Evolutionarily Stable Strategy:-

3 critical changes from classical theory

Strategy: ~~As~~ Agents don't choose from a set. ~~Each~~ agent is endowed with a strategy which they inherit (with possible mutations) → Could be cultural inheritance, genetical inheritance

Interactions: repeated, random pairing of agents to play. → Information available usually doesn't matter as there is no decision making.

Equilibrium: Changed the question to "what is an evolutionarily stable strategy" for the population? → i.e. if the population uses that strategy, it can't be invaded by a small group of mutant agents.

Denote a strategy in the population by σ

$\sigma = (p_1, p_2 \dots p_m)$ $\sum_{i=1}^{m} p_i = 1$ m pure strategies

↳ A particular mixing over pure strategies.

σ is an ESS if

Expected payoff of an agent playing σ against another strategy τ is $E(\sigma,\sigma) > E(\tau,\sigma)$ or if $E(\sigma,\sigma) = E(\tau,\sigma)$ then $E(\sigma,\tau) > E(\tau,\tau)$

$E(\sigma,\sigma) > E(\tau,\sigma)$ → mutant shouldn't perform better against incumbent than incumbent against itself

If the mutant perform equally well $\boxed{E(\sigma,\sigma) = E(\tau,\sigma)}$ then $E(\sigma,\tau) > E(\tau,\tau)$ → then incumbent performs better against mutant than mutant against itself.

So strict Nash equilibriums are ESS.

Since there is no concept of row player, coloumn player etc we can apply ESS to only symmetric games. We can't directly apply the idea of ESS to Battle of Sexes. ~~also~~ we can apply after symmetrising the game.

Note that an ESS could be invaded by ~~multi~~ the entry of multiple mutants at the same time.

ESS $\subset$ NE(G)

ESS mayn't exist for games; eg Rock, Paper, Scissors

Q) What is the ESS of the game Stag hunt

	Stag	Hare
Stag	4,4	0,2
Hare	2,0	1,1

Nash equilibria are (Stag, Stag) (Hare, Hare)

$\sigma^* = ((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3}))$

$$W_1 = 4pq + 0p(1-q) + 2(1-p)q + (1-p)(1-q)$$
$$= 4pq + 2q - 2pq + 1 - q - p + pq$$
$$= 3pq + q + 1 - p \quad = 3p^2 + \cancel{p}$$
$$\frac{\partial W_1}{\partial p} = 3q - 1 = 0 \quad q^* = \frac{1}{3} \quad \text{by symmetry } p^* = \frac{1}{3}.$$

a) Is (Stag, Stag) an ESS

E(Stag, Stag) $\overset{?}{>}$ E(Hare, Stag) Yes. So it is an ESS

b) Is (Hare, Hare) an ESS

E(Hare, Hare) $\overset{?}{>}$ E(Stag, Hare) Yes. So it is an ESS

c) Is σ^* an ESS

$4 \times \frac{1}{9} + \frac{1}{3} \times \frac{2}{3} + 2$

$E((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3})) > E(p', (\frac{1}{3}, \frac{2}{3}))$

$3\frac{1}{9} + \frac{1}{3} + 1 - \frac{1}{3} \overset{?}{>} 3p'\frac{1}{3} + \frac{1}{3} + 1 - p'$

$\frac{4}{3} > \frac{4}{3}$ So they are equal

$E((\frac{1}{3}, \frac{2}{3}), p') \overset{?}{>} E(p', p')$

$\cancel{3} \times \frac{1}{\cancel{3}} p' + p' + 1 - \frac{1}{3} \overset{?}{>} 3p'^2 + p' + 1 - p'$

$2p' + \frac{2}{3} \overset{?}{>} 3p'^2 + 1$

↳ always higher for any p'

Q) Find ESS of Hawk-Dove game

	Dove	Hawk
Dove	2, 2	1, 4
Hawk	4, 1	0, 0

three Nash equilibria
(Dove, Dove), (Hawk, Hawk)
$\sigma^* = \left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$

$W_1 = 2pq + p(1-q) + 4(1-p)q = p + 4q - 3pq$

a) Is (Dove, Dove) an ESS?
$E(\text{Dove}, \text{Dove}) \overset{?}{>} E(\text{Hawk}, \text{Dove})$ No So (Dove, Dove) is not an ESS

b) Is (Hawk, Hawk) an ESS
$E(\text{Hawk}, \text{Hawk}) \overset{?}{>} E(\text{Dove}, \text{Hawk})$ No So (Hawk, Hawk) is not an ESS

$\frac{2}{3}p' + p'\frac{2}{3} + (1-p')\frac{1}{3}4 + (1-p')\frac{2}{3}0$

$\frac{2}{9} + \frac{2}{9} + 4\frac{2}{9}$

c) Is σ^* an ESS

$\cancel{E(\sigma^*, \sigma^*) \overset{?}{>} E(} E\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right) \overset{?}{>} E\left(p', \left(\frac{1}{3}, \frac{2}{3}\right)\right)$

$\frac{1}{3} + 4\frac{1}{3} - \frac{3}{9} > p' + 4\frac{1}{3} - \frac{3}{3}p'$

$4\frac{1}{3} > 4\frac{1}{3}$ So they are equal.

$E\left(\left(\frac{1}{3}, \frac{2}{3}\right), p'\right) \overset{?}{>} E(p', p')$

$\frac{1}{3} + 4p' - \cancel{3}\frac{1}{\cancel{3}}p' \overset{?}{>} p' + 4p' - 3p'^2$

$\frac{1}{3} + 3p' \overset{?}{>} 5p' - 3p'^2$

$3p'^2 \overset{?}{>} 2p' - \frac{1}{3}$

↳ always lower for any p'

The concept of an ESS doesn't involve time or reproduction in an explicit way.

We can give it an evolutionary dynamical interpretation: Start out with a large population involving all possible strategies we are interested in.

Implement payoff dependent random pairwise interactions between strategies.

~~payoff~~ strategies with higher ~~strategies~~ payoff survive and others die out (or they copy successful strategies).

→ Final steady state will be an ESS

Q) Implement dynamics for the Hawk-dove game and S.T it goes to ESS.

How do you find ESS of a game?

Find all symmetric Nash equilibria.

Test for ESS Conditions.

Q) Can we define a dynamic process that will mimick the evolution of strategies? If yes whether the asymptotic state of the dynamic process will be an ESS? → justification of Nash equilibrium or ESS in terms of dynamical systems.

end points of dynamic is often ESS.

There are various ways to specify ~~the~~ an evolutionary dynamic → A representative one is the replicator dynamics.

Replicator dynamics

At time t, a population has $N(t)$ agents. Each agent plays one of k pure strategies.

Let $N_i(t)$ be the number of agents playing pure strategy i at time t.

$$\sum_{i=1}^{k} N_i(t) = N(t)$$

Let $x_i(t)$ be the fraction of agents playing pure strategy i at time t.

$$x_i(t) = \frac{N_i(t)}{N(t)} \qquad \sum_{i=1}^{k} x_i(t) = 1$$

time is treated as continuous. So between times t and $t+\Delta t$, agents pair up randomly and play the game yielding them some payoff u. $\longrightarrow$ expected

[no mutation breeds true]

payoff $\equiv$ no. of progenies per unit time.

$u(i, x(t)) \longrightarrow$ Expected payoff of a single agent playing pure strategy i at time t when population is in state $x \equiv [x_1(t)\, x_2(t) \ldots x_k(t)]$

How do number of agents playing pure strategy i change with time

$$N_i(t+\Delta t) = N_i(t) + u(i, x(t))\, N_i(t)\, \Delta t$$

$$\lim_{\Delta t \to 0} \frac{N_i(t+\Delta t) - N_i(t)}{\Delta t} = u(i, x(t))\, N_i(t)$$

$$\frac{dN_i(t)}{dt} = u(i, x(t))\, N_i(t)$$

We want to write this in terms of $x_i(t)$.

$$\frac{d[x_i(t) N(t)]}{dt} = u(i, x(t))\, x_i(t)\, N(t)$$

$$N(t)\frac{dx_i(t)}{dt} + x_i(t)\frac{dN(t)}{dt} = u(i, x(t))\, x_i(t)\, N(t)$$

but $\displaystyle \frac{dN(t)}{dt} = \sum_{i=1}^{k} u(i, x(t))\, N_i(t) = \underbrace{\sum_{i=1}^{k} u(i, x(t))\, \overset{\frac{N_i(t)}{N(t)}}{x_i(t)}}_{\text{avg no. of progenies per agent per unit time of the entire population } u(x,x)} N(t)$

$\cancel{=}$

or $u(x,x) \equiv$ avg fitness of the population.

$$N(t)\frac{dx_i(t)}{dt} + x_i(t)\,u(x,x)\,N(t) = u(i,x(t))\,x_i(t)\,N(t)$$

$$\frac{dx_i(t)}{dt} = \left[u(i,x(t)) - u(x(t),x(t))\right]x_i(t)$$

fitness of type i > avg fitness $\Rightarrow$ proportion of type i increases in the population.

Replicator dynamics for Hawk-dove game

Dove	2,2	1,4
Hawk	4,1	0,0

$i = \{\text{Dove, Hawk}\}$

Consider a population in which fraction $x_D(t)$ are doves & $x_H(t)$ are Hawks. $x_H(t) = 1 - x_D(t)$

$$\frac{dx_D(t)}{dt} = \left[u(D,x_D(t)) - u(x,x)\right]x_D(t) \quad \text{(1)}$$

$x(t) = [x_D(t), x_H(t)]$

$u(D,x(t)) = 2x_D(t) + (1 - x_D(t)) \rightarrow$ expected payoff of a Dove at time t, assuming random pairing.

$$\begin{aligned}
u(x,x) &= x_D(t)\,u(D,x(t)) + x_H(t)\,u(H,x(t)) \\
&= 2x_D^2(t) + x_D(t)(1 - x_D(t)) + (1 - x_D(t))\left[4x_D(t) + 0\,x_H(t)\right] \\
&= 2x_D^2(t) + x_D(t) - x_D^2(t) + 4x_D(t) - 4x_D^2(t) \\
&= 5x_D(t) - 3x_D^2(t)
\end{aligned}$$

Fixed points of eq (1) are $x_D(t) = 0$ and those given by

$$2x_D + 1 - x_D = 5x_D - 3x_D^2$$

$3x_D^2 = 4x_D - 1 \qquad 3x_D^2 - 4x_D + 1 = 0 \qquad x_D = 1,\ x_D = \frac{1}{3}$

To check stability consider

$$\frac{\partial x_D(3x_D^2 - 4x_D + 1)}{\partial x_D} = \frac{\partial(3x_D^3 - 4x_D^2 + x_D)}{\partial x_D} = 9x_D^2 - 8x_D + 1$$

At $x_D = 0$, $+1$ unstable

$x_D = 1$, $+2$ unstable

$x_D = \frac{1}{3}$, $9 \times \frac{1}{9} - 8\frac{1}{3} + 1 = -\frac{2}{3}$ stable

Replicator Dynamics for Stag-Hunt

Stag	4,4	0,2
Hare	2,0	1,1

fraction of Stag x_S

fraction of Hare $x_H = 1 - x_S$

$$\frac{dx_S}{dt} = \left[u(\text{stag}, x) - u(x, x)\right] x_S$$

$$u(\text{stag}, x) = 4x_S + 0x_H$$

$$u(x,x) = x_S\left[4x_S + 0x_H\right] + (1-x_S)\left[2x_S + 1(1-x_S)\right]$$

$$= 4x_S^2 + (1-x_S)(x_S+1)$$

$$= 4x_S^2 - x_S^2 + 1 = 3x_S^2 + 1$$

Fixed points are $x_S = 0$ and solution of

$4x_S = 3x_S^2 + 1$ $\quad$ $3x_S^2 - 4x_S + 1 = 0$

$x_S = 1$ & $x_S = \frac{1}{3}$

Checking stability $\frac{\partial x_S(4x_S - 3x_S^2 - 1)}{\partial x_S}$

$$= \frac{\partial(4x_S^2 - 3x_S^3 - x_S)}{\partial x_S} = 8x_S - 9x_S^2 - 1$$

$x_S = 0$ $\quad -1$ $\quad$ stable

$x_S = 1$ $\quad -2$ $\quad$ stable

$x_S = \frac{1}{3}$ $\quad 8 \times \frac{1}{3} - 9 \times \frac{1}{9} - 1 = \frac{2}{3}$ unstable