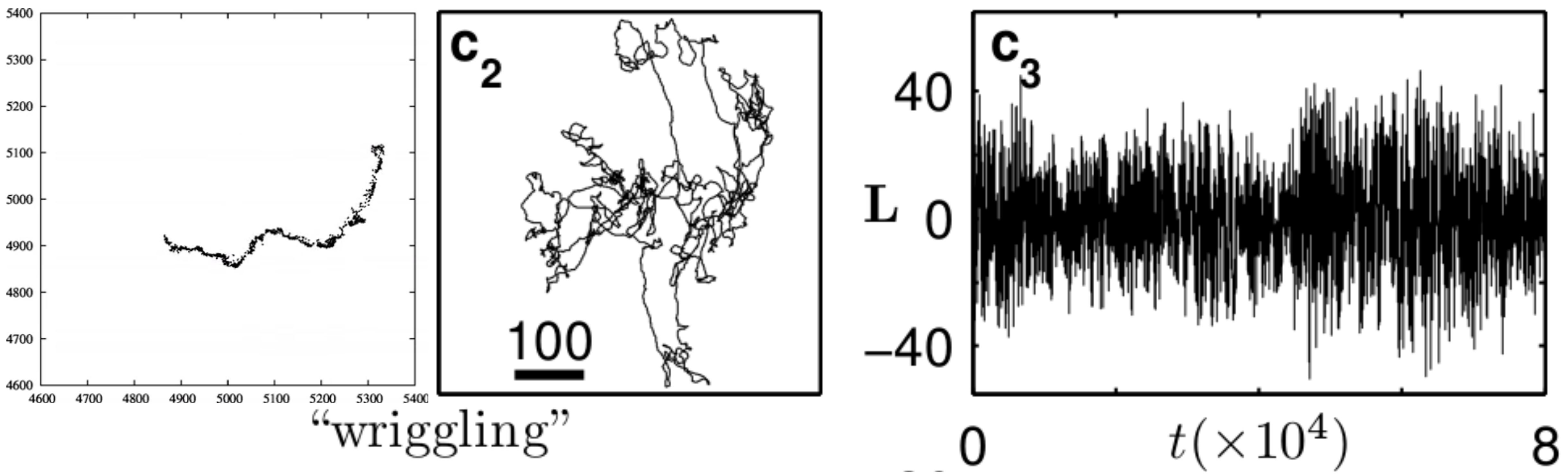
“milling”



“closed trail”



“meandering”



“wriggling”

J Toner & Y Tu, Phys Rev Lett, **75**(23), 4326 (1995).

NAVIER–STOKES EQUATIONS FOR FLOCKS!

- In the same year as the Vicsek model was published, Toner and Tu (1995) proposed a field theory for flocks based on symmetry arguments and conservation laws.
- The isotropic version of their model is:

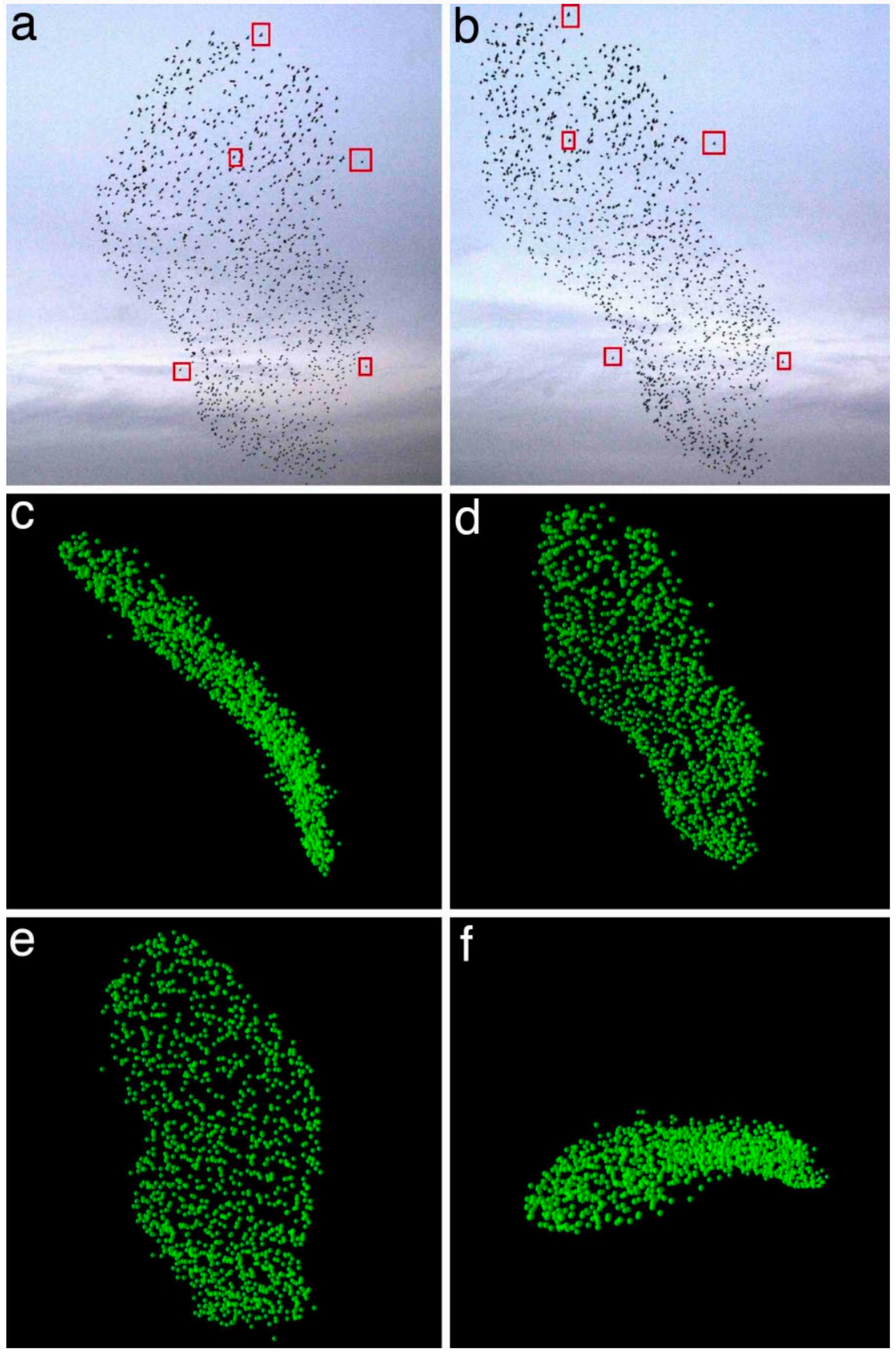
$$\partial_t \mathbf{v} + \underbrace{\lambda_1(\mathbf{v}\cdot\nabla)\mathbf{v} + \lambda_2(\nabla\cdot\mathbf{v})\mathbf{v} + \lambda_3\nabla(|\mathbf{v}|^2)}_{\text{active advection}} = \underbrace{\alpha\mathbf{v} - \beta|\mathbf{v}|^2\mathbf{v}}_{\text{propulsion}} - \nabla P + \underbrace{D_B\nabla(\nabla\cdot\mathbf{v}) + D_T\nabla^2\mathbf{v} + D_2(\mathbf{v}\cdot\nabla)^2\mathbf{v}}_{\text{viscous damping}} + \mathbf{f}$$

$$\partial_t\rho + \nabla\cdot(\mathbf{v}\rho) = 0$$

- Here $\rho(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ are the coarse-grained *density* and *velocity* fields, P is the pressure, and $\mathbf{f}$ is an additive noise term.
- The additional propulsion terms are introduced because momentum is not conserved and Galilean invariance is broken.
- The active advection terms describe how transport is affected by local variations in the field.
- The damping terms relate to transverse, bulk and anisotropic viscosities, and account for the resistance to misalignment.
- Their main result was to show that this “dynamical XY model” does not follow the Mermin-Wagner theorem, and in fact **long-range order** can arise at non-zero temperatures.

M Ballerini et al, PNAS **105**(4): 1232-1237 (2008).

ARE INTERACTIONS METRIC OR TOPOLOGICAL?



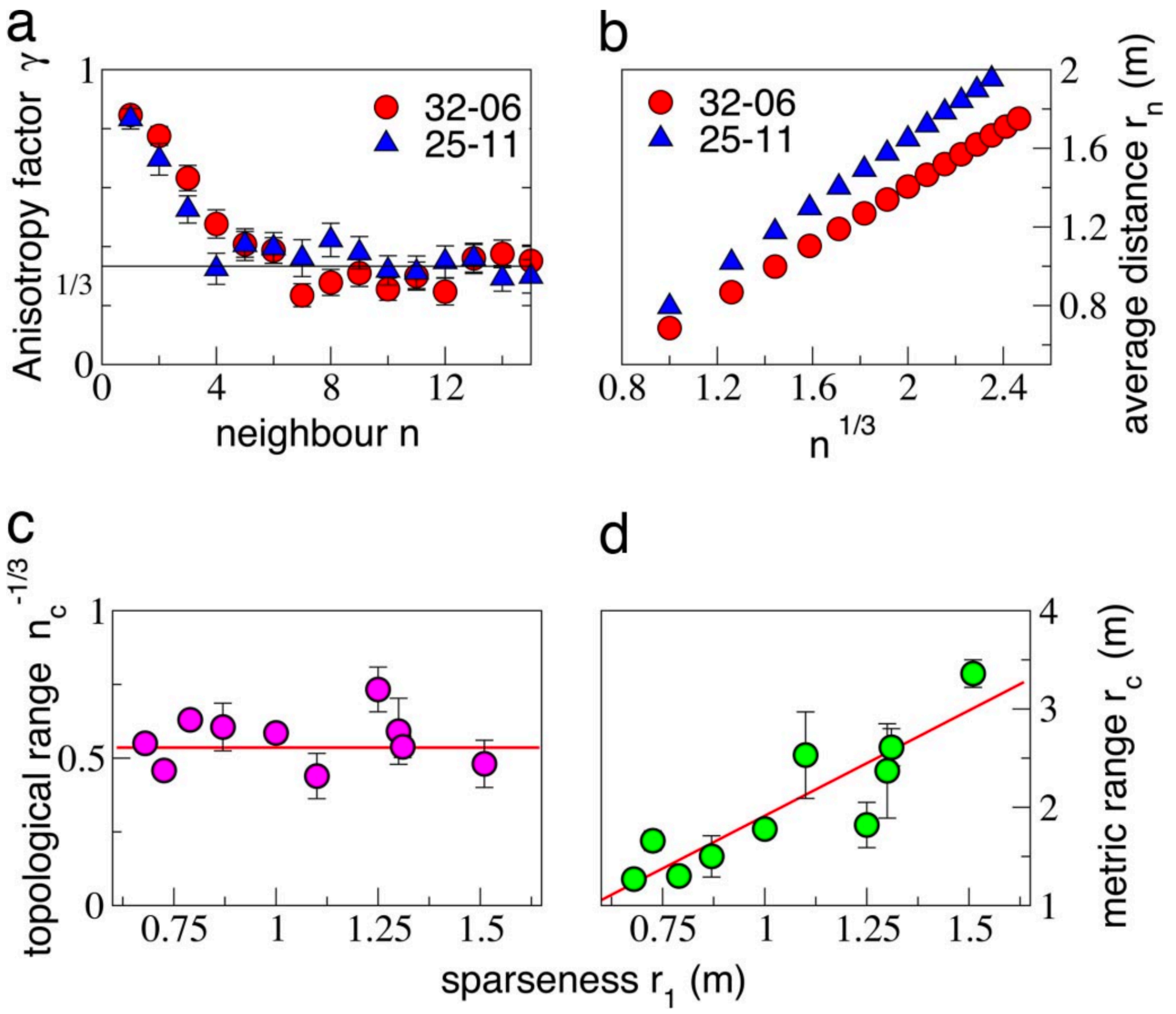
- To directly address the question, Ballerini (2008) et al obtained empirical data from a number of starling flocks. Using several cameras they obtained 3D reconstructions of the flocks.

- They computed an anisotropy coefficient $\gamma(n)$ which captures the extent to which the nth nearest neighbour is likely to be aligned to the average velocity of the flock $\mathbf{V}$. This is computed as: $\gamma(n) = (\mathbf{W}^{(n)} \cdot \mathbf{V})^2$, where $\mathbf{W}^{(n)}$ is the unitary eigenvector of the smallest eigenvalue of $M^{(n)}_{\alpha\beta} = \frac{1}{N}\sum \mathbf{u}^{(n)}_{i,\alpha}\mathbf{u}^{(n)}_{i,\beta}$ and $\mathbf{u}^{(n)}_{i}$ is the unit vector pointing in the direction of the nth nearest neighbor of bird i $(\alpha, \beta = x, y, z)$.

- For an isotropic, noninteracting distribution of points $\gamma = 1/3$, whereas the anisotropy increases towards $\gamma = 0{,}1$. So, they identify the value of n where γ reaches $\sim 1/3$.

M Ballerini et al, PNAS **105**(4): 1232-1237 (2008).

ARE INTERACTIONS METRIC OR TOPOLOGICAL?



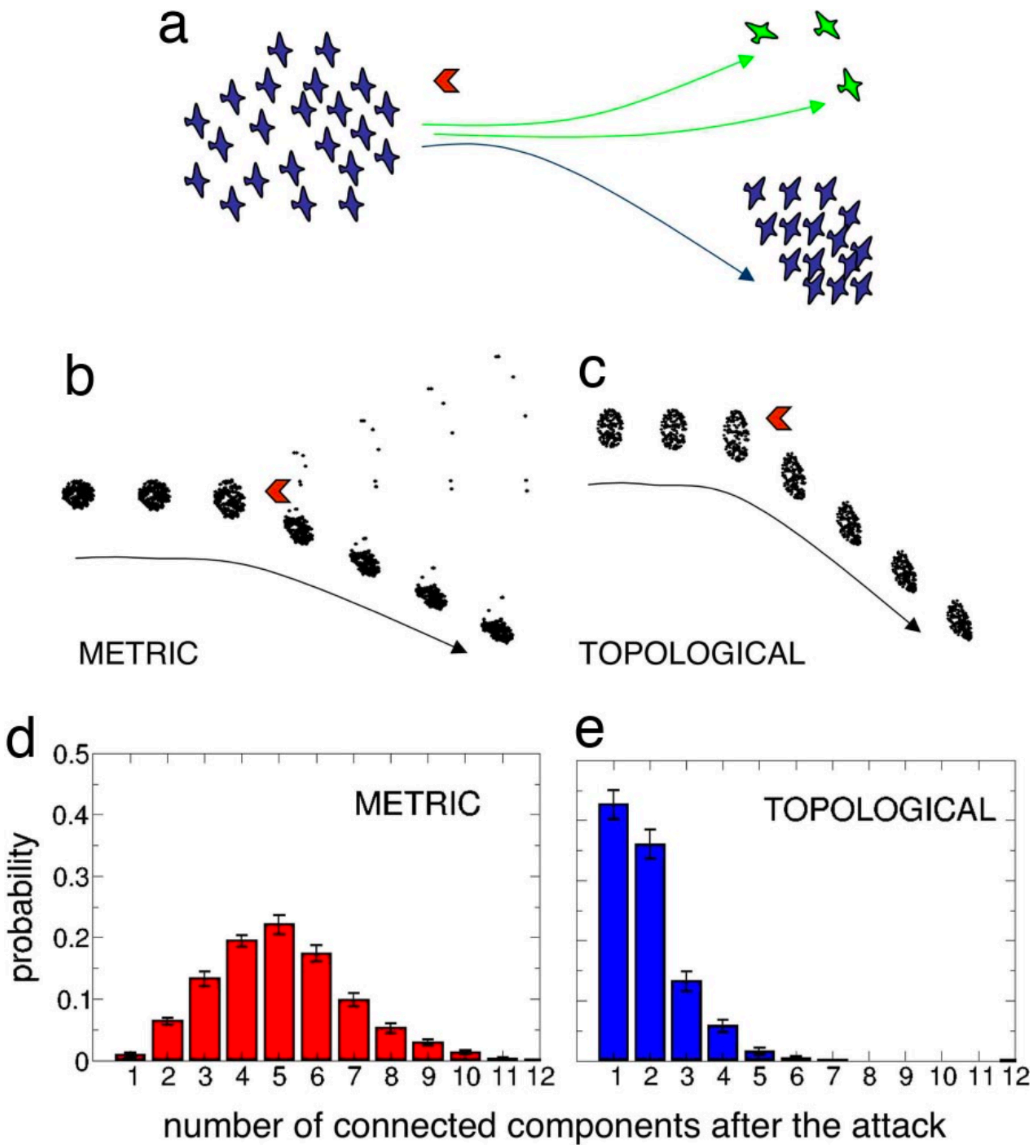



photo: Frans Lanting

FURTHER READING

Articles/Reviews

- T. Viscek et al, *Novel Type of Phase Transition in a System of Self-Driven Particles*, Phys. Rev. Lett. **75**, 1226-1229 (1995).
- M. Ballerini et al, I*nteraction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study*, Proc. Natl. Acad. Sci. USA **105**(4), 1232-1237 (2008).
- A. Cavagna & I. Giardina, *Bird Flocks as Condensed Matter*, Annu. Rev. Condens. Matter Phys. **5**, 183-207 (2014).
- S. Ramaswamy, *The Mechanics and Statistics of Active Matter*, Annu. Rev. Condens. Matter Phys. **1**: 323-345

THANK

YOU!

source: https://www.youtube.com/watch?v=tDQw21ntR64